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Two implicit finite difference methods for time fractional diffusion equation with source term

Yan Ma[1]

Abstract

Time fractional diffusion equation currently attracts attention because it is a useful tool to describe problems involving non-Markovian random walks. This kind of equation is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0,1)$. In this paper, two different implicit finite difference schemes for solving the time fractional diffusion equation with source term are presented and analyzed, where the fractional derivative is described in the Caputo sense. Numerical experiments illustrate the effectiveness and stability of these two methods respectively. Further, by using the Von Neumann method, the theoretical proof for stability is provided. Finally, a numerical example is given to compare the accuracy of the two mentioned finite difference methods.

Mathematics Subject Classification: 65M06

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[1] College of Technology and Engineering, Lanzhou University of Technology, Lan Zhou 730050, China.

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1 Introduction

As an extension of the classical integer order differential equation, fractional differential equation is a kind of equation which is formed by changing integer order derivatives in a standard differential equation into fractional order derivatives. It provides a valuable tool for describing materials with memory and hereditary properties as well as non-locality and dynamic transmission process of anomalous diffusion [1]. Because researching fractional differential equation has important scientific significance and great application prospect, so finding some effective methods to solve it is an actual and important problem. Various ways to solve fractional differential equation analytically have been proposed [2], including Green function method, Laplace and Fourier transform method, but most of fractional differential equations cannot be solved analytically. Therefore, to develop numerical methods for solving fractional differential equation seems to be necessary and important. Scholars have put forward many effective numerical methods : such as finite difference method, finite element method, random walk approach, spectral method, the decomposition method, the homotopy perturbation method, the integral equation method, reproducing kernel method, the variational iteration method and so many others[3]. In this paper, we will use finite difference method to examine the numerical solution of one kind of important fractional differential equation----time fractional diffusion equation. The diffusion equation describes the spread of particles from a region of higher concentration to a region of lower concentration due to collisions of the molecules and Brownian motion. While time fractional diffusion equation is a generalization of the classical diffusion equation, which is obtained from the standard diffusion equation by replacing the first-order time derivative with a fractional derivative of order α , with $0<\alpha<1$. It can be used to treat sub-diffusive flow process, in which the net motion of the particles happens more slowly than Brownian motion [4].

Consider following time fractional diffusion equation with source term :

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial^{2}u(x,t)}{\partial x^{2}}+f(x,t) \qquad 0\le x\le L,\quad 0\le t\le T,\quad 0<\alpha<1, \tag{1}$$

with initial condition

$$u(x,0)=g(x), \qquad 0\le x\le L, \tag{2}$$

and Dirichlet boundary conditions

$$u(0,t)=L(t),\quad u(L,t)=R(t), \qquad 0\le t\le T\ . \tag{3}$$

with $L(0)=g(0)$ and $R(0)=g(L)$ for consistency.

Here $f(x,t),\ g(x),\ L(t),\ R(t)$ are known functions, while the function $u(x,t)$ is unknown. $\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}$ in (1) is defined as the Caputo fractional derivative of order α, given by [5]

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-\tau)^{-\alpha}\frac{\partial u(x,\tau)}{\partial\tau}d\tau \qquad 0<\alpha<1. \tag{4}$$

In view of the research objective of this paper, we investigate the current research status of time fractional diffusion equation with source term. We mainly focus on the discretization technique of time fractional derivative and stability proof method. Karatay et al.[6] proposed a method for solving inhomogeneous nonlocal fractional diffusion equation, in which time fractional derivative is defined by Caputo definition. This method was based on the modified Gauss elimination method. It was proved using the matrix stability approach that the method was unconditionally stable. Lin and Xu [7] constructed and analyzed a stable and high order scheme to efficiently solve the same model as Karatay et al.[6], but with the standard initial condition. The proposed method was based on a finite difference scheme in time and Legendre spectral methods in space. Wei et al.[8] presented and analyzed an implicit scheme, which is based on a finite difference method in time and local discontinuous Galerkin methods in space. Al-Shibani et al.[9] discussed a numerical scheme based on Keller box method for one dimensional time fractional diffusion equation. The fractional derivative term was replaced by the Grünwald-Letnikov formula. Unconditional stability was shown by means of the Von Neumann method. Gao et al.[3] considered fractional anomalous sub-diffusion equations on an unbounded domain. This paper’ main contribution lies in the reduction of fractional differential equations on an unbounded domain by using artificial boundary conditions and construction of the corresponding finite difference scheme with the help of method of order reduction. The stability of the scheme were proved using the discrete energy method.

