Journal of Applied Mathematics & Bioinformatics, vol.1, no.2, 2011, 57-xxx
ISSN: 1792-6602 (print), 1792-6939 (online)
International Scientific Press, 2011

Individual behavior and epidemiological model

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Abstract

In this paper, some results of existence and uniqueness of solution of a semi-linear parabolic problem describing the evolution of a population subjected to a disease are presented. The population is structured into two compartments the healthy individuals and the infected individuals who interact between them. A continuous variable representing a behavioral risk is introduced. The asymptotic in time of the problem is studied, and the existence of a non zero stationary state is proved. The question we would like to investigate is does it exist a segmentation of the population (here a safety group and a risky group) in the same way as in food models with two nutriments. To answer this question, some numerical results concerning the distribution of the population according to the variable representing a behavioral risk are presented within the disease of the HIV-AIDS in Mali.

Mathematics Subject Classification: 37N30, 37N25

Keywords: Epidemiological model, dynamical system, asymptotic behavior, numerical simulations

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Article Info: *Revised* : September 26, 2011. *Published online* : November 30, 2011

1 Introduction

Most mathematical models of infectious disease assume homogeneous mixing, and so each individual is equally likely to infect any other individual. However, real populations do not mix homogeneously: individuals tend to mix in small groups. This has a significant impact on the spread of diseases. In this work, we investigate the spread of an infectious disease with a simple model aiming to describe a population structured with respect to a behavioral risk. More precisely, we represent a population by its density of individuals $f(t,x) \geq 0$, which is structured with a risk variable x. Here x is a continuous variable, $x \in X = (0,1)$, and time interval is denoted by $(0,T)$ where $T > 0$). For $x = 0$, the behavior is considered without any risk, and for $x = 1$, the behavior is considered highly risky. It is supposed that $x \leq 1/2$ characterizes the compartment of healthy individuals, and that $x > 1/2$ characterizes the compartment of the infected individuals. Similar models were considered in epidemiology in [11], [4] for example.

We assume that the density $f(t,x)$ is subjected to a phenomenon of isotropic diffusion due to the interactions between individuals. This phenomenon of diffusion is represented by the term $D\partial_x^2 f(t,x)$ in which D, a constant, represents the coefficient of diffusion. To keep the presentation as much clear as possible, we do not treat the case where the diffusion is not isotropic (as in [11] for example).

Finally, we consider a phenomenon of interaction between the individuals, represented by the logistical term:

$$af(t,x)[K - \int_X B(x-y)f(t,y)dy]$$

where a is the rate of infection, K the number of infected, $\int_X B(x-y)f(t,y)dy$ the term of interaction of usual usage in epidemiology.

Combining the effects of diffusion and those due to interaction between the individuals of different behaviors, we obtain the following equation:

$$\partial_t f(t,x) = D\partial_x^2 f(t,x) + af(t,x)[K - \int_X B(x-y)f(t,y)dy]. \tag{1}$$

So that the model to be complete, we need to specify the kernel B. The individuals live in quite homogeneous communities described with the variable

x and these communities interact depending on their neighborhood of order d. In this paper we consider that the individuals mainly interact except in their neighborhood of order d. Which means that individuals are distant from a distance at least d. The kernel of interaction B is defined in the following way. For $0 < B_1 << 1$ and $0 < d < \frac{1}{2}$ be fixed, define the following functions for everything $x \in [0,1]$

$$b_1(z) = \begin{cases} 1 \text{ if } z \in [d, x] \\ B_1 \text{ elsewhere;} \end{cases}$$

$$b_2(z) = \begin{cases} 1 \text{ if } z \in [x-1, -d] \\ B_1 \text{ elsewhere;} \end{cases}$$

define

$$B(z) = \frac{1}{1-2d}[b_1(z) + b_2(z)] \text{ pour } z \in [-1, 1].$$

The following estimates are verified:

$$B_2 = \frac{2B_1}{1-2d} \leq B(z) \leq \frac{1+B_1}{1-2d} = B_3 \, \forall z \in [-1, 1].$$

Let us notice that Equation (1) can be considered as a particular regime of presented models in [11] in the case of an asexual reproduction by taking:
$\mu = D$ a constant; $n(t, x, v) = aK$ a constant;
$\sigma(t, x, v, \int L(t, x, v, v')f(t, x, v')dv') = a \int_X B(x-y)f(t, y)dy.$

By using the definition of the convolution product, Equation (1) reads:

$$\partial_t f(t, x) = D\partial_x^2 f(t, x) + af(t, x)[K - B * f(t, x)]. \tag{2}$$

The obtained model is a semi-linear parabolic equation. We then add the following initial condition, the value of density f at time $t = 0$.

