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Recursive Empirical bayes analysis for the parameter of truncation distribution

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Abstract

Recursive empirical Bayes test rules for parameter of the one-side truncation distribution family are constructed with asymmetric loss functions and the asymptotically optimal property is obtained. It is shown that the convergence rates of the proposed EB test rules can arbitrarily close to $O(n^{-\frac{1}{2}})$under suitable conditions.

Mathematics Subject Classification: 62C12
Keywords: Asymmetric loss functions; Empirical Bayes test; Asymptotic optimality; Convergence rates

1 Introduction

Since Robbin's pioneering papers [1,2], empirical bayes (EB) approach has been studied extensively in recent years, the readers are referred to literature [3-9].

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Usually, EB methods are considered with kernel-type density estimation, However, in fact we need to use recursive kernel-type density estimation. Thus, we only calculate addition item when size of the past samples is increased. It reduce calculation in a certain degree.

Consider the following model

$$f(x|\theta) = u(x)\varphi(\theta)I_{(a<\theta<x<b)}, \tag{1}$$

where $f(x|\theta)$ denotes the conditional probability density function(pdf) of random variable(r.v)X, given θ, $-\infty \leq a < \theta < b \leq +\infty, u(x)$ is a nonnegative integrable functin in (a,b), $\varphi(\theta) = [\int_\theta^b u(x)dx]^{-1}$.

Assume that $G(\theta)$ is the unknown prior distribution of θ. The marginal density function of X is given by

$$f_G(x) = u(x)\int_a^x \varphi(\theta)dG(\theta) = u(x)v(x), \tag{2}$$

where $v(x) = \int_a^x \varphi(\theta)dG(\theta)$.

In this paper, we discuss the following hypothesis test problem

$$H_0 : \theta \leq \theta_0 \Leftrightarrow H_1 : \theta > \theta_0, \tag{3}$$

where θ_0 is a given constant.

To avoid the influence of two types of possible errors , we adopt asymmetric loss functions[3]

$$L(\theta, d_0) = L_0 I_{(\theta<\theta_0)}, L(\theta, d_1) = L_1 I_{(\theta\geq\theta_0)},$$

where $L_0 = k_1(\theta-\theta_0)^2, L_1 = k_2(\theta-\theta_0)^2 + k_3(\theta-\theta_0), k_i > 0, i = 1,2,3$. If $k_3 = 0, k_1 = k_2$, the asymmetric loss functions are degenerated squared loss functions. $d = \{d_0, d_1\}$ is action space, d_0 and d_1 imply acceptance and rejection of H_0.

Let randomized decision function be defined by

$$\delta(x) = P(\ accept\ \ H_0|X = x). \tag{4}$$

Then, the Bayes risk of test $\delta(x)$ is shown by

$$\begin{aligned} R(\delta(x), G(\theta)) &= \int_a^b\int_a^x [L(\theta,d_0)f(x|\theta)\delta(x) + L(\theta,d_1)f(x|\theta)(1-\delta(x))]dG(\theta)dx \\ &= \int_a^b \beta(x)\delta(x)dx + C_G, \end{aligned} \tag{5}$$

where

$$C_G = \int_\Theta L_1(\theta, d_1) dG(\theta),$$

$$\begin{aligned} \beta(x) &= \int_a^x [L(\theta, d_0) - L(\theta, d_1)] f(x|\theta) dG(\theta) \\ &= \int_a^{\theta_0} (L_0 + L_1) f(x|\theta) dG(\theta) - \int_a^x L_1 f(x|\theta) dG(\theta) \\ &= 2k_2 u(x) v_1(x) + (k_3 - 2k_2) u(x) v_2(x) - [k_2(x-\theta_0)^2 + k_3(x-\theta_0)] f_G(x) \\ &\quad -2(k_1+k_2) u(x) v_1(\theta_0) + (k_3 - 2k_2\theta_0 - 2k_1\theta_0) u(x) v_2(\theta_0). \end{aligned} \tag{6}$$

where $v_1(x) = \int_a^x t v(t) dt$, $v_2(x) = \int_a^x v(t) dt$, $f_G(x)$ is defined by (1.2).