In this paper, we will try to use two different discretization formulas to estimate time fractional derivative, which are cited from papers Karatay et al.[6], Lin and Xu [7] respectively. For the second-order space derivative in this equation, we will adopt the classical central difference approximation. Then using the basic algebra knowledge to derive two different implicit finite difference schemes, which are both effective for solving our problem. Among them, for the first scheme, it’s same with the one proposed in paper [6], but [6] considered the nonlocal condition and used the idea on the modified Gauss-Elimination method based on matrix form, while we will consider the general case and use the algebra knowledge to derive the final implicit scheme. And in paper [6], authors proved stability using matrix stability approach, while we will use Von Neumann method. For the second scheme, compared with paper [7], we adopt the same formula to discretize time fractional derivative, but for estimating space derivative, Lin and Xu [7] used Legendre spectral methods, while we will use central difference approximation. During stability analysis, we will adopt Von Neumann method based on mathematical induction to give the proof according to our own cases and try to work out the properties about the coefficients of schemes, which will play an important role in proving stability. At last, we will make a comparison between the exact solutions and the numerical solutions given by these two methods to conclude which method is more accurate.

The structure of this article is as follows: in section 2 and section 3, we respectively discuss two different finite difference methods for solving time fractional diffusion equation with source term, including their implicit schemes, reliability and stability proof. In section 4, numerical results are shown to compare the accuracy of the two mentioned methods.

2 First Finite Difference Method for Time Fractional Diffusion Equation with Source Term

2.1 Construction of finite difference scheme

In this part, we will discuss a finite difference approximation according to the following ways to discretize time fractional derivative and space second order derivative in time fractional diffusion equation (1) (2) (3). To do this,

Let $t_n = n\Delta t \quad (n = 0,1,2,\cdots,N)$, where $\Delta t = \frac{T}{N}$ is the time step.

$x_i = i\Delta x \quad (i = 0,1,2,\cdots,M)$, where $\Delta x = \frac{L}{M}$ is the space step.

Suppose that $u(x_i, t_n)$ is the exact solution of equation (1) (2) (3) at grid point (x_i, t_n), u_i^n denotes the numerical approximation to $u(x_i, t_n)$.

The time fractional derivative of order α is discretized by using Caputo finite difference formula, which is a first order approximation appeared in Karatay et al.[6] :

$$\frac{\partial^\alpha u(x_i, t_n)}{\partial t^\alpha} = \Delta t^{-\alpha} \sum_{j=0}^{n} v_j (u_i^{n-j} - u_i^0) + o(\Delta t)$$

where

$$v_0 = 1, v_j = (1 - \frac{\alpha + 1}{j}) v_{j-1} \quad j = 1,2,\cdots. \tag{5}$$

For the spatial second derivative, central difference approximation is used:

$$\frac{\partial^2 u(x_i, t_n)}{\partial x^2} = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{\Delta x^2} + o(\Delta x^2) \tag{6}$$

Substitute (4) and (5) into equation (1)

$$\frac{\partial^\alpha u(x_i, t_n)}{\partial t^\alpha} = \frac{\partial^2 u(x_i, t_n)}{\partial x^2} + f(x_i, t_n) \tag{7}$$

The following finite difference scheme can be obtained :

$$\left(u_i^n - u_i^0\right) + \sum_{j=1}^{n} v_j (u_i^{n-j} - u_i^0) = \frac{\Delta t^\alpha}{\Delta x^2} (u_{i+1}^n - 2u_i^n + u_{i-1}^n) + \Delta t^\alpha f(x_i, t_n)$$

For the sake of simplification, let us introduce the notation :

$$r = \frac{\Delta t^{\alpha}}{\Delta x^2} (> 0)$$

then we can get

$$-ru_{i+1}^{n} + (1+2r)u_i^n - ru_{i-1}^{n} = u_i^0 - \sum_{j=1}^{n} v_j (u_i^{n-j} - u_i^0) + \Delta t^{\alpha} f(x_i, t_n)$$

where

$$i = 1,2,\cdots, M-1, \quad n = 1,2,\cdots N, \quad v_0 = 1, v_j = (1 - \frac{\alpha+1}{j}) v_{j-1}, \quad j = 1,2,\cdots. \quad (8)$$

