Journal of Applied Mathematics & Bioinformatics, vol.4, no.2, 2014, 147-162
ISSN: 1792-6602 (print), 1792-6939 (online)
Scienpress Ltd, 2014

Periodic Solutions for a Class of Nonautonomous Subquadratic Second order Hamiltonian System

Song He[1] and An Tianqing[2]

Abstract

This paper studies the periodic solution for non-autonmous second order Hamiltonian system by using the Saddle Point Theorem. Some new results are obtained under suitable conditions which are extension of the corresponding results in the literatures.

Mathematics Subject Classification: 37K99

Keywords: Saddle point theorem; Second order Hamiltonian system; Periodic solutions; (PS) condition； Sobolev's inequality; Wirtinger's inequality

---

[1] College of Science, Hohai University, Nanjing 210098, China.
E-mail:songheengeng@163.com

[2] College of Science, Hohai University, Nanjing 210098, China.
E- mail:antq@hhu.edu.cn

Article Info: *Received* : April 9, 2014. *Revised* : May 24, 2014.
Published online : June 15, 2014.

1 Introduction

Consider the second-order Hamiltonian system

$$\begin{cases} \ddot{u}(t)+Au(t)-\nabla F(t,u)=h(t), \\ u(0)-u(T)=\dot{u}(0)-\dot{u}(T)=0. \end{cases} \qquad t\in[0,T]. \tag{1}$$

where A is a ($N\times N$)-symmetric matrix , $h\in L^1([0,T],R^N)$, F(t,.) is continuously differentiable for a.e. $t\in[0,T]$ and F(.,u) is measurable on $[0,T]$ for each $u\in R^N$

F: $[0,T]\times R^N\to R$ satisfies the following assumption:

(A) : $F(t,x)$ is measurable in t for every $x\in R^N$ and continuously differentiable in x for a.e. $t\in[0,T]$, and there exist $a\in c[R^+,R^+]$, $b\in L^1([0,T],R^+)$

such that

$$|F(t,x)|+|\nabla F(t,x)|\le a(|x|)b(t) .$$

Under assumption (A), the existence of periodic solutions is investigated for the problem (1) when $A(t)=0$, $h(t)=0$ (see[1-4][6-13][15]). Many solvability conditions are given, such as the boundedness condition (see[8]), the coercivity conditions (see[10]), the convexity condition (see[11]), the sub additive condition (see[12]), the periodicity condition (see[15]).

In the case $A(t)=k^2\omega^2 I$, $h(t)=0$, where k is a nonnegative integer, $\omega=\dfrac{2\pi}{T}$ and I is the unit matrix of order N, it has been proved by Mawhin and Willem in [7] that problem (1) has at least one solution under the condition that

$$|\nabla F(t,x)| \le g(t)$$

for some $g \in L^1(0,T)$, $\forall x \in R$, and a.e. $t \in [0,T]$. When

$$\int_0^T F(t, a\cos m\omega t + b\sin m\omega t)dt \to \infty \text{ as } |(a,b)| \to \infty \text{ in } R^{2N}.$$

Tang in [9] consider problem (1), where $A = 0$, $h(t) = 0$, under the sublinear nonlinearity condition, that is, $\exists f, g \in L^1(0,T;R^+)$ $\alpha \in [0,1)$, such that

$$|\nabla F(t,x)| \le f(t)|x|^\alpha + g(t).$$

for all $x \in R$, and a.e. $t \in [0,T]$. The author proved that problem (1) has at least one solution when

$$|x|^{-2\alpha} \int_0^T F(t,x)dt \to +\infty \text{ as } |x| \to \infty \text{ in } R^N.$$

Han proved the problem (1) in [14], where $A(t) = k^2\omega^2 I$, $h(t) = 0$, has at least one solution under the sublinear nonlinearity condition when

$$|(a,b)|^{-2\alpha} \int_0^T F(t, a\cos m\omega t + b\sin m\omega t)dt \to \infty \text{ as } |x| \to \infty \text{ in } R^N.$$

Tang proved the problem (1) in [5], where $A(t)$ is a continuous symmetric matrix of order N, $h(t) = 0$, has at least one solution.

In the present paper, $h \ne 0$ instead of $h = 0$ is considered, and A is a $(N \times N)$-symmetric , which is more general than the previous condition.

