Theoretical Mathematics & Applications, vol. 4, no. 4, 2014, 19-28
ISSN: 1792- 9687 (print), 1792-9709 (online)
Scienpress Ltd, 2014

A fixed point theorem on compact metric space using hybrid generalized φ - weak contraction

L. Shambhu Singh[1,*], Th. Chhatrajit Singh[2] and M. Krishnanando Singh[3]

Abstract

In this paper we obtain common fixed points of two self mappings defined on a compact metric space using the hybrid generalized φ-weak contractions.

Mathematics Subject Classification: 54E45

Keywords: Common fixed point; compact metric space; contractive mapping; hybrid generalized φ - weak contraction

1. Introduction

Many mathematical researcher obtained fixed points and fixed point results on contractive mappings viz, Song [12, 13, 14], Al-Thagafi and Shahzad [10], Reich [4], Kalinke [5], Paliwal [7] Rhoades [8], Hussain and Jungck [11] and

[1] D.M.College of Science Imphal, Manipur, India.
* Corresponding author. E-mail: lshambhu2008@rediffmail.com
[2] National Institute of Technology, Manipur, India.
[3] D.M.College of Science Imphal, Manipur, India.

Article Info: *Received:* July 21, 2014. *Revised:* August 30, 2014.
Published online : November 25, 2014.

others. The concept of weak contraction in fixed point theory was introduced by Alber and Guerre – Delabriere [6] in 1997 for single – value mappings in Hilbert space and proved existence of fixed points using φ- weak contraction on a complete metric space. He also highlighted the relation between φ - weak contraction with that of Boyd and Wong type [2] and the Reich type contraction [4]. Qingnian Zhang, Yisheng Song [16] proved the existence of fixed point in complete metric space for generalized φ - weak contraction.

The aim of this paper is to prove the existence of a unique common fixed point of a hybrid generalized φ- weak contraction mappings for a pair a pair of self-mappings in compact metric space.

Before proving the main results we need the following definitions for our main results.

Definition 1.1 Let (X,d) be a metric space and $f:X\to X$ be a mapping *f* is said to be contractive if for each $x,y\in X$ there exists $k\in(0,1)$ such that $d(x,y)\le k\ \ d(x,y)$.

Definition 1.2 A self – mapping *g* from a metric space (X,d) into itself is called a φ- weak contraction if for each $x,y\in X$ there exists a function $\varphi:[0,\infty)\to(0,\infty)$ such that $\varphi(t)>0$ for all $t\in[0,\infty)$ and $\varphi(0)=0$ and

$$d(gx,gy)\le d(x,y)-\varphi(d(x,y)).$$

Definition 1.3 Let (X,d) be a metric space and $f:X\to X$ be a self-mapping *f* is said to have a fixed point on *X* if there exists $x\in X$ such that $fx=x$.

Example 1.1 Let $(X,d), X = R$ be the usual metric space. For $f : R \to R$ defined by $fx = x^3 \ \forall x \in R$ the fixed points of R are 0, 1 and – 1. This shows that the fixed point, if exists may not be unique.

Definition 1.4 Let S and T be two self-mappings from a metric space (X,d) into itself A point $x \in X$ will be a coincident point of *S* and *T* if $Sx = Tx$.

Example 1.2 Consider $Sx = x^2$, $Tx = x^3$ defined on (R,d). The points $x = 0$ and $x = 1$ are coincident points of the mappings *S* and *T*. Since $STx = S(Tx) = S(x^3) = x^6$ and $TSx = T(Sx) = T(x^2) = x^6$, therefore $STx = TSx$ $\forall x \in R$ and hence *S* and *T* are also commutative on R.

2. Main results

Theorem 2.1 Let (X,d) be a compact metric space and $S,T : X \to X$ be two mappings such that for all $x, y \in X$:

(i) $S(X)$ or $T(X)$ is a closed subspace of *X* and ….. (2.1.1)

(ii) $d(Sx,Ty) \le C(x,y) - \varphi(C(x,y))$ (2.1.2)

where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semi-continuous with $\varphi(t) > 0$ and $\varphi(0) = 0$ and,

$$C(x,y) = \max\left\{ d(x,y), \frac{d(Sx,x) + d(y,T_y)}{2}, \frac{d(x,T_y) + d(y,S_x)}{2} \right\} \qquad (2.1.3)$$

then *S* and *T* have coincident points $z \in X$ which are also the unique common fixed points of the mappings *S* and *T*.

