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Stochastic Inflation and the Term Structure of Interest Rates: a Simple Diffusion Model

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Abstract

The present paper addresses the problem of valuing contingent claims on the term structure in a single good economy under uncertain inflation. In the context of arbitrage-free valuation, a simple diffusion model for pricing inflation -indexed securities is proposed . A martingale characterization of nominal and real prices is given and a stochastic generalization of the Fisher equation is provided . An example of two -factor model which can be used to value inflation-linked securities in practical applications, is also discussed.

JEL classification numbers: G00, G10, G30.
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1 Introduction

In a seminal paper, Cox, Ingersoll, and Ross (1981) obtained, within the context of a general equilibrium model of the economy, a functional relation between nominal and real interest rates. Such a relation can be considered as a stochastic generalization of the Fisher equation which has been derived in a deterministic framework at the end of the last century Fisher (1896). In a single good economy, Cox, Ingersoll and Ross proposed an equilibrium multi-factor model in which the price level is taken as one of the economic fundamentals used to value nominal and real securities in presence of uncertain inflation. Under the hypothesis that the state variables follow diffusion processes, they found that the nominal spot rate is decomposable into the sum of the real interest rate and the expected rate of inflation plus some terms depending on the stochastic nature of the price level process, and on the market prices for risk Cox, Ingersoll, and Ross (1988a) and Cox, Ingersoll, and Ross (1988b). Such extra terms disappear if the price level is assumed to evolve in time in a deterministic way so that the Fisher effect returns to its original version in which the nominal spot rate is equal to the sum of the real interest rate and the rate of inflation.

A completely different approach is followed in this paper, where a no-arbitrage, multi-factor model of the term structure of interest rates incorporating the effects of stochastic inflation is proposed. As pointed out by Pennacchi (1991), a basis for this research is the presumption that a suitable model of the term structure of interest rates requires multiple sources of uncertainty. Empirical evidence supports this view. Stambaugh (1988) considers a latent variable model of the term structure and finds that a two or three state variables model is sufficient to characterize Treasury-bill returns. Although his analysis does not attempt to identify these variables, he suggests that variables that determine the real interest rate and the expected rate of inflation would be obvious candidates. Moreover, Brown and Schaefer (1994) emphasize that several countries (the United Kingdom in particular) have issued inflation-linked securities, and that therefore, a theory of the term structure of nominal and real interest rates is central to the demand for assets and investments. This topic is of great interest as evidenced by many recent contributions to the literature Mkaouar, Prigent, and Abid (2017) and Singor et al. (2013). Having simple models with a good trade-off between financial significance and mathematical tractability is central to operational applications for pricing inflation-linked securities.

In this paper, we model a single good economy in which (default-free) contingent claims on the term structure are traded at nominal prices. The dynamics of the model is determined by defining the stochastic movements of the *nominal* forward rate and the price process of the consumption good taken as the price level to characterize the inflation process. The proposed approach is based on the pricing methodology proposed by Heath, R. Jarrow, and Morton (1992) and Heath, R. Jarrow, and Morton (1990) and aims to solve the problem of valuing contingent claims on the term structure under uncertain inflation conditions. Our approach differs from that proposed by R.A. Jarrow and Yildirim (2003) because in our model the dynamics of the real forward rate is endogenous. This fact allows us to build parsimonious models that are well suited for operational applications.

The purpose is to value nominal and real securities, taking as given the prices of the nominal zero coupon bonds and the price level process. Within this context, a martingale characterization of nominal and real prices is obtained. In fact, it will be shown that discounted (at the nominal rate) nominal prices, as well as discounted (at the real rate) real prices, are martingales with respect to well defined *risk-neutral* measures Evans (2003). Furthermore, the paper provides a simple and original derivation of the stochastic Fisher effect as a natural consequence of the no-arbitrage principle and of the martingale representation of nominal and real prices. Our result is in agreement with that derived by Cox, Ingersoll, and Ross (1981) within the context of a general equilibrium model of the economy and generalizes that obtained by Richard (1978). In addition, it will be shown that the nominal spot-rate can be decomposed into the sum of the real interest rate plus the expected rate of inflation, where the expectation is taken under the nominal risk-neutral measure. From this point of view, the Fisher effect survives in an economy with risk-neutral individuals Lioui and Poncet (2004).

In the second part of the paper we propose a two-factor model which can be used for practical applications. In fact, most pricing formulas show a simpler functional form. In particular, the nominal discount function factorizes into the product of the *real dis-*

count function and of a sort of *inflation discount function*. Furthermore, the nominal term structure, expressed in terms of the yield to maturity, can be represented as two superimposed term structures, namely the real yield plus the inflation yield.

The proposed methodology can be generalized to prices contingent claims on foreign currencies Amin and R.A. Jarrow (1991). In such a case, the analogy is obtained by replacing the prices level process with that of the spot exchange rate.