Denote that

(i)suppose $N(A) = \{x \in R^N | Ax = 0\}$, then $\dim N(A) = m \ge 1$, and $\frac{4\pi^2 k^2}{T^2} \notin \delta(A)$;

(ii) $N(A) = span\{\alpha_1, \alpha_2, \cdots \alpha_m\}$ $j = 1,2,\cdots,m$ $\int_0^T < h(t), \alpha_j > dt = 0$;

(iii) $\forall x \in R^N \quad a.e. \quad t \in [0,t] \quad |x|^{-2\alpha}\int_0^T F(t,x)dt \to -\infty \quad as \quad |x| \to \infty \quad \alpha \in (0,1]$.

Set H_T^1 be the Hilbert space, the inner product can be defined as follows

$$<u,v>=\int_0^T <\dot{u}(t),\dot{v}(t)> dt + \int_0^T <u(t),v(t)>dt .$$

Denote the norm by $\|u\|$. It follows from assumption (A) that the functional φ on H_T^1 given by

$$\varphi(u) = \frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt - \frac{1}{2}\int_0^T <Au,u>dt + \int_0^T F(t,u)dt + \int_0^T <h,u> dt .$$

It is well known that the critical points of φ are the solutions of the problem (1).

2 Main results and proof

In this section , main results are given by using Saddle Point Theorem.

2.1 Properties

Theorem 2.1.1: Suppose that $F(t,x) = G(x) + H(t,x)$ Satisfying assumption (A) and (i), (ii), (iii).There exists $r < -\frac{4\pi^2}{T^2}$, $f, g \in L^1(0,T;R^+)$, and $\alpha \in [0,1)$ such that

$$(\nabla G(x) - \nabla G(y), x-y) \geq -r|x-y|^2 \tag{2}$$

for all $x, y \in R^N$ and

$$|\nabla H(t,x)| \le f(t)|x|^{\alpha} + g(t) \tag{3}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$. Assume that there exists $M \ge 0$, $N \ge 0$, such that

$$|\nabla G(x) - \nabla G(y)| \le M|x-y| + N. \tag{4}$$

for all $x, y \in R^N$. Then problem (1) has at least one solution in H_T^1.

Theorem 2.1.2: Suppose that $F(t,x) = G(x) + H(t,x)$ satisfying assumptions (A) (i), (ii), (iii) and (2), and there exist $B \in C(R^N, R)$,

such that

$$|\nabla G(x) - \nabla G(y)| \le B(x-y). \tag{5}$$

for all $x, y \in R^N$. Assume that there exists $g \in L^1(0,T;R^+)$ such that

$$|\nabla H(t,x)| \le g(t). \tag{6}$$

for all $x \in R^N$ and a.e. $t \in [0,T]$. Then problem (1) has at least one solution in H_T^1.

2.2 Proof of theorem

For $u \in H_T^1$, let $\overline{u} = \frac{1}{T}\int_0^T u(t)dt$ and $\widetilde{u}(t) = u(t) - \overline{u}(t)$. The one has

$$\|\widetilde{u}\|_\infty^2 \le \frac{T}{12}\int_0^T |\dot{u}(t)|^2 dt \qquad \text{(Sobolev inequality)}.$$

and

$$\int_0^T |\tilde{u}(t)|^2 \le \frac{T^2}{4\pi^2}\int_0^T |\dot{u}(t)|^2 dt \qquad \text{(Wirtinger inequality)} .$$

By counting, it can be obtained that

$$<\varphi'(u),v>=\int_0^T (<\dot{u},\dot{v}>-<Au,v>+<\nabla F(t,u),v>+<h,v>)dt .$$

let $q(u)=\frac{1}{2}\int_0^T |\dot{u}(t)|^2 dt-\frac{1}{2}\int_0^T <Au,u> dt$

$$=\frac{1}{2}\|u\|^2-\frac{1}{2}\int_0^T <(A+I)u(t),u>dt=\frac{1}{2}<(I-K)u,u>.$$

where $K: H_T^1 \to H_T^1$ is the linear self-ad joint operator defined, using Riesz representation theorem, by

$$\int_0^T <(A+I)u(t),v(t)> dt =<Ku,v>$$

$(u,v\in H_T^1)$. The compact imbedding of H_T^1 into $C([0,T],R^N)$ implies that K is compact. By classical spectral theory, H_T^1 can be decomposed into the orthogonal sum of invariant subspaces for $I-K$

$$H_T^1 = H^- \oplus H^0 \oplus H^+$$

where $H^0 = N(I-N)$ and H^+ and H^- are such that, for some $\delta > 0$,

$$q(u) \le -\frac{\delta}{2}\|u\|^2 \quad \text{if } u \in H^-$$

$$q(u) \ge \frac{\delta}{2}\|u\|^2 \quad \text{if } u \in H^+$$

that is

$$|q(u)| \ge \frac{\delta}{2}\|u\|^2 \quad \text{if } u \in H_T^1 .$$

2.2.1Proof of Theorem 1.