By (1.5), Bayes test function is obtained as follows

$$\delta_G(x) = \begin{cases} 1, & \beta(x) \le 0 \\ 0, & \beta(x) > 0 \end{cases} \tag{7}$$

Further, the minimum Bayes risk of $\delta_G(x)$ is

$$R(G, \delta_G) = \int_a^b \beta(x) \delta_G(x) dx + C_G. \tag{8}$$

When the prior distribution $G(\theta)$ is known and $\delta(x) = \delta_G(x)$, $R(G)$ is achieved. However, $G(\theta)$ is unknown, so $\delta_G(x)$ can not be made use of, we need to introduce EB method. The rest of this paper is organized as follows. Section 2 presents an EB test. In section 3, we obtain asymptotic optimality and the optimal rate of convergence of the EB test.

2 Construction of EB Test Function

Under the following condition, we need to construct EB test function:let $X_1, X_2, \cdots, X_n, X$ be random variable sequence with the common marginal density function $f_G(x), where X_1, X_2, \cdots, X_n$ are historical sample and X is current sample. Suppose $C_{s,\alpha}$ is a class of probability density function and $f_G(x) \in C_{s,\alpha}, x \in R^1$, where $C_{s,\alpha}$ is continuous and bounded function and has s-th order derivative, satisfying $|C_{s,\alpha}| \le \alpha, s \ge 4$.first construct estimator of $\beta(x)$.

Let $K_r(x) (r = 0, 1, \cdots, s-1)$ be a Borel measurable real function vanishing off $(0, 1)$ such that

$$(A1) \quad \frac{1}{t!} \int_0^1 v^t K_r(v) dv = \begin{cases} (-1)^t, & t = r \\ 0, & t \ne r, t = 0, 1, \cdots, s-1 \end{cases}$$

Denote $f_G^{(0)}(x) = f_G(x), f_G^{(r)}(x)$ is the r order derivative of $f_G(x)$. For $r = 0, 1, \cdots, s-1$. similar to ([7]), kernel estimation of $f_G^{(r)}(x)$ is defined by

$$f_n^{(r)}(x) = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{h_i^{1+r}}K_r(\frac{x-X_i}{h_i}), \tag{9}$$

where $h_i > 0$, and $\lim_{n\to\infty} h_i = 0$.

Let $F(x)$ is distribution function of random variable X. Denote $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I(a < X_i \leq x)$,
where $F_n(x)$ is empirical distribution function of random variable X.

The estimator of $v_1(x), v_2(x)$ are defined as follows

$$v_{1n}(x) = \frac{1}{n}\sum_{i=1}^{n} X_i \frac{1}{u(X_i)} I_{(a<X_i\leq x)}, \tag{10}$$

$$v_{2n}(x) = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{u(X_i)} I_{(a<X_i\leq x)}, \tag{11}$$

Thus, the estimator of $\beta(x)$ is obtained by

$$\begin{aligned}\beta_n(x) &= 2k_2u(x)v_{1n}(x) + (k_3 - 2k_2)u(x)v_{2n}(x) - [k_2(x-\theta_0)^2 + k_3(x-\theta_0)]f_n(x) \\ &\quad -2(k_1+k_2)u(x)v_1(\theta_0) + (k_3 - 2k_2\theta_0 - 2k_1\theta_0)u(x)v_2(\theta_0).\end{aligned} \tag{12}$$

Hence, EB test function is defined by

$$\delta_n(x) = \begin{cases} 1, & \beta_n(x) \leq 0, \\ 0, & \beta_n(x) > 0. \end{cases} \tag{13}$$