The scheme of the paper is the following. In Section 2, we illustrate the main assumptions about the underlying economy, and then we discuss an arbitrage-free multifactor model of the term structure of interest rates which can be used to value nominal and real securities under uncertain inflation. A martingale characterization of nominal and real prices is provided and discussed. From this, a stochastic generalization of the Fisher equation is obtained. In Section 3, a two-factor model which can be used for practical purposed is discussed. Securities such as price-index linked bonds, pension funds etc., are correctly valued in an economic framework in which both the nominal and the real components of the term structure must be accounted for. In Section 4, an example of two-factor model is proposed to value arbitrage-free prices of well defined inflation-linked securities. Finally, in Section 5, some comments conclude the paper.

2 The model

2.1 The general economic setting

We consider a single good, continuous time-state economy with finite horizon $[0, \tau]$, in which (default free) contingent claims on the term structure are traded at nominal prices. Market are ideal in the sense that they are frictionless and competitive, i.e. there are no taxes or transaction costs and all investors act as price takers. Furthermore, markets are open continuously and there are no constraints on traded quantities and on short sales. The internal consistency of the model is assured by requiring the absence of arbitrage profit opportunities. Within this framework we denote by $f(t, T)$, $t < T$, the (instantaneous) nominal forward rate and by

$$v(t,T) = \exp\left[-\int_t^T f(t,u)du\right], \tag{1}$$

the nominal price at time t, of zero coupon bond with nominal face value $F = 1$, maturing at time T. The nominal spot-rate is then given by

$$r(t) = f(t,t). \tag{2}$$

Furthermore, we denote by $p(t)$ the *price level*, i.e. the nominal price of one unit of the consumption good. As we will see later, under the no-arbitrage assumption, the dynamics of $f(t,T)$, and $p(t)$, determines the price processes of all (default-free) zero coupon bonds and of all contingent claims under stochastic inflation. Within the context of a pure arbitrage model, the processes specifying the evolution of $f(t,T)$, and $p(t)$ are exogenously given.

Assumption 1. The dynamics of the model is specified by the following system of diffusion-type stochastic differential equations

$$\frac{dp(t)}{p(t)} = y(t)dt - \sum_{i=1}^{n} \sigma_{pi}(t)dw_i(t), \tag{3}$$

$$df(t,T) = \mu(t,T)dt + \sum_{i=1}^{n} \sigma_i(t,T)dw_i(t), \tag{4}$$

where $w_1(t), ..., w_n(t)$ are n independent Brownian motions on a given filtered probability space $(\Omega, \Im, \{\Im_t\}, P)$, where $\{\Im_t\}$ denotes the natural filtration. The model functions μ and σ_i $(i = 1, 2, ..., n)$ may also depend on $f(t,T)$ and on $p(t)$; the functions y and σ_{pi} $(i = 1, 2, ..., n)$ on $p(t)$, and on $r(t)$. All the dynamical functions, namely the drift and the diffusion coefficients, are assumed to be $\Im_t$ measurable and continuous functions of their arguments[1]. $\diamond$

We assume that Condition A of Friedman (1975) holds. This implies that the unique solution to the system of stochastic differential equation (3) is a diffusion process belonging to $M_w^2[0,\tau]$, where $M_w^2[0,\tau]$ denotes the class of all non-anticipative functions $f(t)$ satisfying

$$E\left[\int_0^\tau |f(t)|^2 dt\right] < \infty.$$

It is to be noted that, by definition, $y(t)$ is the expected rate of inflation, namely

$$y(t) = E_t\left[\frac{dp(t)}{p(t)dt}\right]. \tag{5}$$

A straightforward application of Ito's lemma to expression (1) gives the dynamics of the nominal discount function $v(t,T)$,

$$\frac{dv(t,T)}{v(t,T)} = \left[r(t) - \int_t^T \mu(t,u)du + \frac{1}{2}\sum_{i=1}^n \sigma_{vi}^2(t,T)dt\right] - \sum_{i=1}^n \sigma_{vi}(t,T)dw_i(t), \tag{6}$$

where

$$\sigma_{vi}(t,T) = \int_t^T \sigma_i(t,u)du \qquad (i = 1, 2, ..., n). \tag{7}$$

Assumption 2. Fix $T_1, T_2, ..., T_n$ such that $0 < T_1 < T_2 < ... < T_n < \tau$ and $t < T_1$. We assume that

$$\hat{\sigma}(t) = \begin{pmatrix} \sigma_{v1}(t,T_1) & \sigma_{v2}(t,T_1) & ... & \sigma_{vn}(t,T_1) \\ \sigma_{v1}(t,T_2) & \sigma_{v2}(t,T_2) & ... & \sigma_{vn}(t,T_2) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{v1}(t,T_n) & \sigma_{v2}(t,T_n) & ... & \sigma_{vn}(t,T_n) \end{pmatrix} \tag{8}$$

is nonsingular a.e. in the product measure $P \times L[0,T_1]$, where $L[0,T_1]$ is the Lebesgue measure of the interval $[0,T_1]$. This assumption implies that the market is complete, in the sense that any random variable $(X_T, T < T_1)$ specifying the payout of a contingent claim, is attainable by using admissible self-financing trading strategies Harrison and Pliska (1981) involving the basis bonds with maturities $T_1, T_2, ..., T_n$. $\diamond$

The above assumptions clarify in a formal way the distinction between nominal interest rate risk and real interest rate risk Bomfim (2001) and Laubach and Williams (2003). In fact, Assumption 1 leaves open the possibility that some of the diffusion coefficients of the consumption good price process σ_{pi} are zero (see Section III, and Section IV for an example). This means that possibly m $(m < n)$ sources of risk drive the evolution of the price level $p(t)$; nevertheless Assumption 2 requires that the information carried by $p(t)$ is contained in the dynamics of the bond prices where $n-m$

additional sources of risk are also present.

