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On Diagonal Case for Matrix Exponential

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Abstract

In this article, we present special cases by using similar matrices of computing the matrix exponential with some examples.

Keywords: Similar matrices, the matrix exponential

1. Introduction

In this case of 2×2real matrices we have a simplistic way of computing the matrix exponential. The eigenvalues of matrix A are the roots of the characteristic polynomial $\lambda^2 - tr(A)\lambda + \det(A) = 0$. The discriminant will be used to differentiate between the three cases which are used to compute the matrix exponential of a 2×2 matrix.

Case 1: $D > 0$

The matrix A has real distinct eigenvalues λ_1,λ_2 with eigenvectors v_1,v_2;

$$e^{At} = [v_1 v_2] \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix} [v_1 v_2]^{-1}$$

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Case 2: $D = tr(A)^2 - 4detB = 0$

The matrix A has a real double eigenvalueλ. If $A = \lambda I$

Then $e^{At} = e^{\lambda t} I$

Otherwise

$$e^{At} = [v \;\; w] e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} [v \;\; w]^{-1},$$

Where v an eigenvector of A and w satisfies$(A - \lambda I)w = v$

Case 3: $D = tr(A)^2 - 4detA < 0$

The matrix B has conjugate eigenvalues $\lambda, \overline{\lambda}$ with eigenvectors $u, \overline{u}$.

$$e^{At} = [u \;\; \overline{u}] \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\overline{\lambda} t} \end{pmatrix} [u \;\; \overline{u}]^{-1}$$

Or writing $\lambda = \sigma + iw$, $u = v + iw$,

$$\boldsymbol{e^{At} = [v \;\; w] e^{\sigma t} \begin{pmatrix} \cos wt & -\sin wt \\ \sin wt & \cos wt \end{pmatrix} [v \;\; w]^{-1}}$$

Let $B = P^{-1}AP$,

Where $P = diag(r_1, \dots, r_n)$s.t $r_i \in \mathbb{R}^+ \; \forall i = 1, \dots, n$

Then

$$B = P^{-1}AP = \begin{bmatrix} \frac{1}{r_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{r_n} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_n \end{bmatrix}$$

In this article we compute the matrix exponential for any matrix B.

2. New Results

2.1 Definition 1

Let A,B be two similar matrices and $A = [a_{ij}] \in M_n, \forall i,j = 1, \dots, n,$

and let $B = P^{-1}AP,$

where $P = diag(r_1, \dots, r_n)$s.t $r_i \in \mathbb{R}^+ \ \forall i = 1, \dots, n.$

Then $$B = P^{-1}AP = \begin{bmatrix} \frac{1}{r_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \frac{1}{r_n} \end{bmatrix} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} r_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & r_n \end{bmatrix}$$

So, we have the following results.

2.2 Diagonal case 2

Suppose that A is a $n \times n$ real or complex matrix, and that A is diagonalizable over $\mathbb{C}$, that is, that there exists an invertible complex matrix P such that $A = P^{-1}DP,$

with $$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

Observe that e^D is the diagonal matrix with eigenvalues e^{λ_1},...,e^{λ_n},

we have $$e^A = P^{-1} \begin{pmatrix} e^{\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{pmatrix} P$$

We can consider the matrix B as the following $= P^{-1}AP;$

where A is any matrix and $P = diag(r_1, \dots, r_n)$ for $r_i > 0$ with $1 \le i \le n.$

2.3 Example.1

Let $$A = \begin{pmatrix} 5 & 1 \\ -2 & 2 \end{pmatrix}$$

Then we can evaluate

e^B s.t $B = P^{-1}AP,\ P = diag(r_1, \dots, r_n)$s.t $r_i \in \mathbb{R}^+ \ \forall i = 1, \dots, n,$

as following

Now
$$B=\begin{bmatrix}\frac{1}{r_1} & 0\\ 0 & \frac{1}{r_2}\end{bmatrix}\begin{bmatrix}5 & 1\\ -2 & 2\end{bmatrix}\begin{bmatrix}r_1 & 0\\ 0 & r_2\end{bmatrix}$$

$$B=\begin{bmatrix}5 & \frac{r_2}{r_1}(1)\\ \frac{r_1}{r_2}(-2) & 2\end{bmatrix}$$