So, the first implicit finite difference scheme we've derived to solve time fractional diffusion equation (1) (2) (3) can be written as follows :

When $n = 1$

$$-ru_{i+1}^{1} + (1+2r)u_i^1 - ru_{i-1}^{1} = u_i^0 + \Delta t^{\alpha} f(x_i, t_1). \quad (9)$$

When $n \geq 2$ $\quad -ru_{i+1}^{n} + (1+2r)u_i^n - ru_{i-1}^{n} = -\sum_{j=1}^{n-1} v_j u_i^{n-j} + (\sum_{j=0}^{n-1} v_j) u_i^0 + \Delta t^{\alpha} f(x_i, t_n)$

where

$$i = 1,2,\cdots, M-1, \quad n = 2,3,\cdots N. \quad (10)$$

2.2 Numerical experiments for effectiveness

In this part, we shall illustrate several experiments to show the effectiveness and stability of the method presented above. We will check the agreement behavior between numerical solution and exact solution by using fixed space step Δx and different time step Δt.

Let us consider following time fractional diffusion equation [10]:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^2 u(x,t)}{\partial x^2}+\left(\frac{2e^x t^{2-\alpha}}{\Gamma(3-\alpha)}-t^2e^x\right) \quad \text{for} \quad \alpha=0.5. \tag{11}$$

with the initial condition

$$u(x,0)=0, \qquad 0\le x\le 1. \tag{12}$$

and the boundary conditions

$$u(0,t)=t^2, \qquad u(1,t)=et^2, \quad 0\le t\le T. \tag{13}$$

The exact solution of this fractional diffusion equation is given by

$$u(x,t)=t^2e^x. \tag{14}$$

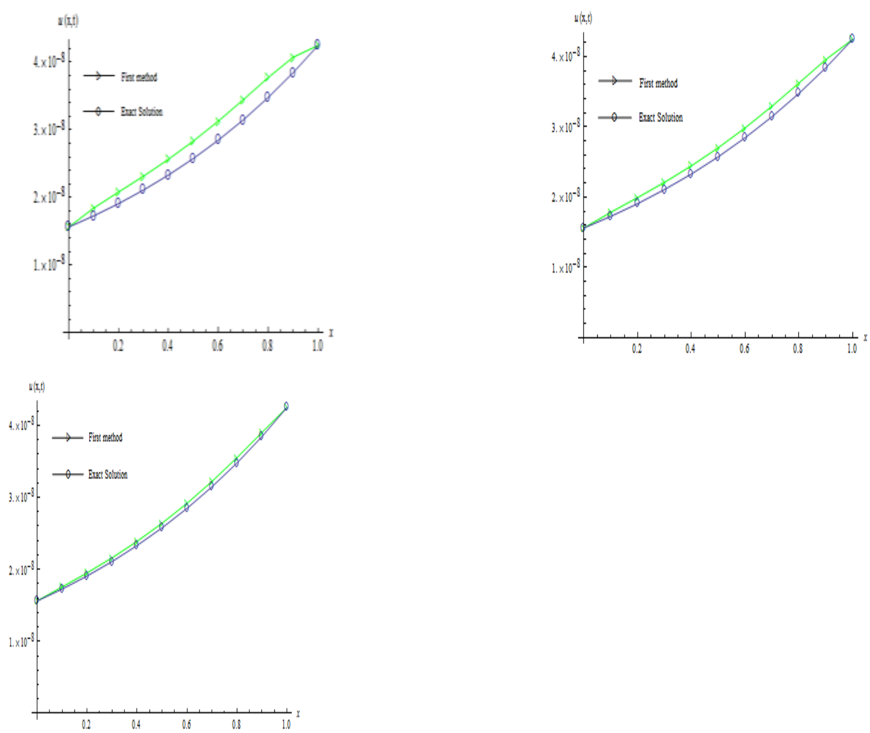

Figure 1 : Comparison between numerical solution and exact solution at

$T=1.25\times10^{-4}$ $\Delta x=0.1, \Delta t=2.5\times10^{-5}$, $\frac{1}{2}\Delta t=1.25\times10^{-5}$, $\frac{1}{4}\Delta t=0.625\times10^{-5}$

From the figures above, we can see that a relatively good agreement can be achieved between numerical solution and exact solution for this particular example. This means this method is feasible for the case we consider. In addition, from the error results under different time step, we observe that our computation is stable.