Proof: Let $x \in X$ be any arbitrary point. There exists point $x_1, x_2, x_3, ...$ in X such that

$$x_1 = Tx_0, x_2 = Sx_1, x_3 = Tx_2, ...,$$

Inductively, we have,

$$x_{2n+2} = Sx_{2n+1}, x_{2n+1} = Tx_{2n}, \text{ for all } n \geq 0.$$

Choosing x_{n+1}, x_n for *x* and *y* when *n* is odd

$$\begin{aligned} d(x_{n+1}, x_n) &= d(Sx_n, Tx_{n-1}) \\ &\leq C(x_n, x_{n-1}) - \varphi(C(x_n, x_{n-1})) \qquad \text{(using (ii))} \\ &\leq C(x_n, x_{n-1}) \\ &\leq \max\left\{ d(x_n, x_{n-1}), \frac{d(Sx_n, x_n) + d(Tx_{n-1}, x_{n-1})}{2}, \right. \\ &\qquad \left. \frac{d(x_n, Tx_{n-1}) + d(x_{n-1}, Sx_n)}{2} \right\} \\ &\leq \max\left\{ d(x_n, x_{n-1}), \frac{d(x_{n+1}, x_n) + d(x_n, x_{n-1})}{2}, \right. \\ &\qquad \left. \frac{d(x_n, x_n) + d(x_{n-1}, x_{n+1})}{2} \right\} \end{aligned}$$

If $d(x_{n+1}, x_n) > d(x_n, x_{n-1})$

then $C(x_n, x_{n-1}) = d(x_{n+1}, x_n)$

Also, $d(x_{n+1}, x_n) = d(Sx_n, Tx_{n-1})$

$$\begin{aligned} &\leq C(x_n, x_{n-1}) - \varphi(C(x_n, x_{n-1})) \\ &= d(x_{n+1}, x_n) - \varphi(C(x_n, x_{n-1})) \end{aligned} \tag{2.1.4}$$

which is a contradiction. Similarly, we have another contradiction if *n* is taken an even number. Thus,

$$d(x_{n+1}, x_n) \leq C(x_n, x_{n-1}) \leq d(x_n, x_{n-1}) \quad \forall \ n \geq 0 \tag{2.1.5}$$

Since $d(x_n, x_{n-1}) \le d(x_{n-1}, x_{n-2})$ therefore, the sequence $\{d(x_n, x_{n-1})\}_{n\ge 0}$ is non-increasing on R^+, the set of all positive real which is bounded from below i.e., there exists a positive real number $r \ge 0$ such that

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = \lim_{n\to\infty} C(x_n, x_{n-1}) = r \tag{2.1.6}$$

Now,

$$\varphi(r) = \varphi\left(\lim_{n\to\infty} C(x_n, x_{n-1})\right)$$

$$\le \liminf_{n\to\infty} \varphi\left(C(x_n, x_{n-1})\right).$$

From (2.1.4), we have,

$$\lim_{n\to\infty} d(x_{n+1}, x_n) \le \lim_{n\to\infty} C(x_n, x_{n-1}) - \liminf_{n\to\infty} \varphi\left(C(x_n, x_{n-1})\right)$$

i.e., $r \le r - \varphi(r)$

i.e., $\varphi(r) \le 0$

i.e., $\varphi(r) = 0$

Thus,

$$r = 0 \qquad \Rightarrow \lim_{n\to\infty} d(x_{n+1}, x_n) = 0 \tag{2.1.7}$$

To prove that $\{x_n\}$ is a Cauchy sequence. If not, there exists some $m, n \ge N(\in)$ a large number for a given $\in > 0$ such that

$$d(x_{n+1}, x_{m+1}) > \in$$

Now,
$$\in < d(x_{n+1}, x_{m+1})$$
$$= d(Sx_n, Tx_m)$$
$$\le C(x_n, x_m) - \varphi\left(C(x_n, x_m)\right)$$

