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Numerical Treatment for Solving Nonlinear Integral Equation of The Second Kind

R. T. Matoog[1]

Abstract

In this work, we use the Toeplitz matrix method (TMM) as a numerical method to solve the nonlinear integral equation of the second kind when the kernel of the integral equation takes the logarithmic form and Carleman function form, respectively. The solution has a computing time requirement of 0(N2), where $(2N+1)$ is the number of discretization points used. Also, the error estimate is computed.

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[1] Department of Mathematics, Faculty of Applied Sciences,Umm Al-Qura University, Makkah, Saudi Arabia, P.O. Box 171. E-mail: r.matoog@yahoo.com

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1 Introduction

Many problems of mathematical physics, theory of elasticity, viscodynamics fluid and mixed problems of continues media lead to an integral equation with a kernel takes one of the following forms.

$$K_{n,m}^{\mu\lambda} = \frac{x^{\lambda}}{y^{\varepsilon+\lambda-1}} W_{n,m}^{\mu}\ (x, y), \qquad W_{n,m}^{\nu} = \int_0^{\infty} J_n\,(tx) J_m\,(ty) t^{\nu}\, dy\,. \tag{1.1}$$

where, J_n (.) is the Bessel function of the first kind of order n.

The kernel (1.1) takes some of the following different forms:

[1] When in (1.1) $n = m = \pm\frac{1}{2}$, $\lambda = \frac{1}{2}$, and $\nu = \varepsilon = 0$, we have a logarithmic function form $k(x - y) = \ell n \frac{1}{|x - y|}$ for symmetric and skew symmetric respectively.

[2] When in (1.1) $n = m = \pm\frac{1}{2}$, $\lambda = \frac{1}{2}$, $\varepsilon = 0$ and $0 \le \nu < 1$, we have Carleman function for symmetric and skew symmetric respectively $k\ (x, y) = |x - y|^{-\nu}$.

Many different analytic methods are established to solve the **FIE** of the first kind with logarithmic kernel, (see [1]) and with Carleman kernel (see [2]).

At the same time, different methods are used and established to solve the **FIE** of the second kind numerically, when the kernel takes a logarithmic form or Carleman from (see [3, 7]), Arutiunian [8] has shown that the plane contact problem of the nonlinear theory of plasticity, in its first approximation, can be reduced to **FIE** of the first kind with Carleman kernel.

[3] When in (1.1) $n = m = \varepsilon = \nu = 0$ and $\lambda = \frac{1}{2}$, we have an elliptic integral kernel $k(x, y) = \frac{2\sqrt{x\,y}}{\pi\,(x + y)}\ E\left(\frac{2\sqrt{x\,y}}{x + y}\right)$. Kovalenko, in his work [4] developed

the **FIF** of the first kind for the mechanics mixed problem of continuous media and obtained its approximate solution when the kernel in the elliptic integral form.

[4] When $\varepsilon = \nu = 0$, $\lambda = \frac{1}{2}$, and $n = m$, we have a potential kernel. Abdou in [9], obtained the solution of **FIE** of the second kind with potential kernel.

[5] When $\varepsilon = 0$, $\lambda = \frac{1}{2}$, $0 \le \nu < 1$, $n = m$, we have the generalized potential kernel (see [10]).

Consider the **NIE**

$$\gamma(x, \varphi(x)) - \mu \int_{-1}^{1} k\left(|x-y|\right) \varphi(y)\, dy = f(x), \qquad (-1 < x < 1), \tag{1.2}$$

Here, in the formula (1.2) we consider that the nonlinear given functional $\gamma(x,\varphi(x))$ is a monotone of the function $\varphi(x)$, so that for every pair of admissible two functions $\varphi_1(x)$ and $\varphi_2(x)$, we have $\{\varphi_1(x) - \varphi_2(x)\}$ $\{\gamma(x,\varphi_1(x)) - \gamma(x,\varphi_2(x))\} \ge 0$, ($|x| \le 1$). The given function f(x) is continuous with its derivatives for $x \in [-1,1]$ and μ is a constant may be complex. The integral formula (1.2) is considered in the work of Nemat – Nasser [6, 11], when the kernel $k(x, y)$ has a strong singularity.