2.3 Theoretical proof for stability

Lemma 2.3.1 The coefficients $v_j = (-1)^j \binom{\alpha}{j}$ $(j = 0,1,2,\cdots)$ satisfy

(1) $v_0 = 1$, $v_j < 0$ $j = 1,2,3,\cdots$;

(2) $\sum_{j=0}^{k-1} v_j > 0$ $k = 2,3,\cdots$.

Theorem 2.3.1 Implicit finite difference scheme defined by (9) (10) is unconditionally stable.

Proof : Assume that discretization of initial condition introduces the error ε_i^0 . Let $\tilde{g}_i^0 = g_i^0 + \varepsilon_i^0$, u_i^n and $\tilde{u}_i^n$ are the numerical solutions of scheme(9) (10) with respect to initial datas g_i^0 and $\tilde{g}_i^0$, respectively.

Suppose that the calculation of $f(x_i, t_n)$ is accurate, then the error is defined as :

$\varepsilon_i^n = \tilde{u}_i^n - u_i^n$

which satisfies the finite difference equations (9) and (10), and this gives :

When $n = 1$:

$$-r\varepsilon_{i+1}^1 + (1+2r)\varepsilon_i^1 - r\varepsilon_{i-1}^1 = \varepsilon_i^0 \tag{15}$$

When $n \ge 2$:

$$-r\varepsilon_{i+1}^{n} + (1+2r)\varepsilon_{i}^{n} - r\varepsilon_{i-1}^{n} = -\sum_{j=1}^{n-1} v_j \varepsilon_i^{n-j} + (\sum_{j=0}^{n-1} v_j)\varepsilon_i^{0} \tag{16}$$

Here we use Von Neumann method and apply mathematical induction to investigate the stability of the first finite difference scheme (9) (10).

To do this, we suppose that ε_i^n can be expressed in the form

$$\varepsilon_i^n = \lambda^n e^{\sqrt{-1}\theta i} \tag{17}$$

Substituting (17) into (15) and (16) gives :

When $n = 1$:

$$-r\lambda e^{\sqrt{-1}\theta(i+1)} + (1+2r)\lambda e^{\sqrt{-1}\theta i} - r\lambda e^{\sqrt{-1}\theta(i-1)} = e^{\sqrt{-1}\theta i} \tag{18}$$

When $n \ge 2$:

$$-r\lambda^n e^{\sqrt{-1}\theta(i+1)} + (1+2r)\lambda^n e^{\sqrt{-1}\theta i} - r\lambda^n e^{\sqrt{-1}\theta(i-1)} = -\sum_{j=1}^{n-1} v_j \lambda^{n-j} e^{\sqrt{-1}\theta i} + (\sum_{j=0}^{n-1} v_j) e^{\sqrt{-1}\theta i} \tag{19}$$

Begin with $n = 1$, from (18) to get

$$-r\lambda e^{\sqrt{-1}\theta(i+1)} + (1+2r)\lambda e^{\sqrt{-1}\theta i} - r\lambda e^{\sqrt{-1}\theta(i-1)} = e^{\sqrt{-1}\theta i}$$

$$\lambda = \frac{1}{1+2r(1-\cos\theta)}$$

Obviously, $$\left|\lambda^1\right| = \frac{1}{1+2r(1-\cos\theta)} \le 1.$$

Now, suppose that $\left|\lambda^m\right| \le 1$, $m = 1,2,\cdots,n-1$,

then from the Lemma 2.3.1 and (19)

$$-r\lambda^n e^{\sqrt{-1}\theta(i+1)} + (1+2r)\lambda^n e^{\sqrt{-1}\theta i} - r\lambda^n e^{\sqrt{-1}\theta(i-1)} = -\sum_{j=1}^{n-1} v_j \lambda^{n-j} e^{\sqrt{-\theta} i} + (\sum_{j=0}^{n-1} v_j) e^{\sqrt{-1}\theta i}$$

So, $$\lambda^n = \frac{1}{1+2r(1-\cos\theta)}[-\sum_{j=1}^{n-1} v_j \lambda^{n-j} + \sum_{j=0}^{n-1} v_j]$$