The characteristic equation is $P(\lambda)=|B-\lambda I|=0$ and it yields the eigenvalues $\lambda_1=4,\ \lambda_2=3,$

$$e^B=\alpha_0 I+\alpha_1 B$$

$$e^3=\alpha_0+3\alpha_1$$

$$e^4=\alpha_0+4\alpha_1$$

Or $\alpha_0=4e^3-3e^4$

and $\alpha_1=e^4-e^3$

So that,
$$e^B=(4e^3-3e^4)I+(e^4-e^3)B$$

$$e^B=(4e^3-3e^4)I+(e^4-e^3)\begin{bmatrix}5 & \frac{r_2}{r_1}(1)\\ \frac{r_1}{r_2}(-2) & 2\end{bmatrix}$$

$$e^B=\begin{pmatrix}2e^4-e^3 & (e^4-e^3)\frac{r_2}{r_1}\\ (2e^3-2e^4)\frac{r_1}{r_2} & 2e^3-e^4\end{pmatrix}$$

3. Corollary

3.1 Corollary 1

If we let $r_1=r_2=1,$

then $$B = \begin{bmatrix} 5 & 1 \\ -2 & 2 \end{bmatrix}$$

And hence, $$e^B = \begin{pmatrix} 2e^4 - e^3 & (e^4 - e^3) \\ (2e^3 - 2e^4) & 2e^3 - e^4 \end{pmatrix}$$

$$e^B = \begin{pmatrix} 89.1108 & 34.5126 \\ -69.0252 & -14.4271 \end{pmatrix} = e^A$$

3.2 Corollary 2

If we let $r_j = r^j \; \forall \; j = 1, \dots, n$ and $r > 0$,

then we have $$B = \begin{pmatrix} 5 & \frac{r^2}{r^1}\,(1) \\ \frac{r^1}{r^2}\,(-2) & 2 \end{pmatrix} \qquad (*)$$

So we have $$e^B = \begin{pmatrix} 2e^4 - e^3 & (e^4 - e^3)r \\ (2e^3 - 2e^4)\frac{1}{r} & 2e^3 - e^4 \end{pmatrix}$$

Put $r = 2$ in (*);

we obtain the following $$B = \begin{pmatrix} 5 & 2 \\ -1 & 2 \end{pmatrix}$$

And hence, $$e^B = \begin{pmatrix} 2e^4 - e^3 & 2(e^4 - e^3) \\ e^3 - e^4 & 2e^3 - e^4 \end{pmatrix}$$

$$e^B = \begin{pmatrix} 89.1108 & 69.0252 \\ -34.5126 & -14.4271 \end{pmatrix}$$

Lemma 1

Let A,B$\in M_n$, if B is similar to A. Then A and B have the same characteristic polynomial.

Proof.

Compute

$$P_B(t) = \det(tI - B)$$

$$= \det(tS^{-1}S - S^{-1}AS)$$

$$= \det(S^{-1}(tI - A)S)$$

$$= detS^{-1}\det(tI - A)\, detS$$

$$= (detS)^{-1}(detS)\det(tI - A)$$

$$= \det(tI - A)$$

$$= P_A(t)$$

Theorem 1

Let A and B be two similar matrices and A, B is an upper or lower triangular matrix. Then the eigenvalues of A and the eigenvalues of B are its diagonal entries.

Proof.

Case 1: Let

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

Then characteristic polynomial is

$$f(\lambda) = \det(A - \lambda I_3) = det\begin{pmatrix} a_{11} - \lambda & a_{12} & a_{13} \\ 0 & a_{22} - \lambda & a_{23} \\ 0 & 0 & a_{33} - \lambda \end{pmatrix}$$

This is also an upper-triangular matrix, so that the determinant is the product of its diagonal entries:

$$f(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)$$

And hence the zeros of this polynomial are exactly a_{11},a_{22},a_{33}

Case 2: Let
$$B = \begin{pmatrix} a_{11} & \frac{r_2}{r_1}a_{12} & \frac{r_3}{r_1}a_{13} \\ 0 & a_{22} & \frac{r_3}{r_2}a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}$$

Then characteristic polynomial is

$$f(\lambda) = \det(B - \lambda I_3) = det\begin{pmatrix} a_{11}-\lambda & \frac{r_2}{r_1}a_{12} & \frac{r_3}{r_1}a_{13} \\ 0 & a_{22}-\lambda & \frac{r_3}{r_1}a_{23} \\ 0 & 0 & a_{33}-\lambda \end{pmatrix}$$

This is also an upper-triangular matrix, so the determinant is the product of its diagonal entries:

$$f(\lambda) = (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)$$

And hence, the zeros of this polynomial are exactly a_{11},a_{22},a_{33}.
We can consider the eigenvalue of A are the eigenvalue B and equal its diagonal entries for a matrix A or B.

4. Using similar matrices by using 2×2 case

In this case of 2×2 real matrices we have a simplistic way of computing the matrix exponential. The eigenvalues of matrix A and matrix B are the roots of the characteristic polynomial of A $\lambda^2 - tr(A)\lambda + \det(A) = 0$ or the roots of the characteristic polynomial of B $\lambda^2 - tr(B)\lambda + \det(B) = 0$.The discriminant will be used to differentiate between the three cases which are used to compute the matrix exponential of a 2 × 2 matrix.

