Theoretical Mathematics & Applications, vol.3, no.4, 2013, 23-39
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2013

Oblique Derivative Problems for Nonlinear Parabolic Equations of Second Order in High Dimensional Domains

Guochun Wen[1], **Yanhui Zhang**[2] **and Dechang Chen**[3]

Abstract

This article mainly deals with the oblique derivative problem for nonlinear nondivergent parabolic equations of second order with measurable coefficients in multiply connected domains. We first derive a priori estimates of solutions for the boundary value problems. Then we use these estimates and the fixed-point theorem to prove the existence of solutions.

Mathematics Subject Classification: 35K60; 35K55; 35K20
Keywords: Oblique derivative problems; nonlinear parabolic equations; high dimensional domains

[1] School of Mathematical Sciences, Peking University, Beijing 100871, China.
E-mail: wengc@math.pku.edu.cn
[2] Math. Dept., Beijing Technology and Business University, Beijing 100048, China.
E-mail: zhangyanhui@th.btbu.edu.cn
[3] Uniformed Services University of the Health Sciences, MD 20814, USA.
E-mail: dechang.chen@usuhs.edu

Article Info: *Received* : September 19, 2013. *Revised* : November 10, 2013.
Published online : December 16, 2013.

1 Formulation of the oblique derivative problem for parabolic equations

Let Ω be a bounded multiply connected domain in $\mathbf{R}^N$ with the boundary $\partial\Omega \in C^2_\alpha(0 < \alpha < 1)$. And let $Q = \Omega \times I$, where $I = 0 < t \le T$ for $0 < T < \infty$. The boundary of Q is $\partial Q = S = \partial Q_1 \cup \partial Q_2 = S_1 \cup S_2$, where $\partial Q_1 = S_1 = \Omega \times \{t = 0\}$ is the bottom and $\partial Q_2 = S_2 = \partial\Omega \times \bar{I}$ is the lateral boundary. We consider the nonlinear parabolic equation of second order

$$F(x,t,u,D_x u,D_x^2 u) - u_t = 0 \ \text{ in } Q.$$

Under certain conditions, the equation can be written as (see Section 1, Chapter I, [6])

$$\sum_{i,j=1}^{N} a_{ij}u_{x_i x_j} + \sum_{i=1}^{N} b_i u_{x_i} + cu - u_t = f \ \text{ in } Q, \tag{1.1}$$

where $D_x u = (u_{x_i})$, $D_x^2 u = (u_{x_i x_j})$, and

$$a_{ij}=\int_0^1 F_{\tau r_{ij}}(x,t,u,p,\tau r)\mathrm{d}\tau,\quad b_i=\int_0^1 F_{\tau p_i}(x,t,u,\tau p,0)\mathrm{d}\tau,$$
$$c=\int_0^1 F_{\tau u}(x,t,\tau u,0,0)\mathrm{d}\tau,\quad f=-F(x,t,0,0,0),$$

with

$$r = D_x^2 u,\quad p = D_x u,\quad r_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j},\quad p_i = \frac{\partial u}{\partial x_i}.$$

Suppose that the above equation satisfies the following condition.

Condition C. For arbitrary functions $u_1(x,t), u_2(x,t) \in B = C^{1,0}_{\beta,\beta/2}(\overline{Q}) \cap W^{2,1}_2(Q)$, $F(x,t,u,D_x u,D_x^2 u)$ satisfies the condition

$$F(x,t,u_1,D_x u_1,D_x^2 u_1) - F(x,t,u_2,D_x u_2,D_x^2 u_2)$$
$$= \sum_{i,j=1}^{N} \tilde{a}_{ij}u_{x_i x_j} + \sum_{i=1}^{N} \tilde{b}_i u_{x_i} + \tilde{c}u,$$

where $0 < \beta < 1$, $u = u_1 - u_2$, $W^{2,1}_2(Q) = W^{2,0}_2(Q) \cap W^{0,1}_2(Q)$, and

$$\tilde{a}_{ij}=\int_0^1 F_{u_{x_i x_j}}(x,t,\tilde{u},\tilde{p},\tilde{r})\mathrm{d}\tau,\quad \tilde{b}_i=\int_0^1 F_{u_{x_i}}(x,t,\tilde{u},\tilde{p},\tilde{r})\mathrm{d}\tau$$

$$\tilde{c} = \int_0^1 F_u(x,t,\tilde{u},\tilde{p},\tilde{r})\mathrm{d}\tau$$

for

$$\tilde{u} = u_2 + \tau(u_1 - u_2),\ \ \tilde{p} = D_x[u_2 + \tau(u_1 - u_2)],\ \ \tilde{r} = D_x^2[u_2 + \tau(u_1 - u_2)].$$

Here we assume that $\tilde{a}_{ij}, \tilde{b}_i, \tilde{c}, f$ are measurable in Q and meet the following inequalities

$$q_0 \sum_{j=1}^{N} |\xi_j|^2 \le \sum_{i,j=1}^{N} \tilde{a}_{ij}\xi_i\xi_j \le q_0^{-1} \sum_{j=1}^{N} |\xi_j|^2,\ \ 0 < q_0 < 1, \tag{1.2}$$

$$\sup_Q \sum_{i,j=1}^{N} \tilde{a}_{ij}^2(x,t) / \inf_Q [\sum_{i=1}^{N} \tilde{a}_{ii}(x,t)]^2 \le q_1 < \frac{1}{N-1/2}. \tag{1.3}$$

$$|\tilde{a}_{ij}| \le k_0,\ \ |\tilde{b}_i| \le k_0,\ \ i,j = 1,...,N,\ \ |\tilde{c}| \le k_0 \ \text{in } Q,\ \ L_p[f, \overline{Q}] \le k_1, \tag{1.4}$$

in which $q_0, q_1, k_0, k_1, p\,(> N+2)$ are non-negative constants. Moreover, for almost every point $(x,t) \in Q$ and $D_x^2 u$, $\tilde{a}_{ij}(x,t,u,D_xu,D_x^2u)$, $\tilde{b}_i(x,t,u,D_xu)$, $\tilde{c}(x,t,u)$ are continuous in $u \in \mathbf{R}, D_xu \in \mathbf{R}^N$.