Therefore, $$\left|\lambda^n\right| = \frac{1}{1+2r(1-\cos\theta)}[\sum_{j=1}^{n-1}(-v_j)\lambda^{n-j} + \sum_{j=0}^{n-1} v_j]$$

$$\leq \frac{1}{1+2r(1-\cos\theta)}[\sum_{j=1}^{n-1}(-v_j)\left|\lambda^{n-j}\right| + \sum_{j=0}^{n-1} v_j]$$

$$\leq \frac{1}{1+2r(1-\cos\theta)}[\sum_{j=1}^{n-1}(-v_j) + \sum_{j=0}^{n-1} v_j]$$

$$= \frac{1}{1+2r(1-\cos\theta)} \cdot v_0$$

$$= \frac{1}{1+2r(1-\cos\theta)} \leq 1.$$

Hence, for all n, θ, we have $\left|\lambda^n\right| \leq 1$. Therefore, according to Von Neumann's criterion for stability, the implicit finite difference scheme defined by (9) (10) is unconditionally stable.

3 Second Finite Difference Method for Time Fractional Diffusion Equation with Source Term

3.1 Construction of finite difference scheme

In this part, we will introduce another finite difference approximation to solve this time fractional diffusion equation (1) (2) (3).Similarly,

Let $t_n = n\Delta t \quad (n = 0,1,2,\cdots N)$, where $\Delta t = \frac{T}{N}$ is the time step.

$$x_i = i\Delta x \quad (i = 0,1,2,\cdots M)\text{, where } \Delta x = \frac{L}{M} \text{ is the space step.}$$

u_i^n denotes the numerical approximation to the exact solution $u(x_i, t_n)$.

Use the following formula to discretize the time fractional derivative, which is cited from Lin & Xu [7] :

$$\begin{aligned}&\frac{\partial^\alpha u(x_i,t_n)}{\partial t^\alpha}\\ &=\frac{1}{\Gamma(2-\alpha)}\sum_{j=0}^{n-1}\frac{u(x_i,t_{n-j})-u(x_i,t_{n-j-1})}{\Delta t^\alpha}[(j+1)^{1-\alpha}-j^{1-\alpha}]+o(\Delta t^{2-\alpha})\end{aligned} \quad . \tag{20}$$

Use central difference approximation to discretize the space second order derivative :

$$\frac{\partial^2 u(x_i,t_n)}{\partial x^2}=\frac{u(x_{i+1},t_n)-2u(x_i,t_n)+u(x_{i-1},t_n)}{\Delta x^2}+o(\Delta x^2)\,. \tag{21}$$

Substituting (20) and (21) into equation (1)

$$\left.\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}\right|_{\substack{x=x_i\\ t=t_n}}=\left.\frac{\partial^2 u(x,t)}{\partial x^2}\right|_{\substack{x=x_i\\ t=t_n}}+f(x_i,t_n)\,. \tag{22}$$

$$(u_i^n-u_i^{n-1})+\sum_{j=1}^{n-1}[(j+1)^{1-\alpha}-j^{1-\alpha}](u_i^{n-j}-u_i^{n-j-1})$$

$$=\frac{\Delta t^\alpha}{\Delta x^2}\Gamma(2-\alpha)(u_{i+1}^n-2u_i^n+u_{i-1}^n)+\Delta t^\alpha\Gamma(2-\alpha)f(x_i,t_n)\,.$$

For the sake of simplification, we still use the notation:

$$r=\frac{\Delta t^\alpha}{\Delta x^2}$$

then, we can get

$$-r\Gamma(2-\alpha)u_{i+1}^n+(1+2r\Gamma(2-\alpha))u_i^n-r\Gamma(2-\alpha)u_{i-1}^n$$

$$= u_i^{n-1} - \sum_{j=1}^{n-1}[(j+1)^{1-\alpha} - j^{1-\alpha}](u_i^{n-j} - u_i^{n-j-1}) + \Delta t^{\alpha}\Gamma(2-\alpha) f(x_i, t_n).$$

Suppose that $\mu = r\Gamma(2-\alpha)$

We will have

$$-\mu u_{i+1}^n + (1+2\mu)u_i^n - \mu u_{i-1}^n$$
$$= u_i^{n-1} - \sum_{j=1}^{n-1}[(j+1)^{1-\alpha} - j^{1-\alpha}](u_i^{n-j} - u_i^{n-j-1}) + \Delta t^{\alpha}\Gamma(2-\alpha) f(x_i, t_n)$$

where

$$i = 1,2,\cdots, M-1, \quad n = 1,2,\cdots, N. \tag{23}$$

Further :