Under Assumption 2, a no-arbitrage argument implies that $\forall t$, and $\forall T$, $t < T < T_1$, the drift coefficient of $v(t,T)$

$$\mu_v(t,T) = \left[r(t) - \int_t^T \mu(t,u)du + \frac{1}{2}\sum_{i=1}^{n} \sigma_{vi}^2(t,T) \right], \tag{9}$$

can be written in the following form

$$\mu_v(t,T) = r(t) + \sum_{i=1}^{n} q_i(t)\sigma_{vi}(t,T), \tag{10}$$

for some $q_i(t)$ $(i = 1, 2, ..., n)$, the market prices for risk, that result uniquely determined from the dynamics and are independent of the chosen basis bonds, i.e. independent of the vector $(T_1, T_2, ..., T_n)$. The proof of this statement can be found in Heath, R. Jarrow, and Morton (1992). Condition (10) is also equivalent to a restriction on the drift of $f(t,T)$, namely

$$\mu(t,T) = -\sum_{i=1}^{n} \sigma_i(t,T)[q_i(t) - \sigma_{vi}(t,T)], \tag{11}$$

relating the drift coefficient of the nominal forward rate process to the market prices for risk.

2.2 The nominal economy

After the suitable change of the probability measure induced by the following transformation of Brownian motions

$$w_i^*(t) = w_i(t) - \int_0^t q_i(u)du \qquad (i = 1, 2, ..., n), \tag{12}$$

Equations (4), and (6) can be rewritten respectively in the following form

$$df(t,T) = \sum_{i=1}^{n} \sigma_i(t,T)\sigma_{vi}(t,T)dt + \sum_{i=1}^{n} \sigma_i(t,T)dw_i^*(t), \tag{13}$$

$$\frac{dv(t,T)}{v(t,T)} = r(t)dt - \sum_{i=1}^{n} \sigma_{vi}(t,T)dw_i^*(t). \tag{14}$$

Let us call the new probability measure, obtained by the transformation of Brownian motions (12), *nominal risk-neutral measure*. The definition is justified by the fact that if we compute the expected (with respect to the nominal risk-neutral measure) rate of return of a T-maturing bond in excess of the nominal spot rate $r(t)$, we get from equation (14) identically zero. In the nominal economy, it follows, from equation (14), that relative nominal prices, i.e. $r(t)$-discounted prices

$$N(t,T) = e^{-\int_0^t r(u)du} v(t,T), \tag{15}$$

are martingales with respect to the nominal risk-neutral measure.

Contingent claims on the term structure are valued on the same ground Harrison and Kreps (1979) and Harrison and Pliska (1981). If we denote by $C(t, X_T)$ the current nominal value of a security characterized by the non-negative ($\Im_T$ measurable) nominal random payout X_T at time $T < T_1$ with $E^*[\exp(-\int_0^T r(u)du)X_T] < \infty$, in absence of arbitrage, the following theorem 2.1 holds[2].

Theorem 2.1. The processes

$$N(t, X_T) = e^{-\int_0^t r(u)du} C(t, X_T), \tag{16}$$

are martingales with respect to the nominal risk-neutral measure induced by the transformation of Brownian motions

$$w_i^*(t) = w_i(t) - \int_0^t q_i(u)du \qquad (i = 1, 2, ..., n). \tag{17}$$

Proof. See Harrison and Pliska (1981). $\diamond$

According to 2.1, nominal prices of contingent claims on the term structure can be calculated by means of the terminal condition

$$C(t, X_T) = E_t^* \left[e^{-\int_t^T r(u)du} X_T \right]. \tag{18}$$

We have implicitly assumed that no dividends are paid by the securities. This is not restrictive and the generalization to the case of contingent claims that pay dividends is straightforward.

As pointed out by Harrison and Pliska (1981), the nominal prices dynamics is then specified by the following stochastic differential equation that in the nominal risk-neutral measure can be written as

$$\frac{dC(t, X_T)}{C(t, X_T)} = r(t)dt - \sum_{i=1}^{n} \sigma_{ci}(t, X_T) dw_i^*(t), \tag{19}$$

where $r(t)$ is the nominal spot-rate[3].

2.3 The real economy

It is interesting to give a different characterization of the economy. We define, therefore, *real economy* the same economy in which, after a change of the numeraire, the prices of contingent claims are expressed in units of the good. Let us denote, therefore, by $\tilde{C}(t, \tilde{X}_T)$ the real price of a security characterized by a random payout $\tilde{X}_T$ at time $T < T_1$, expressed in units of the good. Since $\tilde{X}_T$ units of the good correspond to a nominal amount of $X_T = p(T)\tilde{X}_T$, the following relation holds

$$\tilde{C}(t, \tilde{X}_T) = \frac{C(t, X_T)}{p(t)}. \tag{20}$$

The theorem 2.2 below furnishes a characterization of the real prices in terms of martingales with respect to a well defined measure.