$$\forall x \in X\,; \quad f(0, x) = g_0(x).$$

Moreover the population is assumed to be isolated. Then we have Neumann's conditions as boundary conditions:

$$\forall t \in [0, T]\,; \quad \partial_x f(t, x) = 0 \quad \text{for } x = 0, \text{ and for } x = 1$$

By introducing the following notations $X =]0,1[$; $Q_T = (0,T) \times X$ the problem we are dealing with reads: find f solution to

$$(P.I)\begin{cases} \partial_t f(t,x) - D\partial_x^2 f(t,x) = af(t,x)[K - B * f(t,x)] \quad \text{in } Q_T \,; \\ \partial_x f(t,0) = \partial_x f(t,0) = 0 \quad \forall t \in [0,T] \\ f(0,x) = g_0(x)\,; \quad \forall x \in X \end{cases}$$

Section 2 is devoted to a mathematical investigation of the model. Arguing in the same way as in [11], a fixed point method yields existence and uniqueness of the solution to Problem $(P.I)$. Positivity of the solution is a consequence of the strong maximum principle. Next, in section 3 the asymptotic in time of the solution to Problem $(P.I)$ is considered. Existence of time independent solutions is proved by using a Leray-Schauder's fixed point Theorem. Section 4 is dedicated to numerical experiments. It is shown that contrary to some food models with two nutriments, it does not appear a structuring in two groups in the population, but it exists two peaks over a mean distribution. The example of the HIV-AIDS disease in Mali is handled with the model, and some predictive results are discussed.

2 Mathematical investigations of the model

In what follows, the hypothesis $H_1; H_2$ are assumed to hold true.

1. (H_1): $a \geq 0$, $K \geq 0$;
2. (H_2): $g_0 \in C^2(0,1) \setminus \{0\}$ nonnegative; and $g'(0) = g'(1) = 0$.

Concerning solutions to Problem $(P.I)$ we have.

Theorem 2.1. *Assume the hypotheses (H_1) and (H_2) are satisfied. Then there is a unique positive function f in $C^1((0,T), C^2(X))$ solution to Problem $(P.I)$.*

Proof.
A classical method of fixed point in $L^\infty(0,T; L^2(X))$ is used. Introduce a linearized Problem $(P.A)$ generating a sequence $\{f_n\}_{n\in\mathbb{N}}$ which converges towards the solution to Problem $(P.I)$. The proof is decomposed in two steps.

First it is proved that the sequence has positive elements, then it converges. Let $f_0 \in C^1[0,1]$ be a nonnegative function, recursively define the sequence $\{f_n\}_{n\in\mathbb{N}}$ of solutions to the following problem:

$$(P.A)\begin{cases}\partial_t f_{n+1}(t,x) - D\partial_x^2 f_{n+1}(t,x) + a(B * f_n(t,x))f_{n+1}(t,x) = aKf_n(t,x) & \text{in } Q_T;\\ \partial_x f_{n+1}(t,0) = \partial_x f_{n+1}(t,1) = 0 \qquad \text{in } [0,T];\\ f_{n+1}(0,x) = g_0(x); \quad \text{in } X.\end{cases}$$

Lemma 2.2. *Let us suppose the hypotheses (H_1) and (H_2) are satisfied, and that $f_0 \in C^1(Q_T)$ is positive. Then for all $0 < n$ there is a unique positive solution to Problem $(P.A)$ $f_n \in C^1((0,T); C^2(X))$. Furthermore it exists $0 < C$ independent of T such as*

$$\|f_{n+1}\|_{L^\infty(0,T;H^1(X))} \leq C\|f_n\|_{L^2(Q_T)}$$

Proof.

Arguing recursively and let $f_n \in C^1(Q_T)$ be positive. Now a technical result which we shall use several times is given.

Lemma 2.3. *Let $O \subset \mathbb{R}^q$ be an open subset, $\psi \in C^1(O;\mathbb{R}^+)$ be given different from zero such as $\|D\psi\|_{L^\infty(O)} \leq C$. Then $\forall y \in O$, there exists $0 < \eta < 1$ such that if we set $R_\eta = \frac{\psi(y)}{\eta\|D\psi\|_{L^\infty(O)}}$, we have:*

$$\frac{\psi(y)}{\eta} \leq \psi(x)\,\forall x \in B(y,R_\eta)\cap O.$$

Let us remark that Lemma 2.3 applies with the positive function f_n, and that there exist η, $(t_\eta, x_\eta) \in Q_T$ such as $B((t_\eta,x_\eta),R_\eta)\cap Q_T = B((t_\eta,x_\eta),R_\eta)$. That leads to the existence of

