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Gambler's Risk of Ruin and Optimal Bet

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Abstract

Classical study of betting and gambling largely argues that, individual is able to make more revenue from a fraction of income wagered, with utility of pleasure for participation being minimal. This explains why, unlike insurance, where people resort to reduce the impact of risk, gambling is sought to bear risk (Ankomah 2015). Reasonably so, the gain in revenue is a proximate avenue for a person to place bets to buy and bear risk of such fluctuated nature. However, inherent in this decision making lie the risk of ruin (RoR) that stares the gambler as he/she struggles to earn an expected level of income from a given bankroll. Akin to the gambler's RoR is the optimal stopping time. The paper discusses the best point of exit for the gambler, and further derives and analyse the possible RoR of the gambler relative to the randomness and uncertain nature that characterise the game of gambling, and further considers the optimal betting strategy, whiles suggesting the best point of exit using Wald's equation. It was found that, the ruin of the gambler at any point in time remains eminent with the randomness of the game of betting and gambling. It was concluded that, optimal betting strategy exists for the adoption of bettors relative to the amount for wagering and best point of exit for reason of ruin avoidance.

Keywords: Gambler's Ruin, Probability of Ruin, Random walk, stopping time, Wald's Equation.

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1. Introduction

Gambling and betting on football scores has assumed recent phenomenon in Ghana. The proliferation of betting centers under the trademarks of *mybet.com, Premier betting, Safari bet, Supa bet* to recently introduced *Euro bet* with potential for more bookmakers in the country and accompanied huge patronage by patrons are testament of this assertion. And the economic implications of this phenomenon though not the focus of this paper is enormous — offer of employment to the jobless, quick money for patrons, tax revenue for government and net resultant growth of the economy.

Betting on football scores to a large extent is to commit portion of wealth to an enterprise of uncertain nature with the intent of seeking upsurge in the wagered funds to resurrect an insolvent financial position to raise enough funds to pay creditors, if any, and surplus to meet budget constraints. To others, it is an avenue for pleasure and excitement making. Truth be told, the individual may possibly lose wagered funds to be insolvent from placed-bets. That is, the possibility of getting ruined remains eminent cognizance of the uncertain and random nature of such enterprise. The gambler is thus exposed to substantial losses and rewards concurrently. This makes betting and gambling a sweet bitter game. Such is the gambling and betting. And probability and statistics which are key tools for actuaries provide elegant mathematical framework for evaluating such dynamics of uncertainty and randomness.

Akin to the likelihood of being ruined lies the question of when to stop. By this, the gambler ought to know, on the balance of revenue loss or gain to bow out of the game to avoid ruin. This makes perfect sense, so that, the gambler does not go home at the close of bet-day broke. The gambler as a decision maker must observe the process of betting that involves some randomness and uncertainty to act fast (Hill, 2009). In fact, this is made possible through observation of current state of affairs, while keeping track of past events (history). This is because; the future cannot be predicted with certainty, but past and present information provides crucial basis for exit, when necessary.

This senario, together with ruin probabilities have not received much attention, though few researches have been made in these areas, with less application papers on gambling markets. Insolvency assessment in the area of gambler's ruin and option pricing optimality via stopping times have also been considered. That is, evidence of application of gambler's ruin theory to evaluating the solvency and/or insolvency of business has been discussed. For example, Wilcox (1971) used the gambler's ruin to develop a framework that predicts the risk of ruin (RoR) of firms. His model assumed that the firm's financial state could be defined as its adjusted cash position at any time. The gambler's ruin idea was thus used to assess the bankruptcy of entities based on the inflows and outflows of liquid resources. Wilcox viewed the firm as a gambler who begins the game with an amount of money equal to its net assets. The firm is then assumed to win an incremental amount of assets with probability p or losses with probability $1-p$, and the firm is said to be insolvent when its asset falls to zero (0). The dynamics of a firm's net assets can thus be described by a stochastic process estimated on the time series of net assets. Coad et al (2012), further modelled the bankruptcy of firms by theorising the gambler's ruin framework through arguing that, a firm's performance is best modelled as a random walk process, and further argued that survival is non random and depends primarily on the stock of accumulated resources. In analysing the growth of firms, Coad and his colleagues averred that, the growth of firms is best captured by the gambler's ruin model and postulated that critical firm size might be analogous to how many chips the gambler has when they start, and hence how long they stay at the table and the likelihood of reaching an outcome that is positive. The gambler ruin theory was in this instance used to assessed the survival of firms relative to how long they survive in a competitive environment given the firm's overall reserves. According to Harick et al (1999), gambler's ruin—which in their words is a "classical solution to random walk" can be used to model population sizing to predict the quality of genetic algorithms based on the increased population at a point in time. Canjar (2000) used traditional analysis to estimate the risk of ruin (RoR) by deriving a working formula that encapsulated the mean and variance of games, and the bankroll of the gambler in placing favourable bets, while making the point that the derived formula perform poorly in large skewed games. Further, Canjar (2000) expressed the RoR in terms of the moments of the probability distribution associated with the game.