Step 1. We prove that φ satisfies the (PS) condition.

Suppose that $\{u_n\}$ is a (PS) sequence for φ; that is, $\varphi'(u_n) \to 0$ as $n \to \infty$ and $\{\varphi(u_n)\}$ is bounded.

From Wirting's inequality that

$$(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{1}{2}} \le \|\tilde{u}_n\| \le (\frac{T^2}{4\pi^2}+1)^{\frac{1}{2}}(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{1}{2}} \quad . \tag{7}$$

Combining (3) and Sobolev's inequality that

$$\begin{aligned}
&\left|\int_0^T (H(t,u(t)) - H(t,\bar{u}))dt\right| \\
&= \left|\int_0^T \int_0^1 < \nabla H(t,\bar{u}+s\tilde{u}),\tilde{u} > dsdt\right| \\
&\le \int_0^T \int_0^1 f(t)|\bar{u}+s\tilde{u}|^\alpha |\tilde{u}(t)| dsdt + \int_0^T \int_0^1 g(t)|\tilde{u}(t)| dsdt \\
&\le \int_0^T 2 f(t)(|\bar{u}|^\alpha + |\tilde{u}|^\alpha)|\tilde{u}(t)| dt + \int_0^T \int_0^1 g(t)|\tilde{u}(t)| dsdt \\
&\le 2(|\bar{u}|^\alpha + \|u\|_\infty^\alpha)\|\tilde{u}\|_\infty \int_0^T f(t)dt + \|\tilde{u}\|_\infty \int_0^T g(t)dt \\
&\le \frac{3(4\pi^2 - rT^2)}{4\pi^2 T}\|\tilde{u}\|_\infty^2 \\
&\quad + \frac{4\pi^2 T}{3(4\pi^2 - rT^2)}|\bar{u}|^{2\alpha}(\int_0^T f(t)dt)^2 + 2\|\tilde{u}\|_\infty^{\alpha+1}\int_0^T f(t)dt + \|\tilde{u}\|_\infty \int_0^T g(t)dt \\
&\le \frac{4\pi^2 - rT^2}{16\pi^2}\int_0^T |\dot{u}(t)|^2 dt + c_1|\bar{u}|^{2\alpha} + c_2(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{\alpha+1}{2}} + c_3(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{1}{2}}
\end{aligned} \tag{8}$$

for all $u \in H_1^T$ and some positive constants c_1 c_2 and c_3.

It follows from (2) and Writinger's inequality, it can be obtained that

$$\begin{aligned}
&\int_0^T (G(u(t)) - G(\bar{u}))dt \\
&= \int_0^T \int_0^1 (\nabla G(\bar{u}+s\tilde{u}(t)),\tilde{u}(t))dsdt \\
&= \int_0^T \int_1^0 (\nabla G(\bar{u}+s\tilde{u}(t)) - \nabla G(\bar{u}),\tilde{u}(t))dsdt
\end{aligned}$$

$$=\int_0^T\int_1^0\frac{1}{s}(\nabla G(\bar{u}+s\tilde{u}(t))-\nabla G(\bar{u}),s\tilde{u}(t))dsdt$$

$$\geq\int_0^T\int_1^0(-rs^2|\tilde{u}(t)|^2)dsdt\geq-\frac{rT^2}{8\pi^2}\int_0^T|\dot{u}(t)|^2dt, \qquad (9)$$

for all $u\in H_1^T$. Hence we obtain

$$\left|\int_0^T<\nabla H(t,u_n(t)),\tilde{u}_n(t)>dt\right|$$

$$\leq\frac{4\pi^2-rT^2}{16\pi^2}\int_0^T|\dot{u}_n(t)|^2dt+c_1|\bar{u}_n|^{2\alpha}+c_2(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}}+c_3(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{1}{2}}.$$