$$0 < B_n = B_2\frac{f_n(t_\eta,x_\eta)}{\eta\|Df_n\|_{L^\infty(Q_T)}} \text{ such as } 0 < B_n \leq B * f_n(t,x) \tag{3}$$

for all $(t,x) \in Q_T$. The existence of a weak solution $f_{n+1} \in L^2(0,T;H^1(X)) \cap C^0(0,T;L^2(X))$ is shown as in [6] Theorem 3 P; 356. Since $f_n \in L^2(Q_T)$ the results of regularity given in Theorem 5 p.360 [6], claim that $f_{n+1} \in L^2(0,T;H^2(X)) \cap L^\infty(0,T;H^1(X))$, $\frac{d}{dt}f_{n+1} \in L^2(0,T;L^2(X))$. Then, Theorem 6 P. 365 in [6] applies when the right hand side of the first equation of Problem $(P.A)$ is $aKf_n(t,x) - a(B * f_n(t,x))f_{n+1}(t,x)$, and we obtain that $\frac{d^2}{dt^2} f_{n+1} \in L^2(0,T;L^2(X)) \cap L^\infty(0,T;H^2(X))$. Finally we deduce that $f_{n+1} \in C^1(\overline{Q}_T) \cap C^1((0,T);C^2(X))$.

Now we are going to prove that f_{n+1} is positive by using a comparison argument. Introduce the constant

$$\overline{B}_n = aB_3 \sup_{(t,x)\in Q_T} |f_n|$$

and let $v \in C^1((0,T);C^2(X))$ be solution to

$$\begin{cases} \partial_t v(t,x) - D\partial_x^2 v(t,x) + \overline{B}_n v(t,x) = aKf_n(t,x) & \text{in } Q_T; \\ \partial_x v(t,0) = \partial_x v(t,1) = 0 \qquad \text{in } [0,T]; \\ v(0,x) = g_0(x); \quad \text{in } X. \end{cases}$$

The reader is referred for example to [8] for the properties of the operator $A = -D\partial_x^2 + \overline{B}_n$ with homogeneous Neumann's boundary conditions. It is the generator of a strongly continuous semi-group, thus there exist $0 < \beta$, and $0 < M$ such that $\|e^{tA}\| \le Me^{-\beta t}$, and we have the following representation formula:

$$v(t,x) = e^{tA} g_0(x) + \int_0^t e^{(t-s)A} aKf_n(s,x)\, ds.$$

The maximum principle claims that v is nonnegative, and Hypothesis (H2) and Lemma 2.3 allow us to conclude that there exists $x_\eta \in (0,1)$ such as for all $0 \le t \le T$ and for all $y \in B(x_\eta, R_\eta)$ we have

$$\frac{1}{\eta} g_0(x_\eta) e^{tA} \le v(t,y).$$

The principle of comparison [8] applies and we have that $v \le f_{n+1}$ in Q_T. So f_{n+1} is nonnegative, and verifies the following estimate in Q_T

$$\frac{1}{\eta} \inf_{0<y<1} g_0(y) e^{tA} \le f_{n+1}(t,x) \le \sup_{0<y<1} g_0(y) + \frac{aKB_2M}{\beta} \sup_{(t,x)\in Q_T} |f_n|. \qquad (4)$$

The strong maximum principle implies that if the minimum of f_{n+1} is reached in Q_T then f_{n+1} is a constant, which contradicts Problem (P.A), since for example, g_0 is not assumed to be a constant. □

Remark 2.4. *If g_0 and f_0 are nonnegative and one is different from zero, then f_{n+1} is positive.*

Now we consider the convergence of the sequence. Let us prove that $(f_n)_{n\in\mathbb{N}}$ is a Cauchy's sequence in $L^\infty([0,T],L^2(X))$. Evaluate the expression $\|(f_{n+1}-f_n)(t)\|_{L^2(X))}$. For all $(t,x)\in]0,T[\times X$ we have:

$$\partial_t f_{n+1}(t,x) - D\partial_x^2 f_{n+1}(t,x) + a(B * f_n(t,x))f_{n+1}(t,x) = aKf_n(t,x), \quad (5)$$

$$\partial_t f_n(t,x) - D\partial_x^2 f_n(t,x) + a(B * f_{n-1}(t,x))f_n(t,x) = aKf_{n-1}(t,x). \quad (6)$$

The difference (5)-(6) reads

$$\begin{aligned}&\partial_t(f_{n+1}(t,x)-f_n(t,x)) - D\partial_x^2(f_{n+1}(t,x)-f_n(t,x)) + a(B*f_n(t,x))f_{n+1}(t,x)\\ &-a(B*f_{n-1}(t,x))f_n(t,x) = aK(f_n(t,x)-f_{n-1}(t,x)).\end{aligned}$$