Theorem 2.2. The processes

$$R(t, T) = e^{-\int_0^t x(u)du} \, \tilde{C}(t, \tilde{X}_T), \tag{21}$$

are martingales with respect to the probability induced by the following transformation of Brownian motions

$$\bar{w}_i(t) = w_i(t) - \int_0^t \bar{q}_i(u)du \qquad (i = 1, 2, ..., n). \tag{22}$$

We assume that the diffusion coefficients σ_{ci} $(i = 1, 2, ..., n)$ belong to $M_w^2[0, T]$.

The process $x(t)$ describes the real interest rate process, and it is given by

$$x(t) = r(t) - y(t) + \sum_{i=1}^{n} \sigma_{pi}^2(t) + \sum_{i=1}^{n} \sigma_{pi}(t)\bar{q}_i(t), \tag{23}$$

where the quantities

$$\bar{q}_i(t) = q_i(t) - \sigma_{pi}(t) \qquad (i = 1, 2, ..., n), \tag{24}$$

are the market prices for risk in the real economy.

Proof. See Appendix. $\diamond$

In analogy with the nominal case, the probability measure induced by the transformation of Brownian motions (22) is called *real risk-neutral measure*. The definition is justified by noticing that if we calculate the expected (under the real risk-neutral measure) rate of return of a real security in excess of the real rate, we get identically zero.

Equation (24) expresses the link between the market prices for risk in the real and in the nominal economy. It specifies that $\bar{q}_i$ is identically equal to q_i if and only if the i-th sources of risk does not affect the dynamics of the price level. We must notice that equation (23) can be viewed as a stochastic generalization of the Fisher effect Fisher (1896). In fact, by rewriting it as

$$r(t) = x(t) + y(t) - \sum_{i=1}^{n} \sigma_{pi}^2(t) - \sum_{i=1}^{n} \sigma_{pi}(t)\bar{q}_i(t), \tag{25}$$

we see that the nominal interest rate decomposes into the sum of the real interest rate plus the expected rate of inflation, minus some terms depending on the diffusion coefficients of the price level process. Depending on the values of the market prices for risk in the real economy, the nominal spot rate may be either greater or less than the sum of the real interest rate and the expected rate of inflation. It is be noted that if we assume a deterministic evolution of the prices process $p(t)$, we recover the original form of the Fisher equation $r(t) = x(t) + y(t)$ in which the nominal spot rate is decomposed into the sum of the real rate plus the rate of inflation. In addition, it is interesting to remark that equation (25) can be rewritten as

$$r(t) = x(t) + y^*(t), \tag{26}$$

where $y^*(t)$ is the expected rate of inflation under the nominal risk-neutral measure

$$y^*(t) = E_t^*\left[\frac{dp(t)}{p(t)dt}\right]. \tag{27}$$

In this case too, the Fisher equation can be achieved in the original functional form. It consents to decompose the nominal interest rate as the sum of the real interest rate plus the expected rate of inflation calculated with respect to the nominal risk-neutral measure. From this point of view, the Fisher effect survives in an economy with risk-neutral individuals.

Before ending this Section, we note that it is possible to express equation (3) in the following useful form

$$\frac{dp(t)}{p(t)} = y^*(t)dt - \sum_{i=1}^{n} \sigma_{pi}(t)dw_i^*(t), \tag{28}$$

thus showing that the process

$$D(t) = e^{-\int_0^t y^*(u)du} p(t), \tag{29}$$

is a martingale under the nominal risk-neutral measure

$$p(t) = E_t^*\left[e^{-\int_t^T y^*(u)du} p(T)\right] \qquad (T > t). \tag{30}$$

3 A two-factor model

In this Section we discuss a two-factor specification of the aforementioned theory which can be used for practical purpose. Let us express the nominal forward rate as

$$f(t,T) = x(t,T) + y^*(t,T), \tag{31}$$

which is reminiscent of the Fisher decomposition of the nominal spot rate. In the case $T = t$, the functions $x(t,T)$ and $y^*(t,T)$ are constrained by the Fisher equation to become $x(t,t) = x(t)$ (the real rate), and $y^*(t,t) = y^*(t)$ (the expected rate of inflation under the nominal risk-neutral measure) respectively. From this point of view $x(t,T)$ and $y^*(t,T)$ can be considered as the real component and the inflation component of the nominal forward rate. We assume that these components are stochastically independent and that the real component is also stochastically independent of the (consumption good) price process. We state therefore the following,