and

$$\int_0^T<\nabla G(u_n(t)),\tilde{u}_n(t)>dt\geq-\frac{rT^2}{8\pi^2}\int_0^T|\dot{u}_n(t)|^2dt,$$

for all n. Hence we have

$$\|\tilde{u}_n\|\geq|<\varphi'(u_n),\tilde{u}_n>|$$

$$=\left|\int_0^T(<\dot{u}_n,\dot{\tilde{u}}_n>-<Au_n,u_n>+<\nabla F(t,u_n),\tilde{u}_n>+<h,\tilde{u}_n>)dt\right|$$

$$=\left|\int_0^T|\dot{u}_n(t)|^2dt-\int_0^T<Au_n,\tilde{u}>dt+\int_0^T(\nabla G(u_n(t)),\tilde{u}_n(t))dt\right.$$

$$\left.+\int_0^T(\nabla H(t,u_n(t)),\tilde{u}_n(t))dt+\int_0^T<h,\tilde{u}_n(t)>dt\right|$$

$$\geq\left|\int_0^T|\dot{\tilde{u}}_n(t)|^2dt-\int_0^T<Au,\tilde{u}>dt\right|+\left|\int_0^T<h,\tilde{u}_n(t)>dt\right|$$

$$-\left|\int_0^T(\nabla H(t,u_n(t)),\tilde{u}_n(t))dt\right|+\int_0^T(\nabla G(u_n(t),\tilde{u}_n(t))dt$$

$$\geq\frac{1}{2}\delta\|\tilde{u}_n\|^2+\|h\|_L\|\tilde{u}_n(t)\|-\frac{4\pi^2-rT^2}{16\pi^2}\int_0^T|\dot{u}_n(t)|^2dt-c_1|\bar{u}_n|^{2\alpha}$$

$$-c_2(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}}-c_3(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{1}{2}}-\frac{rT^2}{8\pi^2}\int_0^T|\dot{u}_n(t)|^2dt.$$

$$\geq\frac{1}{2}\delta\int_0^T|\dot{u}(t)|^2dt+\|h\|_L(\int_0^T|\dot{u}(t)|^2dt)^{\frac{1}{2}}-\frac{4\pi^2-rT^2}{16\pi^2}\int_0^T|\dot{u}_n(t)|^2dt-c_1|\bar{u}_n|^{2\alpha}$$

$$-c_2(\int_0^T |\dot{u}_n(t)|^2 dt)^{\frac{\alpha+1}{2}} - c_3(\int_0^T |\dot{u}_n(t)|^2 dt)^{\frac{1}{2}} - \frac{rT^2}{8\pi^2}\int_0^T |\dot{u}_n(t)|^2 dt$$

$$\geq (\frac{1}{2}\delta - \frac{4\pi^2 + rT^2}{16\pi^2})\int_0^T |\dot{u}(t)|^2 dt - c_1|\overline{u}|^{2\alpha} - c_2(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{\alpha+1}{2}} - c_3'(\int_0^T |\dot{u}_n(t)|^2 dt)^{\frac{1}{2}}$$

$$\geq \frac{1}{2}\delta\int_0^T |\dot{u}(t)|^2 dt - c_1|\overline{u}|^{2\alpha} - c_2(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{\alpha+1}{2}} - c_3'(\int_0^T |\dot{u}_n(t)|^2 dt)^{\frac{1}{2}} \quad (10)$$

for large n. By (7) and (10), it can be obtained that

$$c|\overline{u}_n|^{\alpha} \geq (\int_0^T |\dot{u}_n(t)|^2 dt)^{\frac{1}{2}} - c_4 . \quad (11)$$

for some $c > 0$, $c_4 > 0$, and all large n.