In this paper a numerical method (**TMM**) is used to obtain an approximate solution of Eq. (1.2) in L_2 (-1, 1), where the kernel takes one of the following singularities:

$$k(x, y) = \ell n\, |x - y| \tag{1.3}$$

and

$$k(x, y) = |x - y|^{-\nu}, \quad 0 \le \nu < 1 \tag{1.4}$$

Our main object is to obtain a nonlinear system of $(2N+1)$ equations, where $(2N+1)$ is the number of discretization points used and the coefficients matrix is expressed as the sum of two metrics. One of them is Toeplitz matrix $a_{n,m} = a_{n-m}$, and the other is a matrix which has at least one non zero row (column). The error estimate is computed and some numerical examples are considered.

2 The Integral operator

In order to guarantee the existence of a unique solution of Eq. (1.2) we assume through this work following conditions:

(1) The discontinuous kernel of (1.2) satisfies the Fredholm condition

$$\left\{ \int_{-1}^{1}\int_{-1}^{1} k^2 \ (x-y)\ dx\, dy \right\} = c^2 < \infty.$$

(2) The monotone functional γ (x, ϕ (x)) satisfies the Lipschitiz condition with respect to the second argument ϕ (x).

(3) The function f (x) is continuous in $x \in [-1,1]$ and its norm in L_2 satisfies

$$\left\{ \int_{-1}^{1} \left| f\ (y) \right|^2 dy \right\}^{1/2} \leq A \left\| f \right\|_2, \ A \text{ is a constant.}$$

The *continuity* of the integral operator:

$K\phi = \int_{-1}^{1} k\,(x-y)\ \phi(y)\, dy$ in L_2 (-1, 1) can be proved. For taking $x_1, x_2 \in L_2(-1, 1)$, we have

$$\left| K_1\varphi - K_2\,\varphi \right| = \left| \int_{-1}^{1} k\,(x_1, y)\ \varphi\ \ (y)\ dy - \int_{-1}^{1} k\,(x_2, y)\,\varphi(y)\, dy \right|$$

$$\leq \left(\int_{-1}^{1} \varphi^2\ (y)\ dy \right)^{1/2} \left(\int_{-1}^{1} [k(x_1\ , y) - k(x_2\ , \ y)]^2\ \ dy \right)^{1/2}$$

$$\leq \left\| \varphi \right\|_2\ g\ (x_1, x_2)$$

where g $(x_1, x_2) \to 0$ as $x_1 \to x_2$.

Moreover, the normality of the integral operator can be proved as follows:

$$\|K\varphi\|_2 = \left[\int_{-1}^{1} dx \left(\int_{-1}^{1} k(x,y)\varphi(y)\,dy\right)^2\right]^{1/2}$$

$$\leq \left[\int_{-1}^{1} dx \int_{-1}^{1} \varphi^2(y)\,dy \int_{-1}^{1} k^2\ (x,y)\,dy\right]^{1/2} = C\|\varphi\|_2$$

So, we have $\|K\|_2 = C$, C is a constant.

3 Method of solution

The integral term of Eq. (1.2) can be written in the form

$$\int_{-1}^{1} k(x,y)\ \phi(y)\ dy = \sum_{n=-N}^{N-1} \int_{nh}^{(n+1)h} k(x,y)\phi(y)\,dy \qquad (h = \frac{1}{N}). \tag{3.1}$$

The integral of the right hand side of Eq. (3.1) may be written in the form

$$\int_{a}^{a+h} k(x,y)\ \phi(y)\ dy = A(x)\ \phi\ (a) + B\ (x)\ \phi\ (a+h) + R, \tag{3.2}$$

where A(x) and B(x)are arbitrary functions to be determined, R is the estimated error of order $O(h^2)$, and a = nh. A(x) and B (x) can be determined by putting $\phi(x) = 1$ and $\phi(x) = x$, R = 0, and solving the two resultant equations by putting x = mh and a = nh, $-N \leq m \leq N$, $-N \leq n \leq N-1$, in (3.2). Then Eq. (1.2) takes the form

$$\gamma(\Phi(mh)) - \mu b_{n,m}\ \Phi(mh) = F(mh) \tag{3.3}$$

where, we assume $\gamma(x, \phi(x)) = \gamma(\phi(x))$, $\Phi(mh) = \phi(x)$, $F(mh) = f(x)$, x = mh