There is an explanation on the condition (1.3). Consider the linear case of parabolic equation (1.1), namely

$$\sum_{i,j=1}^{N} a_{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^{N} b_i(x,t)u_{x_i} + c(x,t)u - u_t = f(x,t) \ \text{ in } Q.$$

Divide the above equation by $\Lambda = \tau \inf_Q \sum_{i=1}^{N} a_{ii}$, where τ is an undetermined positive constant. Denote $\hat{a}_{ij} = a_{ij}/\Lambda$, $\hat{b}_i = b_i/\Lambda\,(i,j = 1,\cdots,N)$, $\hat{c} = c/\Lambda$, $\hat{f} = f/\Lambda$. Then the above equation is reduced to the form

$$\hat{L}u = \sum_{i,j=1}^{N} \hat{a}_{ij}(x,t)u_{x_ix_j} + \sum_{i=1}^{N} \hat{b}_i(x,t)u_{x_i} + \hat{c}(x,t)u - u_{\Lambda t} = \hat{f},\ \text{ i.e.}$$

$$Lu = \Delta u - u_{\Lambda t} = -\sum_{i,j=1}^{N} [\hat{a}_{ij}(x,t) - \delta_{ij}]u_{x_ix_j} - \sum_{i=1}^{N} \hat{b}_i(x,t)u_{x_i} - \hat{c}(x,t)u + \hat{f} \ \text{ in } Q,$$

where $\Delta u = \sum_{i=1}^{N} \partial^2 u/\partial x_i^2$, $\delta_{ii} = 1$, $\delta_{ij} = 0\,(i \ne j,\ i,j = 1,...,N)$. We require that the above coefficients satisfy

$$\begin{gathered}\sup_Q [2 \sum_{i,j=1,i<j}^{N} \hat{a}_{ij}^2 + \sum_{i=1}^{N} (\hat{a}_{ii} - 1)^2] = \sup_Q [\sum_{i,j=1}^{N} \hat{a}_{ij}^2 + N - 2\sum_{i=1}^{N} \hat{a}_{ii}] < \frac{1}{2},\ \text{ i.e.} \\ \sup_Q [\sum_{i,j=1}^{N} \hat{a}_{ij}^2 - 2\sum_{i=1}^{N} \hat{a}_{ii}] < \frac{1}{2} - N,\end{gathered} \tag{1.5}$$

which is true for the constant $\tau = 2/(2N-1)$ to be derived below. In fact, consider

$$\sup_Q \sum_{i,j=1}^N \hat{a}_{ij}^2 - 2\inf_Q \sum_{i=1}^N \hat{a}_{ii} < \frac{1}{2} - N, \ \text{ i.e.}$$

$$\frac{\sup_Q \sum_{i,j=1}^N a_{ij}^2}{\tau^2 \inf_Q [\sum_{i=1}^N a_{ii}]^2} < \frac{2}{\tau} + \frac{1}{2} - N, \ \text{ or } \ \frac{\sup_Q \sum_{i,j=1}^N a_{ij}^2}{\inf_Q [\sum_{i,j=1}^N a_{ii}]^2} < f(\tau)$$

for $f(\tau) = 2\tau + (1/2 - N)\tau^2$. It is seen that the maximum of $f(\tau)$ on $[0,\infty)$ occurs at the point $\tau = 2/(2N-1)$, and the maximum equals $f(2/(2N-1)) = 1/(N-1/2)$. The above inequality with $\tau = 2/(2N-1)$ is just the inequality (1.3). For convenience the item $u_{\Lambda t} = u_{t'}$ $(t' = \Lambda t)$ in the equation will be written as u_t later on.

In this paper we mainly consider the nonlinear parabolic equations of second order

$$\sum_{i,j=1}^N a_{ij} u_{x_i x_j} + \sum_{i=1}^N b_i u_{x_i} + cu - u_t = f + G(z,t,u,D_x u) \ \text{ in } Q, \tag{1.6}$$

where $G(z,t,u,D_x u)$ possesses the form

$$G(x,t,u,D_x u) = \sum_{i=1}^N B_i |u_{x_i}|^{\sigma_i} + B_0 |u|^{\sigma_0}. \tag{1.7}$$

In (1.7), we assume

$$|B_i| \le k_0, \ i = 0,1,...,N,$$

where k_0, σ_i $(i = 0,1,...,N)$ are positive constants. The above condition, together with Condition C, will be called Condition C$'$.

Problem O. The so-called oblique derivative boundary value problem (Problem O) is to find a continuously differentiable solution $u = u(x,t) \in B = C^{1,0}_{\beta,\beta/2}(Q) \cap W^{2,1}_2(Q)$ of the equation that satisfies the initial-boundary conditions

$$u(x,0) = g(x), \ \ x \in S_1, \tag{1.8}$$

$$lu = d\frac{\partial u}{\partial \nu} + \sigma u = \tau(x,t), \ \ (x,t) \in S_2, \ \text{ i.e.}$$
$$lu = \sum_{j=1}^N d_j \frac{\partial u}{\partial x_j} + \sigma u = \tau(x,t), \ \ (x,t) \in S_2. \tag{1.9}$$

In (1.8) and (1.9), $g(x), d(x,t), d_j(x,t)(j=1,...,N), \sigma(x,t), \tau(x,t)$ are assumed to satisfy the following requirements:

$$\begin{aligned}&C^2_\alpha[g(x),S_1]\le k_2,\quad C^{1,1}_{\alpha,\alpha/2}[\sigma(x,t),S_2]\le k_0,\\&C^{1,1}_{\alpha,\alpha/2}[d_j(x,t),S_2]\le k_0,\quad C^{1,1}_{\alpha,\alpha/2}[\tau(x,t),S_2]\le k_2,\\&\cos(\nu,\mathbf{n})\ge q_0>0,\quad d\ge 0,\quad \sigma\ge 0,\quad d+\sigma\ge 1,\quad (x,t)\in S_2,\end{aligned}\tag{1.10}$$

where $\mathbf{n}$ is the unit outward normal on $S_2, \alpha(0<\alpha<1), k_0, k_2, q_0(0<q_0<1)$ are non-negative constants.