Combimng (4), Wirtinger’s inequality and Cauchy-Schwarz inequality that

$$\begin{aligned}
&\int_0^t (G(u_n(t)) - G(\overline{u}_n))dt \\
&= \int_0^T \int_0^1 < \nabla G(\overline{u}_n + s\widetilde{u}_n(t)), \widetilde{u}_n(t) > ds dt \\
&= \int_0^T \int_0^1 < \nabla G(\overline{u}_n + s\widetilde{u}_n(t)) - \nabla G(\overline{u}_n), \widetilde{u}_n(t) > ds dt \\
&= \int_0^T \int_0^1 \frac{1}{s} < \nabla G(\overline{u}_n + s\widetilde{u}_n(t)) - \nabla G(\overline{u}_n), s\widetilde{u}_n(t) > ds dt \\
&\leq \int_0^T \int_0^1 (sM|\widetilde{u}_n(t)|^2 + N|\widetilde{u}_n(t)|) ds dt \\
&\leq \frac{M}{2}\int_0^T |\widetilde{u}_n(t)|^2 dt + N\sqrt{T}(\int_0^T |\widetilde{u}_n(t)|^2 dt)^{\frac{1}{2}} \\
&\leq \frac{MT^2}{8\pi^2}\int_0^T |\dot{u}_n(t)|^2 dt + \frac{NT\sqrt{T}}{2\pi}(\int_0^T |\dot{u}_n(t)|^2 dt)^{\frac{1}{2}} \quad . \quad (12)
\end{aligned}$$

for all n. From the boundedness of $\{\varphi(u_n)\}$, (8), (11) and (12) that

$$\begin{aligned}
c_5 \leq \varphi(u_n) &= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2 dt - \frac{1}{2}\int_0^T < Au_n, u_n > dt + \int_0^T F(t, u_n) dt + \int_0^T < h, u_n > dt \\
&= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2 dt - \frac{1}{2}\int_0^T < Au_n, u_n > dt + \int_0^T (G(u_n(t)) - G(\overline{u}_n)) dt
\end{aligned}$$

$$+\int_0^T (H(t,u_n(t))-H(t,\bar{u}_n))dt+\int_0^T F(t,\bar{u}_n)dt+\int_0^T <h,u_n> dt$$

$$\leq -\frac{\delta}{2}\|u_n\|^2+\frac{12\pi^2-rT^2+2MT^2}{16\pi^2}\int_0^T|\dot{u}_n(t)|^2dt+c_1|\bar{u}_n|^{2\alpha}+c_2(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}}$$

$$+\frac{NT\sqrt{T}+2\pi c_3}{2\pi}(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{1}{2}}+\|h\|_L\|u_n\|+\int_0^T F(t,\bar{u}_n)dt$$

$$\leq \frac{-8\pi^2\delta+12\pi^2-rT^2+2MT^2}{16\pi^2}\int_0^T|\dot{u}_n(t)|^2dt+c_1|\bar{u}_n|^{2\alpha}+c_2(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{\alpha+1}{2}}$$

$$+\frac{NT\sqrt{T}+2\pi c_3}{2\pi}(\int_0^T|\dot{u}_n(t)|^2dt)^{\frac{1}{2}}+c'\|\bar{u}_n\|+\int_0^T F(t,\bar{u}_n)dt\,.$$

for all large n and some constant c_5, as $u\in H^-$.

It follows from the boundedness of $\{\varphi(u_n)\}$, (8), (11) and (9) that

$$c_6\geq\varphi(u_n)=\frac{1}{2}\int_0^T|\dot{u}_n(t)|^2dt-\frac{1}{2}\int_0^T <Au_n,u_n> dt+\int_0^T(G(u_n(t))-G(\bar{u}_n))dt$$

$$+\int_0^T (H(t,u_n(t))-H(t,\bar{u}_n))dt+\int_0^T F(t,\bar{u}_n)dt+\int_0^T <h,u_n> dt$$

$$\geq(\frac{1}{2}\delta-\frac{rT^2+4\pi^2}{16\pi^2})\int_0^T|\dot{u}(t)|^2dt-c_2(\int_0^T|\dot{u}(t)|dt)^{\frac{\alpha+1}{2}}-c_3(\int_0^T|\dot{u}(t)|dt)^{\frac{1}{2}}$$

$$+c'\|\bar{u}_n\|+\int_0^T F(t,\bar{u}_n)dt\,.$$

for all large n and some constant c_6, as $u\in H^+$.

Hence $\{\bar{u}_n\}$ is bounded implied by (iii). In fact, if not, we may assume that $|\bar{u}_n|\to\infty$ as $n\to\infty$ without loss of generality.

Then from (9) we have

$$\liminf_{n\to\infty}|\bar{u}_n|^{-2\alpha}\int_0^T F(t,\bar{u}_n)dt>-\infty$$

which contradicts

$$|x|^{-2\alpha}\int_0^T F(t,x)dt \to -\infty \quad .$$

Since H_T^1 is self-reflexive, there exists a subsequence of $\{u_n\}$ which weakly converge u .