When $n = 1$:

$$-\mu u_{i+1}^1 + (1+2\mu)u_i^1 - \mu u_{i-1}^1 = u_i^0 + \Delta t^{\alpha}\Gamma(2-\alpha) f(x_i, t_1). \tag{24}$$

When $n \geq 2$:

$$-\mu u_{i+1}^n + (1+2\mu)u_i^n - \mu u_{i-1}^n$$

$$= u_i^{n-1} - \sum_{j=1}^{n-1}[(j+1)^{1-\alpha} - j^{1-\alpha}]u_i^{n-j} + \sum_{j=1}^{n-1}[(j+1)^{1-\alpha} - j^{1-\alpha}]u_i^{n-j-1} + \Delta t^{\alpha}\Gamma(2-\alpha) f(x_i, t_n)$$

$$-\mu u_{i+1}^n + (1+2\mu)u_i^n - \mu u_{i-1}^n$$

$$= u_i^{n-1} - (2^{1-\alpha} - 1)u_i^{n-1} - \sum_{j=2}^{n-1}[(j+1)^{1-\alpha} - j^{1-\alpha}]u_i^{n-j} +$$
$$\sum_{j=1}^{n-2}[(j+1)^{1-\alpha} - j^{1-\alpha}]u_i^{n-j-1} + [n^{1-\alpha} - (n-1)^{1-\alpha}]u_i^0 + \Delta t^{\alpha}\Gamma(2-\alpha) f(x_i, t_n)$$

$$-\mu u_{i+1}^n + (1+2\mu)u_i^n - \mu u_{i-1}^n$$

$$= (2 - 2^{1-\alpha})u_i^{n-1} + \sum_{j=1}^{n-2}[2(j+1)^{1-\alpha} - (j+2)^{1-\alpha} - j^{1-\alpha}]u_i^{n-j-1} +$$
$$[n^{1-\alpha} - (n-1)^{1-\alpha}]u_i^0 + \Delta t^{\alpha}\Gamma(2-\alpha) f(x_i, t_n) \tag{25}$$

If we let $w_j = (j+1)^{1-\alpha} - j^{1-\alpha} \qquad j = 0,1,2,\cdots,n-1,$

then

$$w_0 = 1, \quad w_1 = 2^{1-\alpha} - 1, \quad \text{so } w_0 - w_1 = 2 - 2^{1-\alpha},$$

$$w_{j+1} = (j+2)^{1-\alpha} - (j+1)^{1-\alpha},$$

$$w_j - w_{j+1} = 2(j+1)^{1-\alpha} - (j+2)^{1-\alpha} - j^{1-\alpha},$$

$$w_{n-1} = n^{1-\alpha} - (n-1)^{1-\alpha},$$

therefore, (25) becomes

$$-\mu u_{i+1}^n + (1+2\mu)u_i^n - \mu u_{i-1}^n$$

$$= (w_0 - w_1)u_i^{n-1} + \sum_{j=1}^{n-2}(w_j - w_{j+1})u_i^{n-j-1} + w_{n-1}u_i^0 + \Delta t^\alpha \Gamma(2-\alpha) f(x_i, t_n). \quad (26)$$

So, the second implicit finite difference scheme we've derived to solve time fractional diffusion equation (1) (2) (3) can be written as follows:

When $n = 1$:

$$-\mu u_{i+1}^1 + (1+2\mu)u_i^1 - \mu u_{i-1}^1 = u_i^0 + \Delta t^\alpha \Gamma(2-\alpha) f(x_i, t_1). \quad (27)$$

When $n \geq 2$:

$$-\mu u_{i+1}^n + (1+2\mu)u_i^n - \mu u_{i-1}^n$$

$$= (w_0 - w_1)u_i^{n-1} + \sum_{j=1}^{n-2}(w_j - w_{j+1})u_i^{n-j-1} + w_{n-1}u_i^0 + \Delta t^\alpha \Gamma(2-\alpha) f(x_i, t_n)$$

where

$$i = 1,2,\cdots,M-1, \quad n = 2,3,\cdots,N. \quad (28)$$

3.2 Numerical experiments for effectiveness

In this part, we will still use the example (Takaci et al.[10]) mentioned in section 2 (11) ~ (14) to check the effectiveness of the second method in the same way.