Assumption 3. In the risk-neutral nominal economy the dynamics is specified by the following system of diffusion-type stochastic differential equations (see equation (13))

$$\frac{dp(t)}{p(t)} = y^*(t)dt - \sigma_p(t)dw_I^*(t), \tag{32}$$

$$dx(t,T) = \sigma_R(t,T)\sigma_{vR}(t,T)dt + \sigma_R(t,T)dw_R^*(t), \tag{33}$$

$$dy^*(t,T) = \sigma_I(t,T)\sigma_{vI}(t,T)dt + \sigma_I(t,T)dw_I^*(t), \tag{34}$$

where

$$\sigma_R(t,T) = \sigma_R(x(t,T),t,T),$$

$$\sigma_I(t,T) = \sigma_I(y^*(t,T),p(t),t,T),$$

$$\sigma_p(t) = \sigma_p(y^*(t),p(t),t),$$

and, according to Equation (7)

$$\sigma_{vi}(t,T) = \int_t^T \sigma_i(t,u)du \qquad (i = R, I). \tag{35}$$

$w_R^*(t)$, and $w_I^*(t)$ are two independent Brownian motions on a given filtered probability space $(\Omega, \Im, \{\Im_t\}, P^*)$, and $\{\Im_t\}$ is the natural filtration. $\diamond$

Under Assumption 3, the dynamics of the model partially decouples thus getting

$$dx(t,T) = \sigma_R(x,t,T)\sigma_{vR}(x,t,T)dt + \sigma_R(x,t,T)dw_R^*(t), \tag{36}$$

and

$$\begin{aligned} \frac{dp(t)}{p(t)} &= y^*(t)dt - \sigma_p(t)dw_I^*(t), \\ dy^*(t,T) &= \sigma_I(t,T)\sigma_{vI}(t,T)dt + \sigma_I(t,T)dw_I^*(t), \end{aligned} \tag{37}$$

where the coefficients $\sigma_I(t,T)$, $y^*(t)$, and $\sigma_p(t)$ do not depend on $x(t,T)$, and $x(t)$.

3.1 The real economy

Under Assumption 3, we will prove that most pricing formulas show a simpler functional form. In particular we will show that the nominal discount function factorizes into the product of the *real discount function* and of the *inflation discount function*, and that the nominal term structure, expressed in terms of the yield to maturity, can be represented as two superimposed term structures, namely the real yield plus the inflation yield.

Let us denote by $b(t,T)$ the real price, at time t, of a real bond paying one unit of the consumption good at time T. Since we have assumed that in our model only one source of uncertainly drives the evolution of the price level process $p(t)$, it follows that $\bar{q}_R(t) = q_R(t)$, i.e. the market price for real rate risk is the same both in the nominal and in the real economy. In addition, by applying the composition (20) and 2.1, we get

$$p(t)b(t,T) = E_t^*\left[e^{-\int_t^T r(u)du}p(T)\right], \tag{38}$$

and by recalling the martingale feature of $p(t)$ we finally obtain

$$b(t,T) = E_t^*\left[e^{-\int_t^T x(u)du}\right], \tag{39}$$

where the Fisher equation $r(t) = x(t) + y^*(t)$ and Assumption 3 have been used. Therefore, $b(t,T)$ can be viewed as the real discount function. In a similar way, the nominal price at time t of a nominal bond paying the face value $F = 1$ at maturity T, is given by

$$v(t,T) = \exp\left[-\int_t^T x(t,u)du - \int_t^T y^*(t,u)du\right] = E_t^*\left[e^{-\int_t^T r(u)du}\right]. \tag{40}$$

Under Assumption 3, the above equation factorizes into

$$v(t,T) = E_t^*\left[e^{-\int_t^T x(u)du}\right] E_t^*\left[e^{-\int_t^T y^*(u)du}\right] = b(t,T)h(t,T), \tag{41}$$

where

$$h(t,T) = E_t^*\left[e^{-\int_t^T y^*(u)du}\right]. \tag{42}$$

The nominal discount factor can be therefore decomposed as the product of the real discount factor $b(t,T)$ and of the inflation discount factor $h(t,T)$. As a consequence, the nominal term structure, expressed in terms of the yield to maturity, can be represented as two superimposed term structures

$$\begin{aligned} Y(t,T) &= -\frac{1}{T-t}\log v(t,T) = -\frac{1}{T-t}\log b(t,T) - \frac{1}{T-t}\log h(t,T) = \\ &= Y_R(t,T) + Y_I(t,T), \end{aligned} \tag{43}$$

which reflects the decomposition $f(t,T) = x(t,T) + y^*(t,T)$ of the nominal forward rate. The assumed stochastic decoupling (see Assumption 3) allows to derive the dynamics of $b(t,T)$ and of $h(t,T)$. In fact, by applying Ito's lemma to equation (41) we easily get

$$\frac{db(t,T)}{b(t,T)} = x(t)dt - \sigma_{vR}(t,T)dw_R^*(t), \tag{44}$$

and

$$\frac{dh(t,T)}{h(t,T)} = y^*(t)dt - \sigma_{vI}(t,T)dw_I^*(t). \tag{45}$$

The solutions of the above stochastic differential equations subject to the terminal conditions $b(T,T) = 1$ and $h(T,T) = 1$ respectively, are given by

$$b(t,T) = \exp\left[-\int_t^T x(t,u)du\right], \tag{46}$$

and

$$h(t,T) = \exp\left[-\int_t^T y^*(t,u)du\right]. \tag{47}$$

$x(t,T)$ can be, therefore, identified with the real forward rate. On the other hand, if the real interest rate process is identically equal to zero, $h(t,T)$ can be viewed as the value at time t of a zero coupon bond paying the face value $F = 1$ at maturity T, in an economy in which only stochastic inflation survives. It will be shown in the next section that $h(t,T)$ can be used as a reference index to build a class of index linked securities which are self-immunizing against the risk of inflation.