In view of $\varphi'(u_n)\to 0$ and $\{u_n - u\}$ bounded, one has $\varphi'(u)(u_n - u)\to 0$ and, hence $<\varphi'(u_n)-\varphi'(u), u_n - u> \to 0 \quad (n\to\infty)$ which implies that $\|\dot{u}_n - \dot{u}\|_{L^2}\to 0$

According to Wirtinger's inquality, we have $\|\dot{u}_n - \dot{u}\|_{H_T^1}\to 0$ as $n\to\infty$.

In H_T^1, $u_n \to u$. Then φ satisfies the (PS) condition..

Step 2. Some properities of φ are discussed on $H^0 \oplus H^+$.

Combining (8) and (9) , it can be obtained

$$\left|\int_0^T (G(u(t))-G(0))dt\right| \ge -\frac{rT^2}{8\pi^2}\int_0^T |\dot{u}(t)|^2 dt$$

and

$$\left|\int_0^T (H(t,u(t))-H(0))\right|dt$$

$$\le \frac{4\pi^2 - rT^2}{16\pi^2}\int_0^T |\dot{u}(t)|^2 dt + c_2\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{\alpha+1}{2}} + c_3\left(\int_0^T |\dot{u}(t)|^2 dt\right)^{\frac{1}{2}}.$$

If $u = u^0 + u^+ \in H^0 \oplus H^+$, then

$$\varphi(u) = \frac{1}{2} < (I-K)u^+, u^+ > + \int_0^T F(t,u)dt + \int_0^T < h, u^+ > dt$$

$$\ge \frac{1}{2}\delta\|u^+\|_{H_T^1}^2 + \int_0^T (G(u)+H(t,u))dt + \int_0^T < h, u^+ > dt$$

$$\geq \frac{1}{2}\delta\int_0^T |\dot{u}(t)|^2\,dt + \int_0^T (G(u)-G(0))dt + \int_0^T (H(t,u)-H(0))dt + \int_0^T < h,u^+ > dt$$

$$\geq \frac{1}{2}\delta\int_0^T |\dot{u}(t)|^2\,dt - \frac{rT^2}{8\pi^2}\int_0^T |\dot{u}(t)|^2 dt - \frac{4\pi^2 - rT^2}{16\pi^2}\int_0^T |\dot{u}(t)|^2 dt - c_2(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{\alpha+1}{2}} - c_3(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{1}{2}}$$

$$= \frac{8\pi^2\delta - rT^2 - 4\pi^2}{16\pi^2}\int_0^T |\dot{u}(t)|^2\,dt - c_2(\int_0^T |\dot{u}(t)|\,dt)^{\frac{\alpha+1}{2}} - c_3(\int_0^T |\dot{u}(t)|\,dt)^{\frac{1}{2}}.$$

and hence φ is bounded below on $H^0 + H^+$.Hence, if $H^- = 0$, φ is bounded below on H_1^T and has a minimum by Proposition 4.4 in [1]. We consider $\dim H^- > 0$.

Step 3. Some properities of φ are discussed on H^-.

$u = u^- \in H^-$, then

$$\varphi(u) = \frac{1}{2} < (I-K)u,u > + \int_0^T F(t,u)dt + \int_0^T < h,u > dt$$

$$\leq -\frac{\delta}{2}\int_0^T |\dot{u}(t)|^2 dt + \frac{rT^2}{8\pi^2}\int_0^T |\dot{u}(t)|^2 dt + \frac{4\pi^2 - rT^2}{16\pi^2} + c_2(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{\alpha+1}{2}} + c_3(\int_0^T |\dot{u}(t)|^2 dt)^{\frac{1}{2}}$$

$$= \frac{-8\pi^2\delta + rT^2 + 4\pi^2}{16\pi^2}\int_0^T |\dot{u}(t)|^2 dt + c_2(\int_0^T |\dot{u}(t)|dt)^{\frac{\alpha+1}{2}} + c_3(\int_0^T |\dot{u}(t)|dt)^{\frac{1}{2}}.$$

and $\varphi(u) \to -\infty$ as $\|u\| \to \infty$ in H^-.

Step 4. Using the Saddle Point Theorem to complish the proof.

Let $X = H_T^1$, $X^- = H^-$, $X^+ = H^0 \oplus H^+$.