There are several special cases of Problem O. Problem O with $\nu=\mathbf{n}$, $\sigma=0$ on S_2 is called Problem N, where $\mathbf{n}$ is the normal vector on S_2. Problem O with $f=0$ in (1.1) and $g(x)=0$, $\tau(x,t)=0$ in (1.8),(1.9) is called Problem O_0.

Theorem 1.1. *If equation* (1.1) *satisfies Condition* C*, then Problem* O_0 *for* (1.1) *only has the trivial solution.*

Proof: Let $u(x,t)$ be a solution of Problem O_0 for (1.1). Then it is easy to see that $u(x,t)$ satisfies the equation and the boundary conditions

$$\sum_{i,j=1}^N a_{ij}u_{x_ix_j}+\sum_{i=1}^N b_iu_{x_i}+cu-u_t=0 \text{ in } Q,\tag{1.11}$$

$$u(x,0)=0 \text{ on } S_1,\tag{1.12}$$

$$lu(x,t)=0,\text{ i.e. } d\frac{\partial u}{\partial\nu}+\sigma u=0 \text{ on } S_2.\tag{1.13}$$

Introducing the transformation $v=u\exp(-Bt)$, where B is an appropriately large number such that $B>\sup_Q c$, we see that the boundary value problem (1.11)–(1.13) is reduced to

$$\sum_{i,j=1}^n a_{ij}v_{x_ix_j}+\sum_{i=1}^n b_iv_{x_i}-[B-c]v-v_t=0 \text{ in } Q,\tag{1.14}$$

$$v(x,0)=0 \text{ on } S_1,\tag{1.15}$$

$$lv(x,t)=0,\text{ i.e. } d\frac{\partial v}{\partial\nu}+\sigma v=0 \text{ on } S_2.\tag{1.16}$$

Since $B - \sup_Q c > 0$, $(x,t) \in Q$, there is no harm assuming that $\sigma(x,t) > 0$ on $S_2 \cap \{(x,t) \in S_2,\ d \neq 0\}$. Otherwise through a transformation $V(x,t) = v(x,t)/\Psi(x,t)$, where $\Psi(z,t)$ is a solution of the equation

$$\Delta v - v_t = 0 \text{ in } Q, \text{ i.e. } \sum_{j=1}^{n} v_{x_j^2} - v_t = 0 \text{ in } Q$$

with the boundary condition $\Psi(z,t) = 1$ on ∂Q, the requirement can be realized and the modified equation satisfies the conditions similar to Condition C. By the extremum principle of solutions for (1.14) (see Theorems 2.5 and 2.7, Chapter I, [6]), we can derive that $v(x,t) = u(x,t) = 0$. □

2 A priori estimates of solutions for oblique derivative problems

In this section, we derive a priori estimates of solutions of Problem O for equations (1.1) and (1.6). We begin with the $C^{1,0}(\overline{Q})$ estimates of solutions $u(x,t)$ of Problem O for (1.1).

Theorem 2.1. *Under Condition* C*, any solution* $u(x,t)$ *of Problem* O *for* (1.1) *satisfies the estimate*

$$C^{1,0}[u,\bar{Q}] = ||u||_{C^{1,0}(\overline{Q})} = ||u||_{C^{0,0}(\bar{Q})} + \sum_{i=1}^{N} ||u_{x_i}||_{C^{0,0}(\overline{Q})} \leq M_1, \tag{2.1}$$

in which $M_1 = M_1(q,p,\alpha,k,Q)$ *is a non-negative constant only dependent on* q,p,α,k,Q *for* $q = q(q_0,q_1)$, $k = k(k_0,k_1,k_2)$.

Proof: Suppose that (2.1) is not true. Then there exist sequences of functions $\{a_{ij}^m\}$, $\{b_i^m\}$, $\{c^m\}$, $\{f^m\}$ and $\{g^m(x)\}$, $\{d^m(x,t)\}$, $\{\sigma^m(x,t)\}$, $\{\tau^m(t,x)\}$, such that

1) these functions meet Condition C and the corresponding requirements in (1.10);

2) $\{a_{ij}^m\}$, $\{b_i^m\}$, $\{c^m\}$, $\{f^m\}$ weakly converge to a_{ij}^0, b_i^0, c^0, f^0, and $\{g^m\}$, $\{d^m\}$, $\{\sigma^m\}$, $\{\tau^m\}$ uniformly converge to g^0, d^0, σ^0, τ^0 on S_1 or S_2 respectively; and

3) the initial-boundary value problem

$$\sum_{i,j=1}^{n} a_{ij}^m u_{x_ix_j}^m + \sum_{i=1}^{n} b_i^m u_{x_i}^m + c^m u^m - u_t^m = f^m \text{ in } Q, \tag{2.2}$$

$$u^m(x,0) = g^m(x) \text{ on } S_1, \tag{2.3}$$

$$lu^m(x,t) = \tau^m(x,t), \text{ i.e. } d^m \frac{\partial u^m}{\partial \nu} + \sigma^m u^m = \tau^m(x,t) \text{ on } S_2 \tag{2.4}$$

has a solution $u^m(x,t)$ with unbounded $||u^m||_{\hat{C}^{1,0}(\overline{Q})} = H_m (m = 1,2,...)$. Clearly, there is no harm in assuming that $H_m \geq 1$, and $\lim_{m\to\infty} H_m = +\infty$. It is easy to see that $U^m = u^m/H_m$ is a solution of the initial-boundary value problem

$$\sum_{i,j=1}^{N} a_{ij}^m U_{x_ix_j}^m - U_t^m = B^m, \quad B^m = -\sum_{i=1}^{N} b_i^m U_{x_i}^m - c^m U^m + \frac{f^m}{H_m}, \tag{2.5}$$

$$U^m(x,0) = \frac{g^m(x)}{H_m}, \quad x \in S_1, \tag{2.6}$$

$$lU^m(x,t) = \frac{\tau^m}{H_m}, \text{ i.e. } d^m \frac{\partial U^m}{\partial \nu} + \sigma^m U^m = \frac{\tau^m}{H_m}, \quad (x,t) \in S_2. \tag{2.7}$$