Multiplying by $f_{n+1}(t,x)-f_n(t,x)$ we obtain:

$$\begin{aligned}&\tfrac{1}{2}\partial_t(f_{n+1}(t,x)-f_n(t,x))^2 - D(f_{n+1}(t,x)-f_n(t,x))\partial_x^2(f_{n+1}(t,x)-f_n(t,x))\\ &+a[(B*f_n(t,x))f_{n+1}(t,x)-(B*f_{n-1}(t,x))f_n(t,x)](f_{n+1}(t,x)-f_n(t,x))\\ &= aK(f_n(t,x)-f_{n-1}(t,x))(f_{n+1}(t,x)-f_n(t,x))\end{aligned}$$

or

$$\begin{aligned}&\tfrac{1}{2}\partial_t(f_{n+1}(t,x)-f_n(t,x))^2 - D(f_{n+1}(t,x)-f_n(t,x))\partial_x^2(f_{n+1}(t,x)-f_n(t,x))\\ &+a[(B*f_n(t,x))(f_{n+1}(t,x)-f_n(t,x)) + B*(f_n(t,x)-f_{n-1}(t,x))]\\ &(f_{n+1}(t,x)-f_n(t,x)) = aK(f_n(t,x)-f_{n-1}(t,x))(f_{n+1}(t,x)-f_n(t,x)).\end{aligned}$$

Integrate on X and use the Green's formula:

$$\begin{aligned}&\frac{1}{2}\int_X \partial_t|f_{n+1}(t,x)-f_n(t,x)|^2dx\\ +\ & D\int_X |\partial_x(f_{n+1}(t,x)-f_n(t,x))|^2dx + a\int_X (B*f_n(t,x))|f_{n+1}(t,x)-f_n(t,x)|^2dx\\ \leq\ & aB_3\|f_n(t,x)-f_{n-1}(t,x)\|_{L^2(X)}\|f_{n+1}(t,x)-f_n(t,x)\|_{L^2(X)}\\ +\ & aK\int_X |f_n(t,x)-f_{n-1}(t,x)||f_{n+1}(t,x)-f_n(t,x)|dx.\end{aligned}$$

By using the bound from below (3) and by integrating on $[0,t]$ we have the following estimate:

$$\begin{aligned}&\tfrac{1}{2}\|(f_{n+1}-f_n)(t)\|^2_{L^2(X)} + D\int_0^t\int_X |\partial_x(f_{n+1}-f_n)(x,s)|^2dxds+\\ &\quad aB_n\int_0^t\int_X |f_{n+1}-f_n)(x,s)|^2dxds\\ &\leq (aB_3+aK)\int_0^t\int_X |(f_{n+1}-f_n)(x,s)||(f_n-f_{n-1})(x,s)|dxds.\end{aligned}$$

For every $0 < \alpha$, he Young's inequality yields:

$$\frac{1}{2}\|(f_{n+1}-f_n)(t)\|^2_{L^2(X)} + aB_n\int_0^t \|(f_{n+1}-f_n)(s)\|^2_{L^2(X)}ds$$
$$-(aB_3+aK)\frac{\alpha^2}{2}\int_0^t \|(f_{n+1}-f_n)(s)\|^2_{L^2(X)}ds$$
$$\leq \frac{1}{2\alpha^2}(aB_3+aK)\int_0^t \|(f_n-f_{n-1})(s)\|^2_{L^2(X)}ds$$

or

$$\frac{1}{2}\|(f_{n+1}-f_n)(t)\|^2_{L^2(X)} + \big[aB_n-(aB_2+aK)\frac{\alpha^2}{2}\big]\int_0^t \|(f_{n+1}-f_n)(s)\|^2_{L^2(X)}ds$$
$$\leq \frac{1}{2\alpha^2}(aB_3+aK)\int_0^t \|(f_n-f_{n-1})(s)\|^2_{L^2(X)}ds.$$

Choose α such that $aB_n-(aB_3+aK)\frac{\alpha^2}{2}>0$, we have:

$$\|(f_{n+1}-f_n)(t)\|^2_{L^2(X)} \leq \frac{1}{\alpha^2}(aB_2+aK)\int_0^t \|(f_n-f_{n-1})(s)\|^2_{L^2(X)}ds.$$

Let $\lambda_1 = \dfrac{1}{\alpha^2}(aB_3+aK)$ we have:

$$\|(f_{n+1}-f_n)(t)\|^2_{L^2(X)} \leq \lambda_1\int_0^t \|(f_n-f_{n-1})(s)\|^2_{L^2(X)}ds.$$