It follows from $\dim X^- < \infty$, there exists $R > 0$, such that

$$\sup_{S_R^-} \varphi < \inf_{X^+} \varphi ,$$

where $S_R^- = \{u \in X^- \| \|u\| = R\}$.

φ can be proved that satisfies the all conditions of the Saddle Point Theorem. Then problem (1) has at least one solution in H_T^1.

2.2.2Proof of Theorem 2:

First we prove the φ satisfies the (PS) condition. Suppose $\{u_n\}$ is a (PS) sequence for φ. That is $\{\varphi(u_n)\}$ is bounded, that is $\varphi'(u_n) \to 0$ as $n \to \infty$

Using (2) and (6), Sobolev's inequality and Wiringer's inequality. We obtain

$$\begin{aligned}
\|\tilde{u}_n\| &\geq |< \varphi'(u_n), \tilde{u}_n >| \\
&= \left| \int_0^T (< \dot{u}_n, \dot{\tilde{u}}_n > - < Au_n, u_n > + < \nabla F(t, u_n), \tilde{u}_n > + < h, \tilde{u}_n >) dt \right| \\
&= \left| \int_0^T |\dot{u}_n(t)|^2 dt - \int_0^T < Au_n, \tilde{u} > dt + \int_0^T (\nabla G(u_n(t)) - \nabla G(\overline{u}_n), \tilde{u}_n(t)) dt \right. \\
&\qquad \left. + \int_0^T (\nabla H(t, u_n(t)), \tilde{u}_n(t)) dt + \int_0^T < h, \tilde{u}_n(t) > dt \right| \\
&\geq \frac{1}{2} \delta \int_0^T |\dot{u}_n(t)|^2 dt - r \int_0^T |\tilde{u}_n(t)|^2 dt - \|\tilde{u}_n\|_\infty \int_0^T g(t) dt + \|\tilde{u}_n\|_\infty \int_0^T h(t) dt \\
&\geq \frac{1}{2} \delta \int_0^T |\dot{u}_n(t)|^2 dt - \frac{rT^2}{4\pi^2} \int_0^T |\dot{u}_n(t)|^2 dt - c_7 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}} \\
&= \left(\frac{1}{2} \delta - \frac{rT^2}{4\pi^2} \right) \int_0^T |\dot{u}_n(t)|^2 dt - c_7 \left(\int_0^T |\dot{u}_n(t)|^2 dt \right)^{\frac{1}{2}} . \qquad (13)
\end{aligned}$$

for large n and some positive constant c_7.

Since $r < -\frac{4\pi^2}{T^2}$, (13) and (7) imply that

$$\|\tilde{u}_n\| \le c_8 . \tag{14}$$

for all n and some positive constant c_8.

Now it follows from the boundedness of $\{\varphi(u_n)\}$, (5)(6)(14) and Sobolev's inequality that

$$\begin{aligned}
c_9 &\le \varphi(u_n) \\
&= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2 dt - \frac{1}{2}\int_0^T < Au, u > dt + \int_0^T F(t,u)dt + \int_0^T < h, u > dt \\
&= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2 dt - \frac{1}{2}\int_0^T < Au, u > dt + \int_0^T F(t,\bar{u}_n)dt \\
&\quad + \int_0^T (F(t,u_n(t)) - F(t,\bar{u}_n))dt + \int_0^T < h, u > dt \\
&= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2 dt - \frac{1}{2}\int_0^T < Au, u > dt + \int_0^T \int_0^1 (\nabla G(\bar{u}_n + s\tilde{u}_n(t)) - \nabla G(\bar{u}_n), \tilde{u}_n(t))dsdt \\
&\quad + \int_0^T \int_0^1 (\nabla H(t, \bar{u}_n + s\tilde{u}_n(t)), \tilde{u}_n(t))dsdt + \int_0^T < h, u > dt \\
&\le -\frac{1}{2}\delta \int_0^T |\dot{u}_n(t)| dt + \int_0^T F(t,\bar{u}_n)dt + \|\tilde{u}_n\|_\infty \int_0^T \int_0^1 B(s\tilde{u}_n(t))dsdt \\
&\quad + \|\tilde{u}_\infty\| \int_0^T g(t)dt + \|\tilde{u}_n\|_\infty \int_0^T h(t)dt \\
&\le -\frac{1}{2}\delta \int_0^T |\dot{u}_n(t)|^2 dt + \int_0^T F(t,\bar{u}_n)dt + c_{10}\left(\int_0^T |\dot{u}_n(t)|^2 dt\right)^{\frac{1}{2}} .
\end{aligned} \tag{15}$$