Noting that $L_p[\sum_{i=1}^N b_i^m U_{x_i}^m + c^m U^m, Q]$ is bounded and using the result in Theorem 2.2 below, we can obtain the estimate

$$\begin{aligned} C_{\beta,\beta/2}^{1,0}[U^m, \overline{Q}] &= ||U^m||_{C_{\beta,\beta/2}^{1,0}(\overline{Q})} \\ &= ||U^m||_{C_{\beta,\beta/2}^{0,0}(\overline{Q})} + \sum_{i=1}^{N} ||U_{x_i}^m||_{C_{\beta,\beta/2}^{0,0}(\overline{Q})} \leq M_2, \end{aligned} \tag{2.8}$$

$$||U^m||_{W_2^{2,1}(Q)} \leq M_2 = M_2(q,p,\alpha,k,Q), \quad m = 1,2,..., \tag{2.9}$$

where $\beta\,(0<\beta\leq\alpha)$, $M_2 = M_2(q,p,\alpha,k,Q)$ are non-negative constants. Hence from $\{U^m\}, \{U_{x_i}^m\}$, we can choose a subsequence $\{U^{m_k}\}$ such that $\{U^{m_k}\}$, $\{U_{x_i}^{m_k}\}$ uniformly converge to $U^0, U_{x_i}^0$ in $\overline{Q}$ and $\{U_{x_ix_j}^{m_k}\}, \{U_t^{m_k}\}$ weakly converge to $U_{x_ix_j}^0, U_t^0$ in Q respectively, where U^0 is a solution of the boundary value problem

$$\sum_{i,j=1}^{N} a_{ij}^0 U_{x_ix_j}^0 + \sum_{i=1}^{N} b_i^0 U_{x_i}^0 + c^0 U^0 - U_t^0 = 0, \quad (x,t) \in Q, \tag{2.10}$$

$$U^0(x,0) = 0, \quad x \in S_1, \tag{2.11}$$

$$lU^0(x,t)=0,\ \ \text{i.e. } d\frac{\partial U^0}{\partial\nu}+\sigma U^0=0,\ \ (x,t)\in S_2. \tag{2.12}$$

According to Theorem 1.1, we know $U^0(x,t)=0,(x,t)\in\overline{Q}$. However, from $||U^m||_{C^{1,0}(\overline{Q})}=1$, there exists a point $(x^*,t^*)\in\overline{Q}$, such that $|U^0(x^*,t^*)|+\sum_{i=1}^N|U^0_{x_i}(x^*,t^*)|>0$. This contradiction proves that (2.1) is true. □

Theorem 2.2. *Under Condition* C*, any solution* $u(x,t)$ *of Problem* O *for* (1.1) *satisfies the estimates*

$$||u||_{C^{1,0}_{\beta,\beta/2}(\overline{Q})}\le M_3=M_3(q,p,\alpha,k,Q), \tag{2.13}$$

$$||u||_{W_2^{2,1}(Q)}\le M_4=M_4(q,p,\alpha,k,Q), \tag{2.14}$$

where $\beta\,(0<\beta\le\alpha)$, $M_j\,(j=3,4)$ *are non-negative constants.*

Proof: First of all, we can find a solution $\hat{u}(x,t)$ of the equation

$$\Delta\hat{u}-\hat{u}_t=0 \tag{2.15}$$

with the boundary conditions (1.8) and (1.9), which satisfies the estimate (see [2,6])

$$||\hat{u}||_{C^{2,1}(\bar{Q})}\le M_5=M_5(q,p,\alpha,k,Q) \qquad (2.16).$$

Thus the function

$$\tilde{u}(x,t)=u(x,t)-\hat{u}(x,t) \tag{2.17}$$

is a solution of the equation

$$L\tilde{u}=\sum_{i,j=1}^N a_{ij}\tilde{u}_{x_ix_j}+\sum_{i=1}^N b_i\tilde{u}_{x_i}+c\tilde{u}-\tilde{u}_t=\tilde{f}, \tag{2.18}$$

$$\tilde{u}(x,0)=0,\ \ x\in S_1, \tag{2.19}$$

$$l\tilde{u}(x,t)=0,\ \ (x,t)\in S_2, \tag{2.20}$$

where $\tilde{f}=f-L\hat{u}$. Introduce a local coordinate system $x=x(\xi)$ on the neighborhood G of a surface $S_0\in\partial\Omega$, i.e.

$$x_i=h_i(\xi_1,...,\xi_{N-1})\xi_N+g_i(\xi_1,...,\xi_{N-1}),\ \ i=1,...,N, \tag{2.21}$$

where $\xi_N=0$ is just the surface $S_0:\,x_i=g_i(\xi_1,...,\xi_{N-1})\,(i=1,...,\,N)$, and

$$h_i(\xi)=\frac{d_i(x)}{d(x)}\bigg|_{x_i=g_i(\xi)},\ \ i=1,...,N,\ \ d^2(x)=\sum_{i=1}^N d_i^2(x).$$

Then the boundary condition (2.20) is reduced to the form

$$\frac{\partial \tilde{u}}{\partial \xi_N} + \tilde{\sigma}\tilde{u} = 0 \text{ on } \xi_N = 0, \tag{2.22}$$

where $\tilde{u} = \tilde{u}[x(\xi), t]$, $\tilde{\sigma} = \sigma[x(\xi), t]$.

Secondly, from [2,6], we can find a solution $v(x,t)$ of Problem N for equation (2.15) with the boundary condition

$$\frac{\partial v}{\partial \xi_N} = \tilde{\sigma} \text{ on } \xi_N = 0, \tag{2.23}$$

such that v satisfies the estimate

$$||v||_{C^{2,1}(\overline{Q})} \le M_6 = M_6(q, p, \alpha, k, Q) < \infty, \tag{2.24}$$

and the function

$$V(x,t) = \tilde{u}\mathrm{e}^{v(x,t)} \tag{2.25}$$

is a solution of the boundary value problem in the form

$$\sum_{i,j=1}^{N} \tilde{a}_{ij} V_{\xi_i \xi_j} + \sum_{i=1}^{N} \tilde{b}_i V_{x_i} + \tilde{c}V - V_t = \tilde{f}, \tag{2.26}$$

$$\frac{\partial V}{\partial \xi_N} = 0, \quad \xi_N = 0. \tag{2.27}$$

On the basis of Theorem 3.3, Chapter III, [6], we can derive the following estimates of $V(\xi, t)$:

$$||V||_{C^{1,0}_{\beta,\beta/2}(\overline{Q})} \le M_7 = M_7(q, p, \alpha, k, Q), \tag{2.28}$$

$$||V||_{W^{2,1}_2(Q)} \le M_8 = M_8(q, p, \alpha, k, Q), \tag{2.29}$$

where $\beta\,(0 < \beta \le \alpha)$, $M_j\,(j = 7, 8)$ are non-negative constants. Combining (2.16), (2.24), (2.28) and (2.29), the estimates (2.13) and (2.14) are obtained. □

The following are the estimates of solutions for (1.6).