*Price-index linked bonds*Schwartz (1982) and Garcia and Rixtel (2007) represent an interesting class of index-linked securities which can be easily valued within the context of the proposed model.

If we denote by F the face value of such a bond, its terminal condition is characterized by the fact that it pays at maturity $F/p(t)$ unites of the consumption good, i.e. the same amount of the good that can be bought at time t with F units of money. From this point of view, these securities can be considered as fully immunized against the risk of inflation. Their prices can be easily calculated by recalling the martingale features of $p(t)$, thus obtaining

$$P(t,F,T) = FE_t^*\left[e^{-\int_t^T r(u)du}\ \frac{p(T)}{p(t)}\right] = Fb(t,T). \tag{48}$$

The price of such a bond is therefore given by the nominal face value discounted at the real rate from time T to time t.

An interesting application of the model is the valuation of price index linked bonds with a guaranteed minimum. To this end, let us consider a security which pays at time T the nominal amount

$$X_T = F \max\left(\frac{p(T)}{p(t)}, m\right), \tag{49}$$

where F is the face value and the ratio $p(T)/p(t)$ between the level price at maturity and the level price at the valuation date, is the reference index. The quantity m is the guaranteed minimum. If we pose $K = mp(t)$, equation (49) becomes

$$X_T = \frac{F}{p(t)}[K + \max(p(T) - K, 0)]. \tag{50}$$

The nominal price of this bond is given by

$$C(t, X_T) = \frac{F}{p(t)} K v(t,T) + \frac{F}{p(t)} CO(t,T,K), \tag{51}$$

where $CO(t,T,K)$ is the price of a European call option on the level price with exercise time T and striking price K. This value can be expressed as

$$CO(t,T,K) = b(t,T) E_t^* \left[e^{-\int_t^T y^*(u)du} \ \max(p(T) - K, 0) \right], \tag{52}$$

which can be computed by numerical methods.

4 A practical model

As an example of application we consider a two-factor model of the economy described by the following system of stochastic differential equations

$$\begin{aligned} dx(t,T) &= \sigma_R(t,T)\sigma_{vR}(t,T)dt + \sigma_R(t,T)dw_R^*(t), \qquad (53) \\ \frac{dp(t)}{p(t)} &= y^*(t)dt - \sigma_p dw_I^*(t), \\ dy^*(t,T) &= \rho^2\sigma_p^2(T-t)dt + \rho\sigma_p dw_I^*(t). \end{aligned}$$

The volatility function $\sigma_R(t,T)$ is assumed to decay exponentially in time according to the relation $\sigma_R(t,T) = \sigma_R e^{-\lambda(T-t)}$, where σ_R and λ are constant. The volatility σ_p of the price level process is assumed constant. Finally, the ρ parameter, which can be considered as a measure of the correlation between the processes $p(t)$ and $y^*(t,T)$, is also constant. Under the above assumptions we easily get the following solutions

$$\begin{aligned} f(t,T) &= f(0,T) + \frac{\sigma_R^2}{2\lambda^2}\left[2e^{-\lambda T}(e^{\lambda t} - 1) - e^{-2\lambda T}(e^{2\lambda t} - 1)\right] + \\ &+ \rho^2\sigma_p^2 t\left(T - \frac{t}{2}\right) + \sigma_R \int_0^t e^{-\lambda(T-u)} dw_R^*(u) + \rho\sigma_p w_I^*(t), \end{aligned}$$

where $f(0,T) = x(0,T) + y^*(0,T)$ is the initial forward rate, and

$$p(t) = p(0)\exp\left[\int_0^t y^*(u)du - \frac{1}{2}\sigma_p^2(t) - \sigma_p w_I^*(t)\right]. \tag{54}$$

This model is characterized by fact that it allows real and nominal rates to become negative with a strictly positive probability. However, as pointed out by Flesaker (1993), the probability to get negative nominal rates may be made arbitrarily small by choosing a suitable form of the market prices for risk. After some lengthy calculations, the nominal discount function can be cast in the following form

$$v(t,T) = \exp\left[-\int_t^T f(t,u)du\right] = A(t,T)e^{-\beta(t,T)x(t) - y^*(t)(T-t)}, \tag{55}$$

where

$$\begin{aligned} A(t,T) &= \frac{v(0,T)}{v(0,t)}\exp\left[\beta(t,T)x(0,t) + (T-t)y^*(0,t)\right] \times \\ &\times \exp\left[-\frac{\sigma_R^2}{4\lambda}(1-e^{-2\lambda t})\beta^2(t,T) - \frac{1}{2}\rho^2\sigma_p^2 t(T-t)^2\right], \end{aligned}$$