Let E stand for mathematical expectation with respect to the joint distribution of $X_1, X_2, \cdots, X_n$.
Hence, we get the overall Bayes risk of $\delta_n(x)$

$$R(\delta_n(x), G) = \int_a^b \beta(x)E[\delta_n(x)]dx + C_G, \tag{14}$$

If $\lim_{n\to\infty} R(\delta_n, G) = R(\delta_G, G)$, $\{\delta_n(x)\}$ is called asymptotic optimality of EB test function, and $R(\delta_n, G) - R(\delta_G, G) = O(n^{-q})$, where $q > 0$, $O(n^{-q})$ is asymptotic optimality convergence rates of EB test function of $\{\delta_n(x)\}$, before proving the theorems, we give a series of lemma.

Let c, c_1, c_2, c_3, c_4 be different constants in different cases even in the same expression.

Lemma 2.1. [10] *Let $\{X_i, 1 \le i \le n\}$ be negative associated random variables, with $EX_i = 0$ and $E|X_i|^2 < \infty, i = 1, 2, \cdots, n$ then*

$$E|\sum_{i=1}^{n} X_i|^2 \le c \sum_{i=1}^{n} E|X_i|^2.$$

Lemma 2.2. *$f_n^{(r)}(x)$ is defined by (9). Let $X_1, X_2, \cdots, X_n$ be independent identically distributed random samples. Assume (A1) holds, $\forall x \in \Omega$.*

(I) When $f_G^{(r)}(x)$ is absolute continuous function, $\max\limits_{1\le i\le n} |h_i| \to 0$ and $n(\min\limits_{1\le i\le n} h_i)^{2r+1} \to \infty$, then

$$\lim_{n\to\infty} E|f_n^{(r)}(x) - f_G^{(r)}(x)|^2 = 0.$$

(II) When $f_G(x) \in C_{s,a}$, taking $h_n = n^{-\frac{1}{2+s}}$, for $0 < \lambda \le 1$, then

$$E|f_n^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} \le c \cdot n^{-\frac{\lambda(s-2r)}{2+s}}.$$

Proof Proof of (I)

$$E|f_n^{(r)}(x) - f_G^{(r)}(x)|^2 \le 2|Ef_n^{(r)}(x) - f_G^{(r)}(x)|^2 + 2Var(f_n^{(r)}(x)) := 2(I_1^2 + I_2), \quad (15)$$

where

$$\begin{aligned} Ef_n^{(r)}(x) &= E[\tfrac{1}{n}\sum_{i=1}^{n} \tfrac{1}{h_i^{1+r}} K_r(\tfrac{x-X_i}{h_i})] \\ &= \tfrac{1}{n}\sum_{i=1}^{n} E[\tfrac{1}{h_i^{1+r}} K_r(\tfrac{x-X_i}{h_i})] \\ &= \tfrac{1}{n}\sum_{i=1}^{n} \tfrac{1}{h_i^{r}} \int_0^1 K_r(u) f_G(x - h_i u) du. \end{aligned}$$

We obtain the following Taylor expansion of $f_G(x - h_i u)$ in x

$$f_G(x-h_iu) = f_G(x) + \frac{f_G'(x)}{1!}(-h_iu) + \frac{f_G''(x)}{2!}(-h_iu)^2 + \cdots + \frac{f_G^{(r)}(x - \xi_i h_i u)}{r!}(-h_iu)^r$$

Since $f_G^{(r)}(x)$ is absolute continuous in x and (A1), it is easy to see that

$$\begin{aligned} |Ef_n^{(r)}(x) - f_G^{(r)}(x)| &= |\tfrac{1}{h_i^r} \int_0^1 K_r(u) f_G(x - h_i u) - f_G^{(r)}(x)| du \\ &\le \tfrac{1}{r!} \int_0^1 |K_r(u)||f_G^{(r)}(x - \xi_i h_i u) - f_G^{(r)}(x)| du \\ &\le c\tfrac{1}{r!} \int_0^1 u^r |K_r(u)||\xi_i h_i u| du \\ &\le c|h_i|. \end{aligned}$$