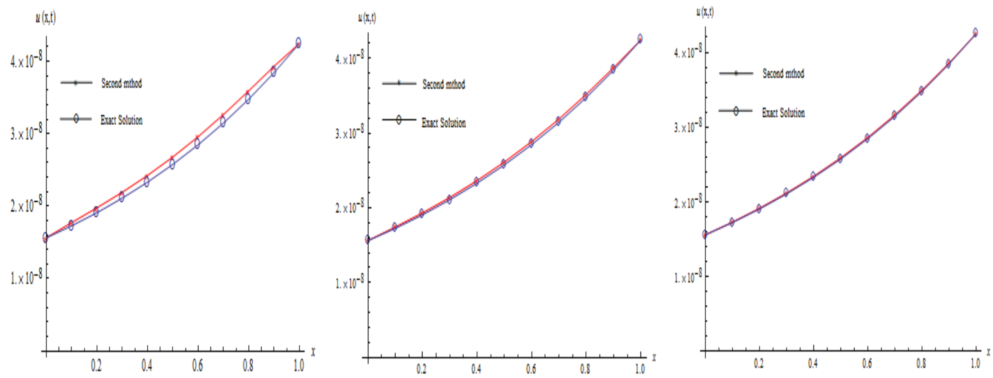

Figure 2 : Comparison of the numerical solution and the exact solution for $\Delta x = 0.1$ $\Delta t = 2.5\times10^{-5}$, $\frac{1}{2}\Delta t = 1.25\times10^{-5}$, $\frac{1}{4}\Delta t = 0.625\times10^{-5}$, at time $T = 1.25\times10^{-4}$

From these figures, we can draw the conclusion that the second scheme can also be accepted to solve this particular example.

3.3 Theoretical proof for stability

Lemma 3.3.1 The coefficients $w_j = (j+1)^{1-\alpha} - j^{1-\alpha}$ ($j = 0,1,2,\cdots$) satisfy :

(1) $w_0 = 1$, $w_j > 0$, $j = 1,2,\cdots$;

(2) $w_j - w_{j+1} > 0$, $j = 0,1,2,\cdots$.

Theorem 3.3.1 Implicit finite difference scheme defined by (27) (28) is unconditionally stable.

Proof : The investigation about stability is completed by Von Neumann method utilizing mathematical induction.

Suppose that discretization of initial condition introduces the error ε_i^0.

Let $\tilde{g}_i^0 = g_i^0 + \varepsilon_i^0$, u_i^n and $\tilde{u}_i^n$ are the numerical solutions of scheme (27), (28) with respect to initial datas g_i^0 and $\tilde{g}_i^0$ respectively .

Assume that the calculation of $f(x_i, t_n)$ is accurate, then the error $\varepsilon_i^n = \tilde{u}_i^n - u_i^n$ satisfies :

When $n = 1$:

$$-\mu\varepsilon_{i+1}^1 + (1+2\mu)\varepsilon_i^1 - \mu\varepsilon_{i-1}^1 = \varepsilon_i^0 . \tag{29}$$

When $n \ge 2$:

$$-\mu\varepsilon_{i+1}^n + (1+2\mu)\varepsilon_i^n - \mu\varepsilon_{i-1}^n = (w_0 - w_1)\varepsilon_i^{n-1} + \sum_{j=1}^{n-2}(w_j - w_{j+1})\varepsilon_i^{n-j-1} + w_{n-1}\varepsilon_i^0 . \tag{30}$$

Suppose ε_i^n can be expressed in the form

$$\varepsilon_i^n = \lambda^n e^{\sqrt{-1}\theta i} \tag{31}$$

Substitute (31) into (29) and (30), we can get

$$-\mu\lambda e^{\sqrt{-1}\theta(i+1)} + (1+2\mu)\lambda e^{\sqrt{-1}\theta i} - \mu\lambda e^{\sqrt{-1}\theta(i-1)} = e^{\sqrt{-1}\theta i} \tag{32}$$

$$-\mu\lambda^n e^{\sqrt{-1}\theta(i+1)} + (1+2\mu)\lambda^n e^{\sqrt{-1}\theta i} - \mu\lambda^n e^{\sqrt{-1}\theta(i-1)}$$

$$= (w_0 - w_1)\lambda^{n-1} e^{\sqrt{-1}\theta i} + \sum_{j=1}^{n-2}(w_j - w_{j+1})\lambda^{n-j-1} e^{\sqrt{-1}\theta i} + w_{n-1} e^{\sqrt{-1}\theta i} \tag{33}$$