Regarding the theory of stopping time, Yoshida (1999) extended the classical results of probability theory to consider the stopping game for sequence of fuzzy random variables. Optimal stopping time game with fuzziness to obtain a saddle point for the game was presented; whiles obtaining an accompanied saddle point for it, the needed rich information in the uncertain and random environment of fuzziness was preserved. Under a regularity assumption, Yoshida (1999) further obtained a minimax theorem and a saddle point for stopping game. Interesting work of the duos'; Dupuis and Wany (2005) considered a class of stopping problems where the ability to stop largely depended on exogenous Poisson signal. By this, the best stop point for the gambler is the Poisson jump times. Discrete and continuous times models of standard Brownian motion versions of optimal stopping times were formulated and solved in the presence of Poisson process. Dong et al (2013), also adopted the idea of stopping time theory to study the best exit strategy of an endowed pre—committing behaviour gambler using the cumulative prospect theory (CPT) of preferences of Barberis (2012). They further stated that, the optimal point of exit for the gambler as a decision is best based on the whole past betting history and not on the current wealth. Most recently, He et al (2014) devised a strategy via stopping time to ascertain the point of exit or continuity relative to past winnings. By this, the gambler as long as he/she wins can continue except when he starts to lose too much relative to earlier wins. Further, He et al (2014) represented the preferences of the gambler through a utility function. For an unrestricted time horizon, they formulated and solved it analytically an optimal stopping problem of a pre—commitment strategy for the gambler.

The remainder of this paper is structured as follows: Firstly, in the next section, a brief perspective of risk is discussed to bring bearing to the topic. In section two, a description and derivation of the gambler's ruin theory is presented, and discussion of its applicability in determining the likelihood of the ultimate ruin of bettor. Section three discusses the optimal betting strategy available to the gambler in the face of associated RoR.

2. Perspectives of Risk

The game of betting and gambling is widely known for its indeterminate outcome making it risk prone. Loosely speaking, risk has a feature of hazard, effect of bad consequence, exposure to misfortune and loses possibility of any kind. Embrechts et al (2005 p.2), define '*risk as the quantifiable likelihood of loss or less than expected returns''*. That is, according to Embrechts and his friends, risk is the fear or possibility of losing an underlining asset in an enterprise of any kind. This is unpleasant if it so happens. Most unpleasant is when all "available assets or funds" are lost to the point of being brought to bankruptcy. To this end, there comes a lot of risks for the attraction and attention of actuaries and financial analysts. This includes credit risk, market risk, operational risk and liquidity risk. In fact these are non—exhaustive lists as many other forms of risks exist for analysis.

However, in this paper we restrict risk to the gambler. To the gambler, risk is the risk of experiencing gambler's ruin, an actuarial concept which states that, the gambler will eventually lose entire bankroll while playing against an opponent. Equally, the risk of the bookmaker—the mandated body to sanction betting and gambling, can also be determined. To the bookmaker, the risk of experiencing losses emanating from claims of gamblers' placed bets, resulting in less-than-expected returns which results in insolvency. In actuarial circles, losses are captured as a function of loss frequency (i.e. the number of losses) and loss severity (i.e. the size or quantum of loss). Essentially, both the gambler and the house are not immune from the unfortunate effects of being at risk. Like the gambler, the bookie's risk of losses over different time periods can be quantified to assess the point bankruptcy. We defer this interesting later assessment of the bookmakers' risk to future work.

3. Methodology

3.1 Gambler's Ruin

Like all investment decisions alike, sports betting and gambling as an investment avenue to seek upsurge in income is not averse to these two criteria:

- The kind of investment mix to achieve the desire goal.
- The strategy to adopt so as not to end up being bankrupt.

The first scenario is deferred to later section, but, the second scenario is presently discussed. The second scenario become very much focused on, presently, relative to the uncertainty and randomness that characterise gambling and betting on football scores. In this case, the gambler is ruined, thus the name gambler's ruin theory.

That is, the probability of a gambler losing sufficient gambling money to the point at which continuity is no longer considered an option to recover loses or recoups initial wagered funds. This takes into consideration the probability of winning, the probability of incurring losses, and the portions of an individual bankroll that is in play or at risk. Prominently, this is known as the probability of ruin (PoR) or risk of ruin (RoR). An actuarial concept which states that, given a finite bankroll, a gambler playing against an adversary with infinite bankroll will eventually be brought to ruined. The solution then lies in finding a betting size amidst chances of win or loses to minimize the risk of gambler's ruin without making insignificant bet to win insignificant amount or earnings.