Theorem 2.3. *Under Condition* C′*, any solution* $u(x,t)$ *of Problem* O *for* (1.6) *satisfies the estimates*

$$C^{1,0}_{\beta,\beta/2}[u, \overline{Q}] = ||u||_{C^{1,0}_{\beta,\beta/2}(\overline{Q})} \le M_9 k_*, \tag{2.30}$$

$$||u||_{W_2^{2,1}(Q)} \le M_{10} k_*, \tag{2.31}$$

where $\beta\,(0<\beta\le\alpha)$, $M_j = M_j(q,p,\beta,k_0,Q)\,(j=6,7)$ *are non-negative constants,* $k_* = k_1+k_2+k_3$ *with* k_1 *and* k_2 *as the constants stated in* (1.4) *and* (1.10) *respectively and* $k_3 = k_0[\sum_{i=1}^N |u_{x_i}|^{\sigma_i} + |u|^{\sigma_0}]$.

Proof: If $k_* = 0$, i.e. $k_1 = k_2 = k_3 = 0$, from Theorem 1.1, it follows that $u(z) = 0$, $z\in Q$. If $k_*>0$, it is easy to see that $U(z) = u(z)/k_*$ satisfies the following equation and boundary conditions:

$$\sum_{i,j=1}^N a_{ij}U_{x_ix_j} + \sum_{i=1}^N b_iU_{x_i} + cU - U_t = \frac{f+G(x,t,u,D_xu)}{k_*} \text{ in } Q, \tag{2.32}$$

$$U(x,0) = \frac{g(x)}{k_*} \text{ on } S_1, \tag{2.33}$$

$$lU = d\frac{\partial U}{\partial \nu} + \sigma U = \frac{\tau(x,t)}{k_*} \text{ on } S_2. \tag{2.34}$$

Noting that $L_p[(f+G)/k_*, \overline{Q}] \le 1$, $C^2_\alpha[g(z)/k_*, S_1] \le 1$, $C^{1,1}_{\alpha,\alpha/2}[\tau, S_2]/k_* \le 1$, and using the proof of Theorem 2.2, we have

$$C^{1,0}_{\beta,\beta/2}[U,\overline{Q}] \le M_9 k_*,\ ||U||_{W_2^{2,1}(Q)} \le M_{10}, \tag{2.35}$$

From the above estimates, it immediately follows that (2.30),(2.31) hold. □

3 Solvability of the oblique derivative problem for parabolic equations

We first consider a special equation of (1.1), namely

$$\begin{gathered}\Delta u - u_t = g_m(x,t,u,D_xu,D_x^2u),\\ g_m = \Delta u - \sum_{i,j=1}^N a_{ijm}u_{x_ix_j} - \sum_{i=1}^N b_{im}u - c_mu + f_m \text{ in } Q,\end{gathered} \tag{3.1}$$

where the coefficients

$$a_{ijm} = \begin{cases} a_{ij}, \\ \delta_{ij}, \end{cases} b_{im} = \begin{cases} b_i, \\ 0, \end{cases} c_m = \begin{cases} c, \\ 0, \end{cases} f_m = \begin{cases} f \text{ in } Q_m, \\ 0 \text{ in } \{\mathbf{R}^N\times I\}\backslash Q_m, \end{cases} \tag{3.2}$$

with $Q_m = \{(x,t) \in Q \,|\, \mathrm{dist}((x,t), \partial Q) \geq 1/m\}$ for a positive integer m. In particular, the linear case of equation (3.1) can be written as

$$\begin{aligned}\Delta u - u_t = g_m(x,t,u,D_x u, D_x^2 u),\quad g_m = \sum_{i,j=1}^{N}[\delta_{ij} - a_{ijm}(x,t)]u_{x_i x_j}\\ -\sum_{i=1}^{N} b_{im}(x,t)u_{x_i} - c_m(x,t)u + f_m(x,t) \text{ in } Q.\end{aligned}\tag{3.3}$$

□

The following theorem provides an expression of solutions of Problem O for equation (3.1).

Theorem 3.1. *Under Condition* C*, if* $u(x,t)$ *is a solution of Problem* O *for equation* (3.1)*, then* $u(x,t)$ *can be expressed in the form*

$$\begin{aligned}&u(x,t) = U(x,t) + \hat{V}(x,t) = U(x,t) + v_0(x,t) + v(x,t),\\ &v(x,t) = H\rho = \int_{Q_0} G(x,t,\zeta,\tau)\rho(\zeta,\tau)d\sigma_\zeta d\tau,\\ &G = \begin{cases} [4\pi(t-\tau)]^{-N/2}\exp[|x-\zeta|^2/4(\tau - t)], & t > \tau, \}\\ 0, \quad t \leq \tau, \quad \text{except } t-\tau = |x-\zeta| = 0,\end{cases}\end{aligned}\tag{3.4}$$

where $\rho(x,t) = \Delta u - u_t = g_m$*. In* (3.4)*,* $\hat{V}(x,t) = v_0(x,t) + v(x,t)$ *is a solution of the Dirichlet problem* (*Problem* D) *for* (3.1) *in* $Q_0 = \Omega_0 \times I$ $(\Omega_0 = \{|x| < R\})$ *with the boundary condition* $\mathrm{Re}\hat{V}(x,t) = 0$ *on* ∂Q_0*, where* R *is a large number such that* $\Omega_0 \supset \overline{\Omega}$*.* $U(x,t)$ *is a solution of Problem* $\tilde{\mathrm{O}}$ *for* $LU = \Delta U - U_t = 0$ *in* Q *with the initial-boundary condition* (3.12) – (3.13) *below, which satisfies the estimates*

$$\begin{aligned}C^{1,0}_{\beta,\beta/2}[U, \overline{Q}] + ||U||_{W_2^{2,1}(Q)} \leq M_{11},\\ C^{1,0}_{\beta,\beta/2}[\hat{V}, \overline{Q_0}] + ||\hat{V}||_{W_2^{2,1}(Q_0)} \leq M_{12},\end{aligned}\tag{3.5}$$

for non-negative constants $\beta\,(0 < \beta \leq \alpha)$*,* $M_j = M_j(q,p,\beta,k,Q_m)\,(j = 11, 12)$ *with* $q = q(q_0,q_1)$ *and* $k = k(k_0,k_1,k_2)$*.*