Now, let us define the following $L^\infty(0,T,\lambda,L^2(X))$ norm by:

$$\|f\|_{L^\infty(0,T,\lambda,L^2(X))} = \sup_{0\leq t\leq T}\||f(t,.)|e^{-\lambda t}\|_{L^2(X)}$$

From the previous estimate we deduce:

$$\|(f_{n+1}-f_n)(t)\|^2_{L^2(X)} \leq \lambda_1\int_0^t \|(f_n-f_{n-1})(s)\|^2_{L^2(X)}e^{\lambda s}e^{-\lambda s}ds,$$

so

$$\|(f_{n+1}-f_n)(t)\|^2_{L^2(X)}e^{-\lambda t} \leq \frac{\lambda_1}{\lambda}(e^{\lambda t}-1)e^{-\lambda t}\|f_n-f_{n-1}\|^2_{L^\infty(0,T,\lambda,L^2(X))}$$

which gives

$$\|f_{n+1}-f_n\|^2_{L^\infty(0,T,\lambda,L^2(X))} \leq \frac{\lambda_1}{\lambda}\|f_n-f_{n-1}\|^2_{L^\infty(0,T,\lambda,L^2(X))}.$$

Choose λ so such that $\frac{\lambda_1}{\lambda} < 1$. Arguing recursively, it comes:

$$\|f_{n+1} - f_n\|^2_{L^\infty(0,T,\lambda,L^2(X))} \leq \lambda^n \|f_1 - f_0\|^2_{L^\infty(0,T,\lambda,L^2(X))}.$$

Finally we deduce:

$$\forall m \in \mathbb{N}\ ,\ \forall p \in \mathbb{N}^*\ ,\ \|f_{m+p} - f_m\|^2_{L^\infty(0,T,\lambda,L^2(X))} \leq \lambda^m \frac{1-\lambda^p}{1-\lambda} \|f_1 - f_0\|^2_{L^\infty(0,T,\lambda,L^2(X))}.$$

If $m \to \infty$ then $\lambda^m \dfrac{1-\lambda^p}{1-\lambda} \|f_1 - f_0\|^2_{L^\infty(0,T,\lambda_3,L^2(X))} \to 0$ because $\|f_n\|_{L^\infty(0,T,L^2(X))}$ is bounded according to (4).
So the sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy's sequence in $L^\infty(0,T,\lambda,L^2(X))$ which is a Banach. Finally the sequence $(f_n)_{n\in\mathbb{N}}$ converges in $L^\infty(0,T,\lambda,L^2(X))$ towards a function noted f. Lemma 2.2 and a continuity argument show that Problem (P.A) has a fixed point f which is the solution to Problem $(P.I)$. $\square$

3 Asymptotic with respect to time of the solution to Problem (P.I)

In this subsection, the behavior for large times of the solution to Problem (P.I) is investigated. First, the limit problem denoted stationary problem, is introduced and we prove the existence of a solution $\overline{f} \in H^2(X)$ for this problem.

Let us consider the following stationary problem $(P.S)$:

$$(P.S) \begin{cases} -D\overline{f}^{''}(x) = a[K - (B * \overline{f}(x))]\overline{f}(x) \quad \text{ in } X \\ \\ \overline{f}^{'}(0) = \overline{f}^{'}(1) = 0. \end{cases}$$

Theorem 3.1. *Assume the hypotheses (H_1) to be satisfied, then there exists a unique nonegative $\overline{f} \in H^2(X)$ solution of the stationary problem(P.S).*

Proof. To get existence of solutions, Leray-Schauder fixed point Theorem is used. Define the following operators:

$$\begin{aligned} A:\ L^\infty(X) &\to H^2(X) \\ \psi &\mapsto A\psi = \theta \end{aligned}$$

solution to

$$-D\theta''(x) + \theta(x) = \psi(x) \qquad \text{in } X$$
$$\theta'(0) = \theta'(1) = 0.$$

$$\begin{aligned} N: \ L^\infty(X) & \to L^\infty(X) \\ \psi & \mapsto \psi + a[K - (B * \psi)]\psi \end{aligned}$$

According to the Rellich-Kondrachov compactness Theorem we have:

Lemma 3.2. *The operator $A \in \mathcal{L}(L^\infty(X); L^\infty(X))$ is compact. The operator N is continuous.*