Here Φ is a vector of (2N +1) elements, and the elements of the matrix $b_{n,m}$ are given by

$$b_{n,m} = \begin{cases} A_{-N}(mh) & n = -N \\ A_n(mh) + B_{n-1}(mh) & -N+1 \leq n \leq N-1 \\ B_{N-1}(mh) & n = N \end{cases}$$

The square matrix $b_{n,m}$ can be written in the form

$$b_{n,m} = b_{n-m} + \begin{Bmatrix} g_{-N,-N} & g_{-N,-N+1} \cdots\cdots & \cdots\cdots & g_{-N,N} \\ 0 & 0 & & 0 \\ d_{-N,-N} & d_{-N,-N+1} \cdots\cdots\cdots & \cdots & d_{-N,N} \end{Bmatrix}$$

The matrix b_{n-m} is the Toeplitz matrix of order $(2N+1)$, where $-N \le n,m \le N$, and the elements of the second matrix are zero except the elements of the first and last rows. Once the matrix $b_{n,m}$ have been obtained, we compute the approximate values $\Phi(mh)$ as solution of the system (3.3). In the linear case the system (3.3) is solved by Toepliz matrix with inner product. In the nonlinear case we compute $\Phi(mh)$ from (3.3) by using the nonlinear system Visual Basic Programming.

4 Convergence analysis

In our convergence analysis we examine the linear test equation

$$\phi(x) - \mu\int_{-1}^{1} k(x-y)\,\phi(y)\,dy = f(x) \qquad -1 \le x \le 1 \tag{4.1}$$

and assume that the forcing function f(x) with its derivatives belong to C[-1,1], and that the kernel k(x- y) is weakly singular of the form (1.3) or (1.4). Then, the Eq. (4.1) has a unique solution $\phi \in C[-1, 1]$. If, for a given mesh $\{x_j\}_{-N}^{N}$, we apply the **TMM** to the test equation (4.1), we obtain an approximate solution $\Phi_N(mh)$ in the form:

$$\phi_N(mh) = f(mh) + \mu \sum_{n=-N}^{N} b_{n,m}\,\Phi_N(nh), \tag{4.2}$$

where $b_{n,m}$ is given by (3.4).

The method is said to be convergent of order ℓ in [-1, 1] if and only if for N sufficiently large there exists a constant C > 0 independent of N such that

$$\|\phi(x) - \phi_N(mh)\|_\infty \le C\, N^{-\ell} \qquad (x = mh) \tag{4.3}$$

Also, in general the local truncation error is defined by

$$t_N\,(k,\phi,x) = \left|\int_{-1}^{1} k(x-y)\;\phi(y)\;dy\; - \sum_{n=-N}^{N} b_{n,m}\;\phi(nh)\right| \tag{4.4}$$

To examine the uniform convergence of the approximate solution $\phi_N(mh)$ to the exact solution $\phi(x)$ of (4.1), notice that

$$\phi(x) - \phi_N\,(mh) = \sum_{n=-N}^{N} b_{n,m}\,\{\phi(nh) - \phi_N\,(nh)\} + t_N\,(k,\phi,x) \tag{4.5}$$

where $t_N(k, \phi, x)$ is given by (4.4).