for all n and some real constants c_9 and c_{10} as $u \in H^-$

$$\begin{aligned}
c_6 \ge \varphi(u_n) &= \frac{1}{2}\int_0^T |\dot{u}_n(t)|^2 dt - \frac{1}{2}\int_0^T < Au_n, u_n > dt + \int_0^T (G(u_n(t)) - G(\bar{u}_n))dt \\
&\quad + \int_0^T (H(t,u_n(t)) - H(t,\bar{u}_n))dt + \int_0^T F(t,\bar{u}_n)dt + \int_0^T < h, u_n > dt \\
&\ge \left(\frac{1}{2}\delta - \frac{rT^2 + 4\pi^2}{16\pi^2}\right)\int_0^T |\dot{u}(t)|^2 dt - c_2\left(\int_0^T |\dot{u}(t)| dt\right)^{\frac{\alpha+1}{2}} - c_3\left(\int_0^T |\dot{u}(t)| dt\right)^{\frac{1}{2}} \\
&\quad + c'\|\bar{u}_n\| + \int_0^T F(t,\bar{u}_n)dt .
\end{aligned}$$

some real constants c_6 as $u \in H^+$.

So using (iii)(7)(14)(15),we obtain $|\overline{u}_n| \leq c_{11}$,

for all n and some positive c_{11}. Furthermore $\{u_n\}$ is bounded by (14). Hence the (PS) condition is satisfied. In a way similer to the proof of the Theorem 1, we can prove that φ satisfies the other conditions of Saddle Point Theorem.

Hence Theorem 2 holds, That is the problem (1) has at least one solution in H_1^T.

References

[1] N. Aizmahin and T.Q. An, The existence of periodic solutions of non-autonomous second-order Hamiltonian systems, Nonlinear Analysis, **74**, (2011), 4862-4867.

[2] Y.W. Ye and C.L. Tang, Periodic solutions for some nonautonomous second order Hamiltonian systems, J. Math. Anal. Appl., **236**, (1999) 227-235.

[3] L. Kun and C.L. Tang, The existence of periodic solutions for some nonautomous second-order Hamilton systems, Journal of Southwest University (Natural Science Edition), **32**, (2010), 110-114.

[4] Y. Wu and T.Q. An, Existence of periodic solutions for non-autonomous second-order Hamiltonian systems, *Electronic Journal of Differential Equations*, **77**, (2013), 1-13.

[5] C.L. Tang and X.P. Wu, Nonautonomous subquadratic second order Hamiltonian systems, *J. Math. Anal. Appl.*, **275**, (2002), 870-882.

[6] M.S. Berger and M. Schechter, On the solvability of semilinear gaidient

operator equations, Adv. Math., **25**, (1997), 97-132.

[7] J. Mawhin and M. Willem, *Critical point theory and Hamiltonian systems*, Springer-Verlag, New York, 1989.

[8] Y.M. Long, Nonlinear oscillations for classical Hamiltonian systems with bi-even subquadratic potentials, *Nonlinear Anal*.,**24**, (1995), 1665-1671.

[9] C.L. Tang, Periodic solutions for nonautonomous second systems with sub linear nonlinearity, *Proc. Amer. Math. Soc.*, **126**, (1998), 3263-3270.

[10] M.S. Berger and M.Schechter, On the solvability of semi linear gradient operator equations, *Adv. Math*., **25**, (1977), 97-132.

[11] J. Mawhin, Semi-coercive monotone variational problems, *Acad. Roy. Belg. Bull. Cl. Sci*., **73**, (1987), 118-130.

[12] C.L. Tang, Periodic solution of non-autonomous second order systems with r-quasisubadditive potential, *J. Math. Anal. Appl*., **189**, (1995), 671-675.

[13] J. Ma and C.L. Tang, Periodic solutions for some nonautonomous second-order systems, *J. Math. Anal. Appl.*, **275**, (2002), 482-494.

[14] Z.Q. Han, 2π -periodic solutions to ordinary differential systems at resonance, Acta Math. Sinica, **43**, (2000), 639-644, (In Chinese).

[15] M. Willem, *Oscillations forcees de systemes Hamiltoniens*, in Public, Semin. Analyse Non Lineaire, Univ. Besancon, 1981.