It is obvious that Problem (P.S) is equivalent to:

$$\varphi = AN(\varphi).$$

Let $Y = \{\psi \in L^\infty(X); 0 \leq \psi\}$. Invoking Theorem 4 p. 504 in ([6]), we have to prove that for all $0 \leq \lambda \leq 1$, $\varphi_\lambda = \lambda AN(\varphi_\lambda)$ remains bounded in Y irrespective of λ.
First observe that for $0 < \lambda \leq 1$ solutions $\varphi_\lambda \in Y$ to

$$\frac{1}{\lambda} A^{-1} \varphi_\lambda = N(\varphi_\lambda)$$

are more regular. Thus by integrating on X the previous equation, from the bounds for function B, we have the following estimate:

$$\frac{K}{B_3} \leq \|\varphi_\lambda\|_{L^1(X)} \leq \frac{K}{B_2}.$$

The following bound from below for $B * \varphi_\lambda$ is deduced;

$$\frac{KB_2}{B_3} \leq B_2 \int_X \varphi_\lambda(x)\, dx \leq B * \varphi_\lambda.$$

Multiply Equation (3) by φ_λ, and integrate by parts, the previous inequality allows us to estimate

$$\frac{1}{\lambda} \int_X (\varphi'_\lambda)^2\, dx + \frac{aKB_2}{B_3} \|\varphi_\lambda\|^2_{L^2(X)} \leq aK \|\varphi_\lambda\|_{L^1(X)} \|\varphi_\lambda\|_{L^\infty(X)} \leq \frac{aK^2}{B_2} \|\varphi_\lambda\|_{L^\infty(X)}.$$

Define $C_m = min(1, \frac{aKB_2}{B_3})$, and let C_i be the the Sobolev embedding constant of $H^1(X)$ into $L^\infty(X)$, then we have:

$$C_m C_i \|\varphi_\lambda\|^2_{L^\infty(X)} \leq \frac{aK^2}{B_2} \|\varphi_\lambda\|_{L^\infty(X)}.$$

Consequently, irrespective of λ, a bound for $\|\varphi_\lambda\|_{L^\infty(X)}$ is obtained. $\square$

In this subsection the convergence of the solution to Problem $(P.I)$ towards a solution to Problem (P.S) is investigated. Let us remind the bound from below given in (3) for the function f:

$$0 < S = B_2 \frac{f^2(t_\eta, x_\eta)}{2_\eta \|Df\|_{L^\infty(Q_T)}} \text{ such that } 0 < S \leq B * f(t,x). \tag{7}$$

The maximum principle gives the following estimation:

$$\|f\|_{L^\infty(Q_T)} \leq \frac{K}{B_2} \tag{8}$$

.

Theorem 3.3. *The solution f to problem $(P.I)$ converges towards $\overline{f} \in H^2(X)$, a solution to Problem $(P.S)$ when t goes to $+\infty$.*

Proof. Take the difference of the two following equations:

$$\partial_t f - D\partial_x^2 f + a(B * f)f = aKf.$$

$$-D(\overline{f})'' + a(B * \overline{f})\overline{f} = aK\overline{f}.$$

The difference $f - \overline{f}$ verifies:

$$\partial_t(f - \overline{f}) - D\partial_x^2(f - \overline{f}) + a(B * f)f - a(B * \overline{f})\overline{f} = aK(f - \overline{f}).$$

or equivalently

$$\partial_t(f - \overline{f}) - D\partial_x^2(f - \overline{f}) + a(B * f)(f - \overline{f}) + aB * (f - \overline{f})\overline{f} = aK(f - \overline{f}).$$

Multiply this equation by $f - \overline{f}$

$$\tfrac{1}{2}\tfrac{d}{dt}(f - \overline{f})^2 - D(f - \overline{f})\partial_x^2(f - \overline{f}) + a(B * f)(f - \overline{f})^2 + aB * (f - \overline{f})\overline{f}(f - \overline{f}) = aK(f - \overline{f})^2.$$

By taking into account the bound from below of $0 \leq (B * \psi)\phi\psi$ for all $0 \leq \phi$, we have the following estimation:

$$\frac{1}{2}\frac{d}{dt}(f - \overline{f})^2 - D(f - \overline{f})\partial_x^2(f - \overline{f}) + a(B * f)(f - \overline{f})^2 \leq aK(f - \overline{f})^2.$$

Integrate on X and use the Green's formula, we have:

$$\frac{1}{2}\frac{d}{dt}\int_X |f-\overline{f}|^2dx+D\int_X |\partial_x(f-\overline{f})|^2dx+a\int_X (B*f)|f-\overline{f}|^2dx \leq aK\int_X |f-\overline{f}|^2dx.$$

or

$$\frac{1}{2}\frac{d}{dt}\int_X |f-\overline{f}|^2dx + a\int_X (B*f)|f-\overline{f}|^2dx \leq aK\int_X |f-\overline{f}|^2dx.$$