Letting $m, n \to \infty,$ we have $\in < 0$ which is a contradiction and hence $\{x_n\}$ is a Cauchy sequence. As $S(X)$ or $T(X)$ is a closed subset of (X, d) which is bounded the sequence (x_n) has a convergent subsequence $\{x_{n(k)}\}$ as $k \to \infty$ converging to some point $z \in X$ as $T(X)$ of $S(X)$ is a compact subspace of (X,d)

$$\lim_{k\to\infty} x_{n(k)+1} \to z. \quad \text{or} \quad \lim_{k\to\infty} Sx_{n(k)} \to z \quad \Rightarrow Sz = z$$

We shall prove that $Tz = z$. If not,

$$d(z,Tz) \le d(Sz,Tx_{2n+1})$$

$$\le C(z,x_{2n+1})$$

$$\le \max.\left\{d(z,x_{2n+1}), \frac{d(z,Sz)+d(x_{2n+1},Tx_{2n+1})}{2}, \frac{d(z,Tx_{2n+1})+d(x_{2n+1},Sz)}{2}\right\}$$

$$\le \max.\left\{d(z,z), \frac{d(z,z)+d(z,Tz)}{2}, \frac{d(z,Tz)+d(z,Sz)}{2}\right\}$$

$$\le \max\left\{d(z,z), \frac{d(z,Tz)}{2}, \frac{d(z,Tz)}{2}\right\}$$

$$\le \frac{d(z,Tz)}{2}$$

which is a contradiction and hence, $Tz = z$. Thus, $Sz = Tz = z \Rightarrow z$ is a common fixed point of the mappings *S* and *T*.

For uniqueness, let there be another fixed point of the mappings *S* and *T* and z' such that $Sz' = Tz' = z'$. To prove $z = z'$

$$d(z,z') = d(Sz,Tz')$$

$$\le C(z,z')$$

$$\le \max.\left\{d(z,z'), \frac{d(Sz,z)+d(z',Tz')}{2}, \frac{d(z,Tz')+d(z',Tz)}{2}\right\}$$

$$= \max.\left\{d(z,z'), \frac{d(z,z)+d(z',z')}{2}, \frac{d(z,z')+d(z',z)}{2}\right\}$$

$$= d(z,z')$$

$\Rightarrow \quad d(z,z') < d(z,z')$

$\Rightarrow$ a contradiction $\Rightarrow \quad z = z'$

Thus, the fixed point is unique. □

Theorem 2.2 Let (X,d) be a compact metric space and $S,T : X \to X$ be two mappings such that for all $x, y \in X$.

(i) $S(X)$ or $T(X)$ is a closed subspace of X and

(ii) $d(Sx,Ty) \le C(x,y) - \varphi(C(x,y))$ where $\varphi : [0,\infty) \to [0,\infty)$

is a lower semi-continuous with $\varphi(t) > 0$ and $\varphi(0) = 0$ and

$$C(x,y) = \max\left\{d(x,y), d(x,Sx), d(y,Ty), \frac{d(x,Ty) + \varphi(y,Sx)}{2}\right\}$$

then S and T have a unique common fixed point $z \in X$ such that $Sz = Tz = z$.

Theorem 2.3 Let (X,d) be a complete metric space and $S,T : X \to X$ be two mappings such that for all $x, y \in X$

$$d(Sx,Ty) \le C(x,y) - \varphi(C(x,y))$$

where $\varphi : (0,\infty) \to [0,\infty)$ is a lower semi-continuous with $\varphi(t) > 0$ and $\varphi(0) = 0$

and $C(x,y) = \max\left\{d(x,y), \frac{d(x,Sx) + d(y,Ty)}{2}, \frac{d(x,Ty) + d(y_2Sx)}{2}\right\}$

they S and T have a unique common fixed point $z \in X$ such that $Sz = Tz = z$.

Theorem 2.4 Let (X,d) be a compact metric space and $S : X \to X$ a mapping defined on X such that for $x, y \in X$.

$$d(Sx,Sy) \le C(x,y) - \varphi(C(x,y))$$

where $\varphi:[0,\infty)\to[0,\infty)$ is a lower semi-continuous function with $\varphi(t)>0, \varphi(0)=0$ and

$$C(x,y) = \max\left\{d(x,y), \frac{d(s,Sx)+d(y,Sy)}{2} \frac{d(x,Sy)+d(y,Sx)}{2}\right\}$$

then there exists a unique point $z \in X$ such that $Sz = z$.