and

$$\beta(t,T) = \frac{1}{\lambda}\left[1 - e^{-\lambda(T-t)}\right]. \tag{56}$$

$v(0,s)$ is the initial discount function. In this model the Fisher equation becomes

$$r(t) = x(t) + y(t) - \sigma_p^2 - \sigma_p \bar{q}_I(t). \tag{57}$$

We assume that $\bar{q}_I(t)$, the market price for *inflation risk* in the real economy, is identically zero Cox, Ingersoll, and Ross (1981), so that, by recalling equation (24), it follows that

$$q_I(t) = \sigma_p, \tag{58}$$

and

$$r(t) = x(t) + y(t) - \sigma_p^2. \tag{59}$$

The quantity σ_p^2 that is equal to zero in the original Fisher equation, is due to the stochastic nature of the process $p(t)$. It is numerically negligible in several practical applications Moriconi (1993). From equation (54), and by noticing that $y^*(t,t) = y^*(t)$, we get

$$y^*(t) = y^*(0,t) + \frac{1}{2}\rho^2\sigma_p^2 t^2 + \rho\sigma_p w_I^*(t), \tag{60}$$

where $w_I^*(t) = w_I(t) - \sigma_p t$, and the expected rate of inflation is therefore given by

$$y(t) = y(0,t) - \rho\sigma_p^2 t\left(1 - \frac{1}{2}\rho t\right) + \rho\sigma_p w_I(t), \tag{61}$$

with

$$y(0,t) = y^*(0,t) + \sigma_p^2. \tag{62}$$

Within the framework of this model, we can discuss the pricing procedure of an interesting class of inflation-linked securities. Let us, therefore, consider a bond which pays at time T the nominal amount

$$X_T = F + F\left[\frac{1}{h(s,s+\tau)} - 1\right] \qquad (t < s,\ \ s+\tau < T), \tag{63}$$

where F is the face value of the bond and the quantity $\frac{1}{h(s,s+\tau)} - 1$ the reference index. Since such securities are characterized by a terminal condition which is given by the face value F multiplied by one plus the rate of return of $h(t,T)$, calculated on the period from time s to time $s+\tau$, they can be considered as a kind of contingent claims which are partially self-immunizing against the risk of inflation. The price at time t can be calculated according to the martingale representation in the following way

$$C(t, X_T) = F E_t^*\left[\frac{e^{-\int_t^T r(u)du}}{h(s,s+\tau)}\right], \tag{64}$$

where

$$h(t,s) = \frac{h(0,s)}{h(0,t)} \exp\left[-\rho^2\sigma_p^2\frac{st}{2}(s-t) - \rho\sigma_p(s-t)w_I^*(t)\right]. \tag{65}$$

$h(0,s)$ can be inferred from the initial discount function $v(0,s) = b(0,s)h(0,s)$, where $b(0,s)$ accounts for the real part of the initial term structure. An explicit calculation, using normal random variables, shows that expression (64) simplifies to

$$C(t, X_T) = Fv(t,T)\frac{h(t,s)}{h(t,s+\tau)}e^{\rho^2\sigma_p^2[\tau^2(s-t)-\tau(T-s)(s-t)]}, \tag{66}$$

so that the price of such a security is factorized in two terms. The first represents the nominal price of a zero coupon bond with face value F, and the second accounts for inflation risk reduction. The case $s = t$, $\tau = T - t$ is also significant, and the above formula becomes

$$C(t,T) = Fb(t,T), \tag{67}$$

which coincides with the price of a fully immunized security (see equation (48)).

5 Conclusions

The modeling procedure proposed in this paper may be viewed as a first insight toward an arbitrage-free, multi-factor theory of the term structure of interest rates incorporating the effects of uncertain inflation. Within this context, a stochastic generalization of the Fisher equation has been derived and a practical example of application of the theory discussed. The single good model can be considered as a first approximation to more sophisticated multi-good descriptions of the economy, in which the price of an opportunely chosen basket of goods can used as reference index for the inflation process.

All the relevant equations are derived by requiring the absence of arbitrage profit opportunities, nevertheless the followed procedure leaves open the question about the possibility to include such models within the context of a general equilibrium model of the economy.

A Appendix

Proof of Theorem 2.2.

It is easy to show that, by applying Ito's lemma to expression (20), we obtain

$$\frac{d[C(t,X_T)/p(t)]}{C(t,X_T)/p(t)} = \Bigg[r(t) - y(t) + \sum_{i=1}^{n} \sigma_{ci}(t,X_T) q_i(t) + \sum_{i=1}^{n} \sigma_{pi}^2(t) + \tag{A.1}$$

$$- \sum_{i=1}^{n} \sigma_{ci}(t,X_T)\sigma_{pi}(t)\Bigg] dt - \sum_{i=1}^{n} \Bigg[\sigma_{ci}(t,X_T) - \sigma_{pi}(t)\Bigg] dw_i(t),$$

where equations (19) and (3) have been used. By recalling equations (22) and (23), we finally get

$$\frac{d[C(t,X_T)/p(t)]}{C(t,X_T)/p(t)} = x(t)dt - \sum_{i=1}^{n} \Bigg[\sigma_{ci}(t,X_T) - \sigma_{pi}(t)\Bigg] d\bar{w}_i(t). \tag{A.2}$$