By using (7) and the L^∞ bounds for f (8) and $\overline{f}$ we have:

$$\frac{1}{2}\frac{d}{dt}\int_X |f-\overline{f}|^2dx + aS\int_X |f-\overline{f}|^2dx \leq 4aK\frac{a^2K^4}{C_m^2C_i^2B_2^2}\frac{a^2K^2}{B_2^2}.$$

Define $\alpha = 2aS$ and $\beta = 8aK\dfrac{a^2K^4}{C_m^2C_i^2B_2^2}\dfrac{a^2K^2}{B_2^2}$, it comes:

$$\frac{d}{dt}\|f-\overline{f}\|^2_{L^2(X)} + \alpha\|f-\overline{f}\|^2_{L^2(X)} \leq \beta.$$

Let us introduce $q(t) = \|f-\overline{f}\|^2_{L^2(X)}$, the function q verifies the following differential inequality:

$$\frac{d}{dt}q(t) + \alpha q(t) \leq \beta \tag{9}$$

or equivalently

$$\frac{d}{dt}[e^{\alpha t}q(t)] \leq \beta e^{\alpha t}.$$

By integrating between 0 and t we have:

$$0 \leq q(t) \leq e^{-\alpha t}q(0) + \beta\int_0^t e^{-\alpha(t-\tau)}d\tau.$$

Now integrate (9) between t and $t+1$ and define the function $h(t) = \displaystyle\int_t^{t+1} q(s)ds$, it is obvious that h verifies the same inequation as q with a different initial condition. Therefore the following estimation holds true:

$$\begin{aligned} h(t) &\leq e^{-\alpha t}h(0) + \beta\int_0^t e^{-\alpha(t-\tau)}d\tau \\ &\leq e^{-\alpha t}h(0) + \frac{\beta}{\alpha}. \end{aligned}$$

Define $c_b = \frac{\beta}{\alpha}$, it comes:

$$h(t) \leq e^{-\alpha t}h(0) + c. \tag{10}$$

Now let us integrate (9) from t to s for $t < s < t+1$

$$\begin{aligned} q(s) &\leq e^{\alpha(t-s)}q(t) + \frac{\beta}{\alpha}e^{-\alpha s}\int_t^s e^{\alpha\tau}d\tau \\ &\leq e^{\alpha(t-s)}q(t) + c_b[1 - e^{\alpha(t-s)}]. \end{aligned}$$

Since $t < s$, we have:

$$q(s) < q(t) + c_b. \tag{11}$$

Integrate (11) with respect to t for $s-1 < t < s$:

$$\int_{s-1}^s q(s)dt \leq \int_{s-1}^s q(t)dt + \int_{s-1}^s c_b dt$$

or

$$q(s) \leq \int_{s-1}^s q(t)dt + \int_{s-1}^s c_b dt. \tag{12}$$

Combining (10) and (12), we have:

$$\begin{aligned} q(s) &\leq h(s-1) + \int_{s-1}^s c_b dt \\ q(s) &\leq e^{-\alpha(s-1)}h(0) + 2\int_{s-1}^s c_b dt. \end{aligned}$$

So we have:

$$\lim_{s\to+\infty} q(s) = 0 \quad \text{because } s \text{ and } s-1 \text{ have the same limit at } +\infty.$$

Consequently the solution of Problem $(P.I)$ converges towards a solution of Problem $(P.S)$ when t goes to $+\infty$. $\square$

4 Numerical results

Within the case the spread of HIV-AIDS in Mali, in this section, a finite difference method is used for the (P.S.) model to investigate numerically the segmentation of the sexually active population of the Mali in two compartments, a risky one and a safety one.

The numerical computations are conducted with non dimensional quantities.

Let $I = 100$, the space step is defined by:

$$\Delta x = \frac{1}{I}.$$

Introduce the family of points $\{x_i\}$ defined by:

$$x_i = i\Delta x \text{ for } 0 \leq i \leq I.$$

The time step $\Delta t = 0.01$ is fixed according to:

$$\frac{D\Delta t}{(\Delta x)^2} \leq 0.5. \tag{13}$$

the family of time points $\{t_n\}$ is defined by:

$$t_n = n\Delta t \text{ for } 0 \leq n.$$

The solution to Problem (P.I) is approximated at points (t_n, x_i) by $f(t_n, x_i) \sim f_i^n$ computed recursively with the following scheme: for $\{f_i^0\}_{i=0}^{i=100}$ to be given with $f_j^0 = f_{j+1}^0$ for $j = \{0, 99\}$;

$$\begin{cases} f_j^{n+1} = f_{j+1}^{n+1} \text{ for } j = \{0, 99\}; \\ f_i^{n+1} = f_i^n + \dfrac{D\Delta t}{(\Delta x)^2}(f_{i+1}^n - 2f_i^n + f_{i-1}^n) + a\Delta t f_i^n (K - \Delta x \displaystyle\sum_{l=0}^{I} B(x_i, x_l) f_l^n) \\ \text{for } 1 \leq i \leq 99 \end{cases} \tag{14}$$

Observe that Condition (13) is a CFL condition for the backward Euler's scheme (14), thus the positivity of the approximated density function is ensured.