Hence, we have

$$\|\,\Phi - \Phi_N\|_\infty \;\le\; \left\|\,(I - D_N\,)^{-1}\right\|_\infty\;\|\,t_N\|_\infty \tag{4.6}$$

where D_N is the linear operator defined by $D_N : C[-1,1] \;\rightarrow C[-1,1]$,

$$D_N\;f(x) = \sum_{n=-N}^{N} b_{n,m}\;f(mh) \qquad ,\; f\;\in C[-1,1],\; x \in[-1,1] \tag{4.7}$$

To investigate the behavior of the term $\left\|\,(I - D_N\,)^{-1}\right\|_\infty$, we follow the way:

Firstly, let $L_N(f, y)$ denote the interpolating polynomial of degree $\le$ (2N +1) that coincide with the function f at the nodes $\{x_j\}_{j=-N}^{N}$, then for the kernel $k(x-y)$ in the form of (1.3) or (1.4), and $f \in C[-1,1]$, we have

$$\underset{N\to\infty}{Lim}\left\|\int_{-1}^{1} k(x-y)\;f(y)\;dy - \int_{-1}^{1} k(x-y)\;L_N\,f(y)\;dy\right\|_\infty = 0 \tag{4.8}$$

Secondly, with the aid of [1] , pp. 12014, we consider the following lemma

$$\underset{N}{Sup}\left\|\,\sum_{n=-N}^{N} b_{n,m}\;f\,(mh)\right\| \le C\|\,f\,\|_\infty \qquad \left(\|\,f\,\|_\infty\; = 1\right) \tag{4.9}$$

Moreover, the kernel k(x, y) of Eq. (1.3) and (1.4) satisfies the following:

For $k(x-y) \in L_g$, g > 0, we have

$$\underset{x^\backslash\to x}{Lim}\|\,k\,(x^\backslash - y) - k(x-y)\|_q = 0,\; x \in[-1,1] \tag{4.10}$$

Hence, we have

$$\underset{m^{\backslash}\to m}{Lim}\ \underset{N}{Sup}\ \sum_{n=-N}^{N}\left|b_{n,m}-b_{n,m^{\backslash}}\right|=0 \tag{4.11}$$

In this aim, we can say, if (4.8), (4.9) and (4.10) hold, then for all N sufficiently large there exist a constant C > 0 indpendent of N such that $\left\|(I-D_N)^{-1}\right\|_{\infty}\leq C$.

Moreover, we can say that the operator D defined by

$$D_N : C\,[-1,1]\to C\,[-1,1],\quad Df\,(x)=\int_{-1}^{1}k(x,\,y)\,f\,(y)\,dy\,, \tag{4.12}$$

is compact operator on $C\,[-1,1]$, if the kernel k(x-y) is defined and continuous for all $x,y\in[-1,1]$, $x\neq y$ and $\left|k\,(x-y)\right|\leq C\left|x-y\right|^{-\nu}$, $0\leq\nu<1$, for $x,y\in[-1,1]$, $x\neq y$. Furthermore, in the case $k(x-y)=\ell n\left|x-y\right|$, one can write

$$\begin{aligned}&\ell n\left|x-y\right|=h(x,y)\,.\,\left|x-y\right|^{-\nu}\,,\\&(h(x-y)=\ell n\left|x-y\right|.\,\left|x-y\right|^{\nu}\,,\,(0\leq\nu<1))\end{aligned} \tag{4.13}$$

where $h(x,y)\in C\,[-1,1]$ for $x\in[-1,1]$.

5 Numerical examples

Example 1. Consider the integral equation

$$\gamma(\varphi\,(x))-\mu\int_{-1}^{1}k\,(x-y)\,\varphi(y)\,dy=f\,(x)\,. \tag{5.1}$$

In Table 1 results from the numerical solution based on the relation between φ(x) and N, when $\gamma(\phi\,(x))=\varphi^2(x)$, where we take N = 20. If N takes large values, for example N = 100, that gives more clarification of the values of φ which is the nearest to the exact solution. Also, we deduce that as ν or μ increases the error is also, increases.