Since the diffusion coefficients in the above relation (A.2) belong to $M_w^2[0,T]$, it follows that the real interest rate discounted processes

$$R(t,T) = e^{-\int_0^t x(u)du} \, \frac{C(t,X_T)}{p(t)}, \tag{A.3}$$

are martingales with respect to the real risk-neutral measure induced by the Brownian motions transformation

$$d\bar{w}_i(t) = dw_i(t) - \bar{q}_i(t)dt \qquad (i = 1, 2, ..., n). \tag{A.4}$$

References

[1] K.I. Amin and R.A. Jarrow. “Pricing foreign currency options under stochastic interest rates”. In: *Journal of International Money and Finance* 10 (1991), pp. 310–329.

[2] A.N. Bomfim. “Measuring equilibrium real interest rates: What can we learn from yields on indexed bonds?” In: *Finance and Economics Discussion Series* 53 (2001).

[3] R.H. Brown and S.M. Schaefer. "The term structure of real interest rates and the Cox, Ingersoll, and Ross model". In: *Journal of Financial Economics* 35 (1994), pp. 3–42.

[4] J.C. Cox, J.E. Ingersoll, and S.A. Ross. "A reexamination of the traditional hypotheses about the term structure of interest rates". In: *Journal of Finance* 36 (1981), pp. 769–799.

[5] J.C. Cox, J.E. Ingersoll, and S.A. Ross. "A Theory of the term structure of interest rates". In: *Econometrica* 53 (1988), pp. 385–407.

[6] J.C. Cox, J.E. Ingersoll, and S.A. Ross. "An intertemporal general equilibrium model of asset prices". In: *Econometrica* 53 (1988), pp. 363–384.

[7] M.D.D. Evans. "Real risk, inflation risk and term structure". In: *Economic Journal* 113 (2003), pp. 245–389.

[8] I. Fisher. "Appreciation and interest". In: *Publications of the American Economic Association* XI (1896), pp. 91–92.

[9] B. Flesaker. "Arbitrage free pricing of interest rate futures and forward contracts". In: *Journal of Futures Markets* 13 (1993), pp. 77–91.

[10] A. Friedman. *Stochastic Differential Equations and Applications*. New York: Academic Press, 1975.

[11] J.A. Garcia and A. van Rixtel. *Inflation-linked bonds from a Central Bank perspective*. May. 2007.

[12] J.M. Harrison and D.M. Kreps. "Martingales and arbitrage in multiperiod security markets". In: *Journal of Economic Theory* 20 (1979), pp. 381–408.

[13] J.M. Harrison and S. Pliska. "Martingales and stochastic integrals in the theory of continuous trading". In: *Stochastic Processes and Their Applications* 11 (1981), pp. 215–260.

[14] D. Heath, R. Jarrow, and A. Morton. "Pricing and the term structure of interest rates: A discrete time approximation". In: *Journal of Financial and Quantitative Analysis* 25 (1990), pp. 419–440.

[15] D. Heath, R. Jarrow, and A. Morton. “Bond pricing and the term structure of interest rates: A new methodology for contingent claims valuation”. In: *Econometrica* 60 (1992), pp. 77–105.

[16] R.A. Jarrow and Y. Yildirim. “Pricing Treasury Inflation Protected Securities and Related Derivatives Using an Hjm Model”. In: *Journal of Financial and Quantitative Analysis* 38 (2003), pp. 337–359.

[17] T. Laubach and J.C. Williams. “Measuring the natural rate of interest”. In: *Review of Economics and Statistics* 85 (2003), pp. 1063–1070.

[18] A. Lioui and P. Poncet. “General equilibrium real and nominal interest rates”. In: *Journal of Banking and Finance* 28 (2004), pp. 1569–1595.

[19] F. Mkaouar, J.L. Prigent, and I. Abid. “Long-term investment with stochastic interest and inflation rates: The need for inflation-indexed bonds”. In: *Economic Modelling* 67 (2017), pp. 228–247.

[20] F. Moriconi. “Analyzing default-free bond markets by diffusion models”. In: *Proceedings of the 3rd AFIR International Colloquium*. Springer-Verlag, 1993.

[21] G.G. Pennacchi. “Identifying the dynamics of real interest rates and inflation: Evidence using survey data”. In: *Review of Financial Studies* 4 (1991), pp. 53–86.

[22] S.F. Richard. “An Arbitrage model of the term structure of interest rates”. In: *Journal of Financial Economics* 6 (1978), pp. 33–57.

[23] E. Schwartz. “The Pricing of commodity-linked bonds”. In: *Journal of Finance* 37 (1982), pp. 525–539.

[24] S.N. Singor et al. “Pricing inflation products with stochastic volatility and stochastic interest rates”. In: *Insurance: Mathematics and Economics* 52 (2013), pp. 286–299.

[25] R. Stambaugh. “The information in forward rates: Implications for models of the term structure”. In: *Journal of Financial Economics* 21 (1988), pp. 41–70.