Proof: The details of the proofs of the Theorems 2.2, 2.3 and also 2.4 are omitted.

□

Example 2.1 Let $X = [0,1] = \mathbb{R}$, a closed and bounded subset of $\mathbb{R}$ equipped with the usual metric $d(x,y) = |x-y|$ consider the mappings $S: X \to X$ and $T: X \to X$ defined by

$$Sx = \frac{1}{2}x^2, Tx = 0$$

For $x, y \in X$,

$$C(x,y) = \max\left\{d(x,y), \frac{d(x,Sx)+d(y,Ty)}{2}, \frac{d(x,Ty)+d(y,Sx)}{2}\right\}$$

$$= \max\left\{|x-y|, \frac{\left|x-\frac{1}{2}x^2\right|+|y|}{2}, \frac{|x|+\left|y-\frac{1}{2}x^2\right|}{2}\right\}$$

$$= x - y, \qquad 0 \le y \le \frac{1}{2}x^2$$

$$= x + y - \frac{1}{2}x^2, \left(\frac{1}{2}x^2 \le y \le x - \frac{1}{2}x^2\right)$$

$$= x + y - \frac{1}{2}x^2, x - \frac{1}{2}x^2 \le y$$

Taking $\varphi(t) = \frac{1}{6}t^2, t > 0$

the mappings S' and T satisfy all the condition of Theorem of 2.1 and hence $x = 0$ is the unique common fixed points of S and T.

References

[1] Kannan R., Some results on fixed points, *Bull. Calcutta Math. Soc.*, **60**, (1968), 71-76.

[2] Boyd D.W., Wong T.S.W., on non linear contractions, Proc. Amer. Soc., **20**, (1969), 458-464.

[3] Reich S., Some fixed point problems, *Atti. Accad. Naz. Lincei*, **57**, (1974), 194-198.

[4] Wong C.S., on Kannan type, *Proc. Amer. Math. Sci*., **105**, – III MR 50# 10929 Zbl 292-47004, (1975).

[5] Kalinke A.K., on a fixed point theorem for Kannan type mappings, *Math. Taponic*.,**33**(5), (1988), 721-723.

[6] Alber Y.J., Guerre-Delabriere S., *Principles of weakly contractive maps in Hilbert Spaces*, in: I Gohberg, Yu lyubich (Eds). New Results in operator Theory, in : Advance and Appl., **98**, Birkhauser, Basel (1997), 7-22.

[7] Paliwal V.C., Fixed point theorems in metric spaces, Bull Cal. Math. Soc., **80**, (1998), 2005-2008.

[8] Rhoades B.E., Some theorems on weakly contractive maps, *Non linear Anal.*, **47**, (2001), 2683-2693.

[9] Shahzad. M., Invariant approximation, generalised I-contractions and R-sub weakly commuting maps, *Fixed Point Theory Appl*., **1**, (2005), 79-86.

[10] Al. Thagafi M.A., Shahzad M., Non commuting self maps and invariant approximations, *Non linear Anal*., **64**, (2006), 2778-2786.

[11] Hussain N., Jungck G., Common fixed point and in variant approximation results for non commuting generalized (f, g) - non expansive maps., *Journal of Math. Anal. Appl.*, **321**, (2006), 851-861.

[12] Song Y., Coincidence points for non commuting f - weakly contractive mappings, *Int. Compute. Appl. Math*. (IJCAM), **2**(1), (2007), 51-57.

[13] Song Y., Common fixed points and invariant approximations for generalized (f-g) non-expansion mappings, *Common. Math. Anal.*, **2**, (2007), 17-26.

[14] Song Y., Xu S., A note on common fixed points for Banach operator pairs, *Int. J. Contemp. Math. Sci.*, **2**, (2007), 1163-1166.

[15] Shambhu L., Mahendra Y., Singh M.R., Some common fixed point theorem in complete in complete metric spaces IJMSEA, *ISSH :* 0973-9424, **12**(IV), (2008), 263-271.

[16] Zhang Q., Song Y., Fixed point Theory for generalized φ - weak contraction, *Applied Mathematics letters*, **22**, (2009), 75-78.