Proof: It is easy to see that the solution $u(x,t)$ of Problem O for equation (3.1) can be expressed by the form (3.4). Noting that $a_{ijm} = 0\,(i \neq j)$, $b_{im} = 0$, $c_m = 0$, $f_m(x,t) = 0$ in $\{\mathbf{R}^N \times I\}\backslash Q_m$ and $\hat{V}(x,t)$ is a solution of Problem D for (3.1) in Q_0, we can obtain that $\hat{V}(x,t)$ in $\hat{Q}_{2m} = \overline{Q}\backslash Q_{2m}$ satisfies the estimate

$$\hat{C}^{2,1}[|\hat{V}(x,t)|^{\sigma_0+1}, \hat{Q}_{2m}] \leq M_{13} = M_{13}(q,p,\alpha,k,Q_m).$$

On the basis of Theorem 2.3, we can see that $U(x,t)$ satisfies the first estimate in (3.5), and then $\hat{V}(x,t)$ satisfies the second estimate in (3.5). □

Theorem 3.2. *Under Condition* C, *Problem* O *for* (3.3) *has a solution* $u(x,t)$.

Proof: We prove the existence of solutions of Problem O for the nonlinear equation (3.1) by using the Larey-Schauder theorem. To begin, we introduce the equation with the parameter $h \in [0,1]$

$$\Delta u - u_t = hg_m(x,t,u,D_x u, D_x^2 u) \text{ in } Q. \tag{3.6}$$

Denote by B_M a bounded open set in the Banach space $B = C^{1,0}_{\beta,\beta/2}(\overline{Q}) \cap W_2^{2,1}(Q)(0<\beta\le\alpha)$, the elements of which are real functions $V(x,t)$ satisfying the inequalities

$$C^{1,0}_{\beta,\beta/2}[V,\bar{Q}] + ||V||_{W_2^{2,1}(Q)} < M_{14} = M_{12}+1, \tag{3.7}$$

in which $W_2^{2,1}(Q) = W_2^{2,0}(Q)\cap W_2^{0,1}(Q)$, M_{12} is a non-negative constant as stated in (3.5). We choose any function $\tilde{V}(x,t) \in \overline{B_M}$ and substitute it into the appropriate positions on the right hand side of (3.6), and then we make an integral $\tilde{v}(x,t) = H\rho$ as follows

$$\tilde{v}(x,t) = H\rho, \ \ \rho(x,t) = \Delta\tilde{V} - \tilde{V}_t. \tag{3.8}$$

Next we find a solution $\tilde{v}_0(x,t)$ of the initial-boundary value problem in Q_0:

$$\Delta\tilde{v}_0 - \tilde{v}_{0t} = 0 \text{ on } Q_0, \tag{3.9}$$

$$\tilde{v}_0(x,t) = -\tilde{v}(x,t) \text{ on } \partial Q_0, \tag{3.10}$$

and denote by $\hat{V}(x,t) = \tilde{v}(x,t)+\tilde{v}_0(x,t)$ the solution of the corresponding Problem D in Q_0. Moreover we can find a solution $\tilde{U}(x,t)$ of the corresponding Problem $\tilde{\text{O}}$ in Q

$$\Delta\tilde{U} - \tilde{U}_t = 0 \text{ on } Q, \tag{3.11}$$

$$\tilde{U}(x,0) = g(x) - \hat{V}(x,0) \text{ on } \Omega, \tag{3.12}$$

$$\frac{\partial\tilde{U}}{\partial\nu} + \sigma(x,t)\tilde{U} = \tau(x,t) - \frac{\partial\hat{V}}{\partial\nu} + \sigma(x,t)\hat{V} \text{ on } S_2. \tag{3.13}$$

Now we consider the equation

$$\Delta V - V_t = hg_m(x, t, \tilde{u}, D_x\tilde{u}, D_x^2\tilde{U} + D_x^2\hat{V}),\ 0 \le h \le 1, \tag{3.14}$$

where $\tilde{u} = \tilde{U} + \hat{V}$.

By Condition C, applying the principle of contracting mapping, we can find a unique solution $V(x,t)$ of Problem D for equation (3.14) in Q_0 satisfying the initial-boundary condition

$$V(x,t) = 0 \ \text{ on } \ \partial Q_0. \tag{3.15}$$

Set $u(x,t) = U(x,t)+V(x,t)$, where the relation between U and V is the same as that between $\tilde{U}$ and $\tilde{V}$. Denote by $V = S(\tilde{V}, h)$, $u = S_1(\tilde{V}, h)\,(0 \le h \le 1)$ the mappings from $\tilde{V}$ onto V and u respectively. Furthermore, if $V(x,t)$ is a solution of Problem D in Q_0 for the equation

$$\Delta V - V_t = hg_m(x, t, u, D_xu, D_x^2(U + V)),\ \ 0 \le h \le 1, \tag{3.16}$$

where $u = S_1(V, h)$, then from Theorem 3.1, the solution $V(x,t)$ of Problem D for (3.16) satisfies the estimate (3.7), and consequently $V(x,t) \in B_M$. Set $B_0 = B_M \times [0,1]$.

In the following, we verify that the mapping $V = S(\tilde{V}, h)$ satisfies the three conditions of Leray-Schauder theorem:

1) For every $h \in [0,1], V = S(\tilde{V}, h)$ continuously maps the Banach space B into itself, and is completely continuous on B_M. Besides, for every function $\tilde{V}(x,t) \in \overline{B_M},\ S(\tilde{V}, h)$ is uniformly continuous with respect to $h \in [0,1]$.