Begin with $n = 1$, from (32), we have

$$\lambda = \frac{1}{1+2\mu(1-\cos\theta)}$$

Hence,

$$\left|\lambda^1\right| = \frac{1}{1+2\mu(1-\cos\theta)} \le 1.$$

Now, suppose that

$$\left|\lambda^m\right| \le 1, \quad m = 1,2,\cdots n-1.$$

From (33) and Lemma 3.3.1, we know that

$$\lambda^n = \frac{1}{1+2\mu(1-\cos\theta)}[(w_0 - w_1)\lambda^{n-1} + \sum_{j=1}^{n-2}(w_j - w_{j+1})\lambda^{n-j-1} + w_{n-1}]$$

So,

$$\left|\lambda^n\right| = \frac{1}{1+2\mu(1-\cos\theta)}\left|(w_0 - w_1)\lambda^{n-1} + \sum_{j=1}^{n-2}(w_j - w_{j+1})\lambda^{n-j-1} + w_{n-1}\right|$$

$$\le \frac{1}{1+2\mu(1-\cos\theta)}[(w_0 - w_1)\left|\lambda^{n-1}\right| + \sum_{j=1}^{n-2}(w_j - w_{j+1})\left|\lambda^{n-j-1}\right| + w_{n-1}]$$

$$\le \frac{1}{1+2\mu(1-\cos\theta)}[(w_0 - w_1) + \sum_{j=1}^{n-2}(w_j - w_{j+1}) + w_{n-1}]$$

$$= \frac{1}{1+2\mu(1-\cos\theta)} \cdot w_0$$

$$= \frac{1}{1+2\mu(1-\cos\theta)} \le 1.$$

Therefore, for all n, θ, we have $\left|\lambda^n\right| \le 1$.

According to the Von Neumann criteria about stability, we can get the conclusion that the implicit finite difference scheme (27) (28) is unconditionally stable.

4 Accuracy comparison of two methods

In this section, we will construct a comparison of the accuracy of the two implicit finite difference schemes discussed in section 2 and section 3 respectively. We will still use the previous example to compare exact solution and numerical solutions obtained using these two different technique to support the theoretical statements.

4.1 Numerical experiment

Remark 4.1.1 The accuracy of the first finite difference scheme is $o(\Delta x^2)$ in the spatial grid size and $o(\Delta t)$ in the fractional time step [6].

Remark 4.1.2 The accuracy of the second finite difference scheme is $o(\Delta x^2)$ in

Table 1 : Relative errors of two methods at $T = 1.25 \times 10^{-4}$,

$\Delta x = 0.1$, $\Delta t = 0.625 \times 10^{-5}$

x	First method	Second method
0.1	1.68980×10^{-2}	3.57302×10^{-3}
0.2	2.25787×10^{-2}	4.86785×10^{-3}
0.3	2.44079×10^{-2}	5.31020×10^{-3}
0.4	2.49380×10^{-2}	5.43977×10^{-3}
0.5	2.49832×10^{-2}	5.45392×10^{-3}
0.6	2.46816×10^{-2}	5.37396×10^{-3}
0.7	2.37376×10^{-2}	5.13588×10^{-3}
0.8	2.12658×10^{-2}	4.54074×10^{-3}
0.9	1.50528×10^{-2}	3.13807×10^{-3}
Average of Errors	2.20604×10^{-2}	4.75927×10^{-3}

the spatial grid size and $o(\Delta t^{2-\alpha})$ in the fractional time step [7].

This means that in theory, the second method will be more accurate than the first one. In fact, the following numerical experiment [10] supports this conclusion.

Clearly, the second method's solution is more accurate than the first one.

5 Conclusions

In this paper, we have presented two methods for solving time fractional diffusion equation with source term. For the second order spatial derivative term in this equation, both methods adopted central difference approximation with second order accuracy. Whilst, for the time fractional derivative term, two different formulae were used to discretize it in these two methods. The first formula is based on the relationship between Caputo fractional derivative and Grünwald-Litnikov fractional derivative, derived from standard Grünwald-Litnikov formula, which can achieve first order accuracy in time. The second formula is derived directly from the definition of Caputo fractional derivative by using numerical integration method, which can achieve $(1 < 2-\alpha < 2)$ order accuracy in time. Based on these two different discrete formulae, two implicit finite difference schemes were derived to solve our target equation. Numerical experimental work examined that these two schemes can both effectively solve our equation, while the second scheme is preferable than the first one. With the aid of mathematical induction, by Von Neumann method, we proved that these two methods are both unconditionally stable.

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