Case 1: $D = tr(B)^2 - 4detB > 0$

The matrix B has real distinct eigenvalues λ_1,λ_2 with eigenvectors v_1,v_2;

$$e^{Bt} = [v_1 v_2]\begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}[v_1 v_2]^{-1}$$

Example 2

$$A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$$

Consider e^{Bt} where $$B = \begin{pmatrix} 4 & \frac{r_2}{r_1}(-2) \\ \frac{r_1}{r_2}(1) & 1 \end{pmatrix}$$

Here $det(B) = 6$ and $tr(B) = 5$, which means $D = 1$. The characteristic equation is

$$\lambda^2 - 5\lambda + 6 = 0.$$

The eigenvalues are 2 and 3, and the eigenvectors are $[1 \;\; 1]^T$ and $[2 \;\; 1]^T$, respectively. Therefor

$$e^B = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} e^2 & 0 \\ 0 & e^3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -e^2 + 2e^3 & 2e^2 - 2e^3 \\ -e^2 + e^3 & 2e^2 - e^3 \end{pmatrix}$$

$$= \begin{pmatrix} 32.7820 & -25.3930 \\ 12.6965 & -5.3074 \end{pmatrix}$$

Case 2: $$D = tr(B)^2 - 4detB = 0$$

The matrix B has a real double eigenvalueλ. If $B = \lambda I$,

Then $$e^{Bt} = e^{\lambda t} I,$$

Otherwise $$e^{Bt} = [v \;\; w] e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} [v \;\; w]^{-1},$$

Where v an eigenvector of B and w satisfies$(B - \lambda I)w = v$.

Example 3

Let $$A = \begin{pmatrix} 6 & -1 \\ 4 & 2 \end{pmatrix} \quad , \; B = \begin{pmatrix} 6 & \frac{r_2}{r_1}(-1) \\ \frac{r_1}{r_2}(4) & 2 \end{pmatrix}$$

Here $det(B) = 16$ and $tr(B) = 8$, therefor $D = 0$. The characteristic equation is

$$\lambda^2 - 8\lambda + 16 = 0$$

Thus $\lambda = 4$. The eigenvector associated with the eigenvalue 4 is

$$v = [1 \;\; 2]^T$$

Solving $$\left(\begin{pmatrix} 6 & \frac{r_2}{r_1}(-1) \\ \frac{r_1}{r_2}(4) & 2 \end{pmatrix} - \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix} \right) w = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

We obtain $w = [1 \;\; 1]^T$. Using the method for 2×2 matrices with a double eigenvalue, we have found

$$e^B = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} e^4 \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}^{-1}$$

$$= e^4 \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 3e^4 & -e^4 \\ 4e^4 & -e^4 \end{pmatrix}$$

$$= \begin{pmatrix} 163.7945 & -54.5982 \\ 218.3926 & -54.5982 \end{pmatrix}$$

Case 3: $D = tr(B)^2 - 4detB < 0$

The matrix B has conjugate eigenvalues $\lambda, \bar{\lambda}$ with eigenvectors $u, \bar{u}$.

$$e^{Bt} = [u \ \ \bar{u}] \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\bar{\lambda} t} \end{pmatrix} [u \ \ \bar{u}]^{-1}$$

Or writing $\lambda = \sigma + iw, \ u = v + iw$

$$e^{Bt} = [v \ \ w] e^{\sigma t} \begin{pmatrix} \cos wt & -\sin wt \\ \sin wt & \cos wt \end{pmatrix} [v \ \ w]^{-1}$$

Example 4

Let $A = \begin{pmatrix} 3 & -2 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & \frac{r_2}{r_1}(-2) \\ \frac{r_1}{r_2}(1) & 1 \end{pmatrix}$

Since $det(B) = 5$ and $tr(B) = 4, \ D = -4$ the characteristic equation is

$$\lambda^2 - 4\lambda + 5 = 0$$

And $\lambda = 2 \pm i$. The eigenvector $u = [2 \quad 1 \quad - i]^T$. Therefore $\sigma = 2, \ w = 1, v = [2 \quad 1]^T$ and $w = [0 \quad -1]^T$.

So $$e^B = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} e^2 \begin{pmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}^{-1}$$

$$= e^2 \begin{pmatrix} \cos 1 - \sin 1 & -2\sin 1 \\ -\sin 1 & \sin 1 + \cos 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2.2254 & -12.4354 \\ 6.2117 & 10.21 \end{pmatrix}$$

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