Population sexually active in 2007 in Mali is taken to be 6357563, the number of infected individuals is taken to be $K = 140000$. The rate of infection is fixed to $a = 0.7$, the coefficient of spread is fixed to $D = 0.0004$, and the size of the neighborhood of the kernel B is $d = 0.02$, the value for the constant $B_1 = 0.000001\mathrm{S}$. Starting from the following initial condition:

$$\begin{cases} f_i^0 = 1 \text{ if } i \in [49, 51] \\ f_i^0 = 0 \text{ if not,} \end{cases}$$

the numerical results are yielded for the three next years by the numerical scheme (14). In the following table, the available datum in literature for the sexually active part of the population (Psexlit), the computed sexually active part of the population (Psexnum) predicted by the presented model (14) and the risky part (corresponding to the left peak in Figures 1-3) are reported.

Year	Psexlit	Psexnum	Risky	Error Pop
2008	6 531 735	6 531 100	20%	$9,72 10^{-5}$
2009	6 713 503	6 719 700	18%	$2,93 10^{-5}$
2010	7 312 068	7 312 700	15%	$8,64 10^{-5}$

In the following figures, the distribution of the sexually active part of the population is depicted according to the behavioral risk variable for the years 2008, 2009 and 2010.

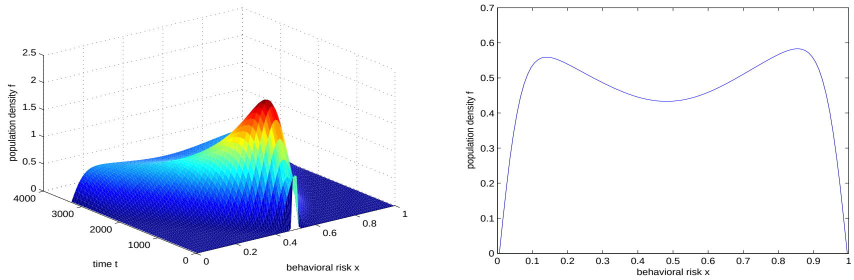

Figure 1: Sexually active population density in Mali for 2008

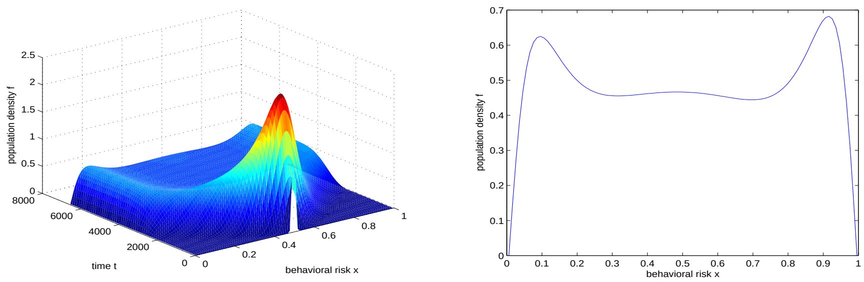

Figure 2: Sexually active population density in Mali for 2009

The comparison of the numerical results presented with the data available in Literature for the population show that model (P.S) can be considered as a predictive one. Moreover, the adults part with a risky sexual behavior is evaluated at 15,7% in [1] for example. If the size of the exclusion neighborhood is double ($d = 0.04$) the numerical results are almost unchanged. As conclusion, the presented model (P.S) for the spread of HIV-AIDS in Mali does not predict a strong segmentation in two compartments of the population, as it is predicted by some food models with two compartments of nutriment.

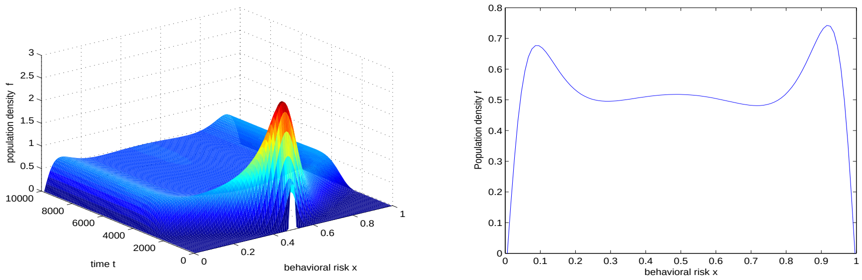

Figure 3: Sexually active population density in Mali for 2010

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