2) For $h = 0$, from Theorem 2.2 and (3.7), it is clear that $V = S(\tilde{V}, 0) \in B_M$.

3) From Theorem 2.2 and (3.7), we see that $V = S(\tilde{V}, h)(0 \le h \le 1)$ does not have a solution $V(x,t)$ on the boundary $\partial B_M = \overline{B_M} \backslash B_M$.

Hence we know that Problem D for equation (3.6) with $h = 1$ has a solution $V(z,t) \in B_M$, and then Problem O of equation (3.6) with $h = 1$, i.e. (3.1) has a solution

$$u(x,t) = S_1(\tilde{V}, h) = U(x,t) + V(x,t) = U(x,t) + v_0(x,t) + v(x,t) \in B.$$

□

Theorem 3.3. *Under Condition* C*, Problem* O *for* (1.1) *has a solution.*

Proof: By Theorems 2.3 and 3.2, Problem O for equation (3.1) possesses a solution $u_m(x,t)$ satisfying the estimates (2.13) and (2.14), where $m = 1, 2, ...$ Thus, we can choose a subsequence $\{u_{m_k}(x,t)\}$, such that $\{u_{m_k}(x,t)\}$, $\{u_{m_k x_i}(x,t)\}(i = 1, ..., N)$ in $\overline{Q}$ uniformly converge to $u_0(x,t)$, $u_{0x_i}(x,t)\,(i = 1, ..., N)$ respectively. Obviously, $u_0(x,t)$ satisfies the boundary conditions of Problem O. On the basis of the principle of compactness of solutions for equation (3.1) (Theorem 4.6, Chapter I, [6]), we see that $u_0(x,t)$ is a solution of Problem O for (1.1). □

Theorem 3.4. *Let the complex equation* (1.6) *satisfy Condition* C′.

(1) *When* $0 < \sigma_0, \sigma_1, ..., \sigma_N < 1$, *Problem* O *for* (1.6) *has a solution* $u(x) \in B = C^{1,0}_{\beta,\beta/2}(Q) \cap W^{2,1}_2(Q)$.

(2) *When* $\min(\sigma_0, \sigma_1, ..., \sigma_N) > 1$, *Problem* O *for* (1.6) *has a solution* $u(x) \in B$, *provided that*

$$M_{17} = L_p[f, \overline{Q}] + C^2_\alpha[g, S_1] + C^{1,1}_{\alpha,\alpha/2}[\tau, S_2] \tag{3.17}$$

is sufficiently small.

Proof: (1) Noting that

$$(M_9 + M_{10})\{L_p[f, \bar{Q}] + \sum_{i=1}^{N} L_\infty[B_i, \bar{Q}]t^{\sigma_i} + L_\infty[B_0, \overline{Q}]t^{\sigma_0} + C^2_\alpha[g, \partial Q] + C^{1,1}_{\alpha,\alpha/2}[\tau, S_2]\} = t, \tag{3.18}$$

where M_9, M_{10} are the positive constant as in (2.30),(2.31).

Because $0 < \sigma_0, \sigma_1, ..., \sigma_N < 1$, the above equation has a unique solution $t = M_{18} > 0$.

Now we introduce a bounded, closed and convex subset B^* of the Banach space $B = C^{1,0}(Q) \cap W^{2,1}_2(Q)$, whose elements are of the form $\{u(x)\}$ satisfying the condition

$$B^* = \{u(x,t)\,|\,C^{1,0}[u, \overline{Q}] + ||u, Q||_{W^{2,1}_2(Q)} \le M_{18}\}. \tag{3.19}$$

We choose any function $\tilde{u}(x,t) \in B^*$ and substitute it into the corresponding positions in the coefficients of (1.6), (1.8), and (1.9) to obtain the following

$$\tilde{F}(x, t, u, D_x u, D^2_u x, \tilde{u}, D_x\tilde{u}, D^2_x\tilde{u}) = \tilde{G}(x, t, u, D_x u, \tilde{u}, D_x\tilde{u}_x), \tag{3.20}$$

$$u(x,0)=g(x)\ \text{ on }\ S_1,$$
$$d(x)\frac{\partial u}{\partial\nu}+\sigma(x,t)u=\tau(x,t)\ \text{ on }\ S_2. \tag{3.21}$$

where

$$\tilde{F}(x,t,u,D_xu,D_x^2u,\tilde{u},D_x\tilde{u},D_x^2\tilde{u})=\sum_{i,j=1}^{N}a_{ij}u_{x_ix_j}+\sum_{i=1}^{N}b_iu_{x_i}+cu-f,$$
$$\tilde{G}(x,t,u,D_xu,\tilde{u},D_x\tilde{u}_x)=\sum_{i=1}^{N}B_i|u_{x_i}|^{\sigma_i}+B_0|u|^{\sigma_0}.$$

In accordance with the method in the proof of Theorem 3.2, we can prove that the boundary value problem (3.20), (3.21) has a unique solution $u(x)$. Denote by $u(x)=T[\tilde{u}(x)]$ the mapping from $[\tilde{u}(x)]$ to $[u(x)]$. Noting that

$$L_p[\sum_{i=1}^{N}b_i\tilde{u}_{x_i},\overline{Q}]\le M_6k_0(k_1+k_2+k_3),\ C_\alpha[c\tilde{u},\overline{D}]\le M_6k_0(k_1+k_2+k_3), \tag{3.22}$$

from Theorem 2.2, we have

$$C^{1,0}_{\beta,\beta/2}[u,\overline{Q}]+||u||_{W_2^{2,1}(Q)}\le M_7\{L_p[f,\overline{Q}]+C^2_\alpha[g,S_1]+C^{2,1}_\alpha[\tau,S_2]+L_\infty[G,\overline{Q}]\}$$
$$\le M_7\{M_{17}+\textstyle\sum_{i=1}^{N}L_\infty[B_i,\overline{Q}]C[u_{x_i}\bar{Q}]^{\sigma_i}+L_\infty[B_0,\overline{Q}]C[\tilde{u},\bar{Q}]^{\sigma_0}\}\le M_{18}. \tag{3.23}$$

This shows that T maps B^* onto a compact subset in B^*.