When $\max_{1\le i\le n}|h_i|\to 0$,we get

$$\lim_{n\to\infty}|Ef_n^{(r)}(x)-f_G^{(r)}(x)|^2=0.$$

That is to say

$$\lim_{n\to\infty}I_1^2=\lim_{n\to\infty}|Ef_n^{(r)}(x)-f_G^{(r)}(x)|^2=0. \tag{16}$$

Again since

$$\begin{aligned}I_2 &= Var[f_n^{(r)}(x)]\\ &= Var[\tfrac{1}{n}\sum_{i=1}^{n}\tfrac{1}{h_i^{1+r}}K_r(\tfrac{x-X_i}{h_i})]\\ &= \tfrac{1}{n^2}\sum_{i=1}^{n}\tfrac{1}{h_i^{2(1+r)}}Var[K_r(\tfrac{x-X_i}{h_i})]\\ &\le \tfrac{1}{n^2}\sum_{i=1}^{n}\tfrac{1}{h_i^{2(1+r)}}E[K_r(\tfrac{x-X_1}{h_i})]^2\\ &\le \tfrac{1}{n^2}\sum_{i=1}^{n}\tfrac{1}{h_i^{2r+1}}\int_0^1 K_r^2(u)f_G(x-h_iu)du\\ &\le \tfrac{c}{n^2}\sum_{i=1}^{n}\tfrac{1}{h_i^{2r+1}}\\ &\le \tfrac{c}{n}\min_{1\le i\le n}h_i^{-(2r+1)}\end{aligned}$$

When $n(\min_{1\le i\le n}h_i)^{2r+1}\to\infty$, we obtain

$$\lim_{n\to\infty}I_2=0. \tag{17}$$

Substituting (15) and (16) into (14), proof of (I) is completed.

Proof of (II) Similar to (14), we can show that

$$\begin{aligned}E|f_n^{(r)}(x)-f_G^{(r)}(x)|^{2\lambda} &\le 2[Ef_n^{(r)}(x)-f_G^{(r)}(x)]^{2\lambda}+2[Varf_n^{(r)}(x)]^{\lambda}\\ &:= 2(J_1^{2\lambda}+J_2^{\lambda}).\end{aligned} \tag{18}$$

We obtain the following Taylor expansion of $f_G(x-h_iu)$ in x

$$f_G(x-h_iu)=f_G(x)+\frac{f'_G(x)}{1!}(-h_iu)+\frac{f''_G(x)}{2!}(-h_iu)^2+\cdots+\frac{f_G^{(s)}(x-\xi_ih_iu)}{s!}(-h_iu)^s$$

where $0<\xi<1$, due to A_1 and $f_G(x)\in C_{s,\alpha}$, we have

$$\begin{aligned}E|f_n^{(r)}(x)-f_G^{(r)}(x)| &\le \tfrac{1}{n}\sum_{i=1}^{n}\int_0^1|K_r(u)|h_i^{s-r}u^s|\tfrac{f_G^{(s)}(x-\xi_ih_iu)}{s!}|du\\ &\le c(\max_{1\le i\le n}h_i)^{s-r}\end{aligned}$$

Therefore, we get $J_1^{2\lambda} = (\max\limits_{1\leq i\leq n} h_i)^{2\lambda(s-r)}$,

$$J_2^{\lambda} = [n(\min_{1\leq i\leq n} h_i)^{2r+1})]^{-\lambda}.$$

When $\max\limits_{1\leq i\leq n} h_i = n^{-\frac{1}{2(1+s)}}$ and $\min\limits_{1\leq i\leq n} h_i = n^{-\frac{2}{2(1+s)}}$,we get

$$E|f_n^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} \leq cn^{-\frac{\lambda(s-2r)}{s+1}}.$$