Table 1: μ=0.01

X	Exact	Logarithm	Err. L.	Carleman(ν=0.2)	Err. C.
-1	1.6000E-05	1.6000E-05	1.396E-11	1.6000E-05	1.7860E-11
-0.7	7.8400E-06	7.8400E-06	1.3216E-11	7.8400E-06	1.7733E-11
-0.6	5.7600E-06	5.7600E-06	1.0297E-11	5.7600E-06	1.7484E-11
-0.4	2.5600E-06	2.5600E-06	6.022E-12	2.5600E-06	1.7127E-11
-0.2	6.4000E-07	6.4000E-07	3.9199E-12	6.3998E-07	1.6954E-11
-0.1	1.6000E-07	1.6000E-07	3.4744E-12	1.5998E-07	1.6918E-11
0	0.0000E+00	3.3339E-12	3.33394E-12	-1.6906E-11	1.6906E-11
0.1	1.6000E-07	1.6000E-07	3.4744E-12	1.5998E-07	1.6918E-11
0.3	1.4400E-06	1.4400E-06	4.738E-12	1.4400E-06	1.7021E-11
0.5	4.0000E-06	4.0000E-06	7.865E-12	4.0000E-06	1.7278E-11
0.6	5.7600E-06	5.7600E-06	1.0297E-11	5.7600E-06	1.7484E-11
1	1.6000E-05	1.6000E-05	1.292E-11	1.6000E-05	1.7760E-11

Table 2: μ=0.2

X	Exact	Logarithm	Err. L.	Carleman(ν=0.3)	Err. C.
-1	4.0000E-04	4.0001E-04	8.6233E-09	3.9999E-04	1.1259E-08
-0.8	2.5600E-04	2.5601E-04	1.00619E-08	2.5599E-04	1.1311E-08
-0.6	1.4400E-04	1.4401E-04	6.4006E-09	1.4399E-04	1.0962E-08
-0.5	1.0000E-04	1.0000E-04	4.8903E-09	9.9989E-05	1.0824E-08
-0.4	6.4000E-05	6.4004E-05	3.7483E-09	6.3989E-05	1.0719E-08
-0.1	4.0000E-06	4.0022E-06	2.17052E-09	3.9894E-06	1.0575E-08
0	0.0000E+00	2.0837E-09	2.08371E-09	-1.0566E-08	1.0566E-08
0.1	4.0000E-06	4.0022E-06	2.17052E-09	3.9894E-06	1.0575E-08
0.3	3.6000E-05	3.6003E-05	2.95239E-09	3.5989E-05	1.0647E-08
0.4	6.4000E-05	6.4004E-05	3.7483E-09	6.3989E-05	1.0719E-08
0.6	1.4400E-04	1.4401E-04	6.4006E-09	1.4399E-04	1.0962E-08
1	4.0000E-04	4.0001E-04	7.9737E-09	3.9999E-04	1.1194E-08

Example 2. Consider the integral equation

$$\sqrt{\varphi(x)} - \mu \int_{-1}^{1} k(x-y)\varphi(y)\,dy = f(x)\,. \tag{5.2}$$

The relation between φ(x) and N= 20 is established in Table 3. The error estimate in each case is obtained.

Table 3: μ=0.2

X	Exact	Logarithm	Err. L.	Carleman(ν=0.2)	Err. C.
-1	4.0000E-04	4.0002E-04	1.6739E-08	3.9998E-04	2.3110E-08
-0.8	2.5600E-04	2.5602E-04	1.9531E-08	2.5598E-04	2.3349E-08
-0.7	1.9600E-04	1.9602E-04	1.5941E-08	1.9598E-04	2.2695E-08
-0.5	1.0000E-04	1.0001E-04	9.4928E-09	9.9978E-05	2.1601E-08
-0.3	3.6000E-05	3.6006E-05	5.7310E-09	3.5979E-05	2.0989E-08
-0.2	1.6000E-05	1.6005E-05	4.7482E-09	1.5979E-05	2.0832E-08
0	0.0000E+00	4.0448E-09	4.0448E-09	-2.0721E-08	2.0721E-08
0.1	4.0000E-06	4.0042E-06	4.2133E-09	3.9793E-06	2.0748E-08
0.2	1.6000E-05	1.6005E-05	4.7482E-09	1.5979E-05	2.0832E-08
0.3	3.6000E-05	3.6006E-05	5.7310E-09	3.5979E-05	2.0989E-08
0.6	1.4400E-04	1.4401E-04	1.2424E-08	1.4398E-04	2.2090E-08
0.7	1.9600E-04	1.9602E-04	1.5941E-08	1.9598E-04	2.2695E-08
1	4.0000E-04	4.0002E-04	1.5478E-08	3.9998E-04	2.2825E-08

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