Next, we verify that T in B^* is a continuous operator. In fact, we arbitrarily select a sequence $\{\tilde{u}_n(z)\}$ in B^*, such that

$$C^{1,0}(\tilde{u}_n-\tilde{u}_0,\overline{Q})+||\tilde{u}_n-\tilde{u}_0)||_{W_2^{2,1}(Q)}\to 0\ \text{ as }n\to\infty. \tag{3.24}$$

By Theorem 2.3, we can see that

$$L_p[\tilde{F}(x,t,u_n,D_x\tilde{u}_n,D_x^2u_n,\tilde{u}_n,D_x\tilde{u}_n,V)-\tilde{F}(x,t,u_0,D_xu_0,D_x^2u_0,\tilde{u}_0,D_x\tilde{u}_0,V),\overline{Q}]\to 0,$$
$$L_p[\tilde{G}(x,t,u_n,D_xu_n,\tilde{u}_n,D_x\tilde{u}_n)-\tilde{G}(x,t,u_0,D_xu_0,\tilde{u}_0,D_x\tilde{u}_0),\overline{Q}]\to 0\ \text{ as }n\to\infty, \tag{3.25}$$

in which $V(x)\in L_p(\overline{Q})$.

Moreover, from $u_n=T[\tilde{u}_n],\ u_0=T[\tilde{u}_0]$, it is clear that u_n-u_0 is a solution of Problem O for the following equation and boundary conditions:

$$\tilde{F}(x,t,u_n,D_xu_n,D_x^2u_n,\tilde{u}_n,D_x\tilde{u}_n,D_x^2\tilde{u}_n)-\tilde{F}(x,t,u_0,D_xu_0,D_x^2u_0,\tilde{u}_0,D_x\tilde{u}_0,D_x^2u_0)+$$
$$G(x,t,u_n,D_xu_n,\tilde{u}_n,D_x\tilde{u}_n)-G(x,t,u_0,D_xu_0,\tilde{u}_0,D_x\tilde{u}_0)=0\ \text{ in }\ Q, \tag{3.26}$$

$$u(x,0)=0 \text{ on } S_1,$$
$$d(x)\frac{\partial(u_n-u_0)}{\partial\nu}+\sigma(x)(u_n-u_0)=0 \text{ on } S_2. \tag{3.27}$$

In accordance with the method in proof of Theorem 2.2, we can obtain the estimate

$$\begin{aligned}&C^{1,0}_{\beta,\beta/2}[u,\overline{D}]+||u_n-u_0||_{W^{2,1}_{p_0}(Q)}\\ &\le M_{19}\{L_p[\tilde{G}(x,t,u_n,D_xu_n,\tilde{u}_n,D_x\tilde{u}_n)-\tilde{G}(x,t,u_0,D_xu_0,\tilde{u}_0,D_x\tilde{u}_0),\overline{Q}]\\ &+L_p[\tilde{F}(x,t,u_n,D_x\tilde{u}_n,D^2_xu_n,\tilde{u}_n,D_x\tilde{u}_n,V)-\tilde{F}(x,t,u_0,D_xu_0,D^2_xu_0,\tilde{u}_0,D_x\tilde{u}_0,V),\overline{Q}]\},\end{aligned} \tag{3.28}$$

in which $M_{19}=M_{19}(q_0,p_0,\beta,k_0,Q)$. From the above estimate, we obtain $C^{1,0}_{\beta,\beta/2}[u_n-u_0,\overline{Q}]+||u_n-u_0||_{W^{2,1}_{p_0}(Q)}\to 0$ as $n\to\infty$. On the basis of the Schauder fixed-point theorem, there exists a function $u(x)\in B^*$ such that $u(x)=T[u(x)]$, and from Theorem 2.3, it is easy to see that $u(x)\in B^*$, and $u(x)$ is a solution of Problem O for the equation (1.6) and the boundary condition (1.8),(1.9) with $0<\sigma_0,...,\sigma_N<1$.

(2) If $\min(\sigma_0,...,\sigma_N)>1$, (3.18) has the solution $t=M_{20}$ provided that M_{17} in (3.17) is small enough. Consider a closed and convex subset B_* in the Banach space $B=C^{1,0}(\overline{Q})\cap W^{2,1}_2(Q)$, i.e.

$$B_*=\{u(x)\,|\,C^{1,0}[u,\overline{Q}]+||u||_{W^{2,1}_2(Q)}\le M_{20}\}.$$

Applying a method similar to that in (1), we can verify that there exists a solution $u(x)\in B_*$ of Problem O for (1.6), when the constant

$$\min(\sigma_0,\sigma_1,...,\sigma_N)>1.$$

□

Acknowledgements.
Yanhui Zhang was supported by the grant PHR(IHLB201106206).
Note: The opinions expressed herein are those of the authors and do not necessarily represent those of the Uniformed Services University of the Health Sciences and the Department of Defense.

References

[1] Alkhutov A.Yu. and Mamedov I.T., The first boundary value problem for nondivergence second order parabolic equations with discontinuous coefficients, *Math USSR Sbornik*, **59**, (1988), 471–95.

[2] Ladyzhenskaja O.A., Solonnikov V.A. and Uraltseva N.N., *Linear and quasilinear equations of parabolic type*, Amer. Math. Soc., Providence, RI, 1968.

[3] Wen G.C., *Conformal mappings and boundary value problems*, Translation of Mathematics Monographs, **106**, Amer Math Soc., Providence RI, 1992.

[4] Wen G.C., *Approximate methods and numerical analysis for elliptic complex equations*, Gordon and Breach, Amsterdam, 1999.

[5] Wen G.C., *Linear and nonlinear parabolic complex equations*, World Scientific, Singapor, 1999.

[6] Wen G.C. and Zou B.T., *Initial-boundary value problems for nonlinear parabolic equations in higher dimensional domains*, Science Press, Beijing, 2002.

[7] Wen G.C., Chen D. and Xu Z.L., *Nonlinear complex analysis and its applications*, Mathematics Monograph Series **12**, Beijing, Science Press, 2008.

[8] Wen G.C., *Recent progress in theory and applications of modern complex analysis*, Beijing, Science Press, 2010.