Obviously, proof of (II) is completed. □

Lemma 2.3. [6] *$R(\delta_G, G)$ and $R(\delta_n, G)$ are defined by (8) and (13), then*

$$0 \leq R(\delta_n, G) - R(\delta_G, G) \leq c\int_a^b |\beta(x)|P(|\beta_n(x) - \beta(x)| \geq |\beta(x)|)dx.$$

Lemma 2.4. *$v_{1n}(x)$ and $v_{2n}(x)$ are defined by* (10), (11), *for* $0 < \lambda \leq 1$, *we get*

$$E|v_{in}(x) - v_i(x)|^{2\lambda} \leq cn^{-\lambda}[w_i(x)]^{\lambda}, i = 1, 2.$$

where $w_1(x) = \int_a^x t^2 \frac{v(t)}{u(t)}dt, w_2(x) = \int_a^x \frac{v(t)}{u(t)}dt.$

Proof Applying moment monotone inequality, we have

$$(E|v_{in}(x) - v_i(x)|^{2\lambda})^{\frac{1}{2\lambda}} \leq (E|v_{in}(x) - v_i(x)|^2)^{\frac{1}{2}}$$

That is to say

$$E|v_{in}(x) - v_i(x)|^{2\lambda} \leq (E|v_{in}(x) - v_i(x)|^2)^{\lambda} := J$$

Since $Ev_{in}(x) = v_i(x)$, we can known $v_{in}(x)$ is an unbiased estimator of $v_i(x)$. By Lemma 2.1, we can easily get

$$J = E|v_{in}(x) - v_i(x)|^{2\lambda} \leq cn^{-\lambda},$$

the proof of Lemma 2.4 is completed. □

3 Asymptotic Optimality and Convergence Rates of Empirical Bayes Test

Theorem 3.1. *$\delta_n(x)$ is defined by (13). Assume (A1) and the following regularity conditions hold*

(1)$\int_a^b x^2 f_G(x)dx < \infty, \int_a^b x f_G(x)dx < \infty, f_G(x)$ is a absolute continuous function of x,

(2)$\int_a^b v_i(x)u(x)dx < \infty, \int_a^b w_i(x)u(x)dx < \infty, \int_a^b u(x)dx < \infty, i = 1, 2,$

(3)$\max\limits_{1\le i\le n} |h_i| \to 0$, $n(\min\limits_{1\le i\le n} h_i) \to \infty$, we have

$$\lim_{n\to\infty} R(\delta_n, G) = R(\delta_G, G).$$

Proof By Lemma 2.3, we have

$$0 \le R(\delta_n, G) - R(\delta_G, G) \le c\int_a^b |\beta(x)|P(|\beta_n(x) - \beta(x)| \ge |\beta(x)|)dx.$$

Note that $\Psi_n(x) = |\beta(x)|P(|\beta_n(x)-\beta(x)| \ge \beta(x)|)$, hence, $\Psi_n(x) \le |\beta(x)|$. By conditions of Theorem 3.1, we have

$$\begin{aligned}\int_a^b |\beta(x)|dx \quad &\le c[\sum_{i=1}^{2} \int_a^b u(x)v_i(x)dx + \int_a^b u(x)dx + \int_a^b x^2 f_G(x)dx \\ &+ \int_a^b x f_G(x)dx + \int_a^b f_G(x)dx] < \infty.\end{aligned}$$

By Markov inequality, we obtain $\Psi_n(x) \le E|\beta_n(x) - \beta(x)|$, hence

$$E|\beta_n(x)-\beta(x)| \le c\{\sum_{i=1}^{2} E[u(x)|v_{in}(x)-v_i(x)|]+(1+|x|+x^2)E|f_n(x)-f(x)|\}.$$

Applying domain convergence theorem, we obtain

$$0 \le \lim_{n\to\infty} R(\delta_n, G) - R(\delta_G, G) \le \int_a^b [\lim_{n\to\infty} \Psi_n(x)]dx. \tag{19}$$

If Theorem 3.1 holds, we only need to prove $\lim\limits_{n\to\infty} \Psi_n(x) = 0$, for $x \in (a, b)$. By Jensen inequality, we have

$$\begin{aligned}\Psi_n(x) \quad &\le c\{\sum_{i=1}^{2} E[u(x)|v_{in}(x) - v_i(x)|] + (1 + |x| + x^2)E|f_n(x) - f_G(x)|\} \\ &\le cu(x)[\sum_{i=1}^{2} E|v_{in}(x) - v_i(x)|^2]^{\frac{1}{2}} + (1 + |x| + x^2)[E|f_n(x) - f_G(x)|^2]^{\frac{1}{2}}.\end{aligned}$$

Again Lemma 2.2(1) and Lemma 2.4, for fixed $x \in (a,b)$ and $\lambda = 1$, we get

$$\begin{aligned} 0 \le \lim_{n\to\infty} \Psi_n(x) \quad &\le cu(x)[\sum_{i=1}^{2} \lim_{n\to\infty} E|v_{in}(x) - v_i(x)|^2]^{\frac{1}{2}} \\ &+(1+|x|+x^2)[\lim_{n\to\infty} E|f_n(x) - f(x)|^2]^{\frac{1}{2}} = 0. \end{aligned} \tag{20}$$

Substituting (3.2) into (3.1), the proof of Theorem 3.1 is completed. □

Theorem 3.2. *$\delta_n(x)$ is defined by (13). Assume (A1) and the following regularity conditions hold*

(4) $f_G(x) \in C_{s.\alpha}$,

(5) $\int_a^b |\beta(x)^{1-\lambda} x^{k\lambda} dx < \infty$, $k = 1, 2$, *for* $0 < \lambda \le 1$,

(6) $\int_a^b |\beta(x)^{1-\lambda} x^{k\lambda} u^\lambda(x)[w_i(x)]^{\lambda/2} dx < \infty$, $i = 1, 2$,

(7) $\max_{1\le i\le n} |h_i| \to 0$, $n(\min_{1\le i\le n} h_i) \to \infty$, *we have*

$$R(\delta_n, G) - R(\delta_G, G) = O(n^{-\frac{\lambda s}{2(s+1)}}),$$

where $s \ge 2$.

Proof By Lemma 2.3, Markov and C_r inequalities, we obtain

$$\begin{aligned} 0 \quad &\le R(\delta_n, G) - R(\delta_G, G) \le c \int_a^b |\beta(x)|^{1-\lambda} E|\beta_n(x) - \beta(x)|^\lambda dx \\ &\le c \int_a^b |\beta(x)|^{1-\lambda}[\sum_{i=1}^{2} u^\lambda(x) E|v_{in}(x) - v_i(x)|^\lambda \\ &+(1+|x|^\lambda + x^{2\lambda})E|f_n(x) - f(x)|^\lambda]dx. \end{aligned} \tag{21}$$

By Lemma 2.2 and 2.4, we get

$$\begin{aligned} 0 \le R(\delta_n, G) - R(\delta_G, G) \quad &\le c[\sum_{i=1}^{2} \int_a^b |\beta(x)|^{1-\lambda} u^\lambda(x) w_i^{\lambda/2}(x) n^{\lambda/2} dx \\ &+ \int_a^b |\beta(x)|^{1-\lambda}(1+|x|^\lambda + x^{2\lambda}) c n^{-\frac{\lambda s}{2(s+1)}}]dx. \end{aligned}$$

By conditions of Theorem 3.1, we have

$$0 \le R(\delta_n, G) - R(\delta_G, G) \le cn^{-\lambda s/[2(s+1)]},$$

hence

$$R(\delta_n, G) - R(\delta_G, G) = O(n^{-\lambda s/[2(s+1)]}).$$

The proof of Theorem 3.1 is completed. □

Remark 1 When $\lambda \to 1$, $O(n^{-\frac{\lambda s}{2(s+1)}})$ is arbitrarily close to $O(n^{-\frac{1}{2}})$.

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