3.2 Derivation of Gambler's Ruin

Suppose a gambler wagers a dollar each time a game is played with specified probabilities p of winning and $1-p$ of losing, and let the total wealth of the gambler after n^{th} gambles be W_n . If the gambler has an objective of reaching a pre-determined total fortune of N without first getting ruined, then the gambler's position can be describe by a simple random walk analysis such that $\{W_{n\,:\,n\geq 0}\}$, and

$$W_{n\,=}\gamma_1+\gamma_2+\cdots+\cdots+\gamma_{\mathrm{n}},\ \ W_{o\,=\,i}.$$

The earnings on successive gamble in this instance is expressed as $p(\gamma=1)=p=p(\gamma=-1)=q=1-p=\frac{1}{2}$. Heuristically, the gambler stops playing when either $W_n=0$ or $W_n=N$ in which case the gambler can be said to have been ruined or attained the desired goal.

Conditioning on the outcome of the first gamble, $\gamma_1=1$ or $\gamma_2=-1$, we have

$$p_i=\ {}_{p}p_{i+1}+\ {}_{q}p_{i-1} \tag{2.1}$$

If $\gamma_1=1$, then the gambler wins and the total wealth increases to $W_1=i+1$ and thus will now wins with probability p_{i+1}. Similarly, if $W_1=-1$, the fortune of the gambler decreases to $W_1=i-1$ and hence wins with probability p_{i-1} by the ideas and reasoning of Markov property. These probabilities corresponding to the two outcomes are p and q which is the result of equation (2.1), since $p+q=1$ can be re—written as

$${}_{p}p_i+\ {}_{q}p_i=\ {}_{p}p_{i+1}+\ {}_{q}p_{i-1} \tag{2.2}$$

Factorising and simplifying equation (2.2), we have

$$ {}_q p_i - {}_q p_{i-1} = {}_p p_{i+1} - {}_p p_i $$

$$ {}_p(p_{i+1} - p_i) = {}_q(p_i - p_{i+1}) $$

Dividing through by p

$$ p_{i+1} - p_i = \frac{q}{p}(p_i - p_{i-1}) $$

In particular, $p_2 - p_1 = \frac{q}{p}(p_1 - p_0) = (\frac{q}{p})p_1$,
(Note that probability of no initial wealth is 0)

such that $p_3 - p_2 = \frac{q}{p}(p_2 - p_1) = \left(\frac{q}{p}\right)^2 p_1$

More generally,

$$ P_{i+1} - P_i = \left(\frac{q}{p}\right)^i p_1 \qquad 2.3 $$

Such that $0 < i < N$.

We have

$$ p_{i+1} - p_1 = \sum_{k=1}^{i} (p_{k+1} - p_k) $$

$$ p_{i+1} - p_1 = \sum_{k=1}^{i} (\frac{q}{p})^k p_1 $$

This result in

$$ p_{i+1} = p_1 + \sum_{k=1}^{i} (\frac{q}{p})^k p_1 = p_1 \sum_{k=0}^{i} (\frac{q}{p})^k $$

$$ = \begin{cases} p_1 \frac{1-\left(\frac{q}{p}\right)^{i+1}}{1-\frac{q}{p}} & if\ p \neq q; \\ p_{1\,(i+1)} & if\ p = q = 0.5 \end{cases} \qquad 2.4 $$

Choosing $i = N - 1$ and using the fact that, $p_N = 1$ yields

$$1 = p_N = \begin{cases} p_1 \frac{1-\left(\frac{q}{p}\right)^N}{1-\frac{q}{p}} & if\ p \neq q\ ; \\ p_1 N & if\ p = q = 0.5; \end{cases} \qquad 2.5$$

From which we conclude that

$$p_i = \begin{cases} \dfrac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N} & if\ p \neq q; \\ \dfrac{i}{N} & if\ p = q = 0.5; \end{cases} \qquad 2.6$$

Bearing in mind that, $1-p_i$ is the probability of ruin with equation (2.6) being the probability of getting attained result. That is, the probability of getting ruined denoted by q_i is given by

$$q_i = \begin{cases} \frac{\left(\frac{q}{p}\right)^N-\left(\frac{q}{p}\right)^i}{\left(\frac{q}{p}\right)-1} & if\ p \neq q; \\ 1-\frac{i}{N} & if\ p = q = 0.5; \end{cases} \qquad 2.7$$

Where i is the initial gamble of the gambler;

N is the attained wealth after a number of placed-bets or expected fortune to reach from a number of plays, while noting that $p_i + q_i = 1$. That is, the probability of getting ruined plus probability of being successful sums up to 1.

3.3 Consequence of the Gamblers' Ruin for Infinite Wager

The actuarial risk situations of the gambler can be accessed via-a-vis the above for infinite play. Infinitely getting rich or being ruined of a risk—averse gamblers are thus determined

$$\lim_{N\to\infty} p_i = 1-\left(\frac{q}{p}\right)^i > 0, p > 0.5 \qquad 2.21$$

That is, from equation (2.6), the probability of the gambler getting infinitely rich is $\mathbf{1}-(\frac{\boldsymbol{q}}{\boldsymbol{p}})^i$. This works if the probability of win is greater than 0.5 (or $P > 0.5$), in which case $(\frac{q}{p}) < 1$. The scenario however changes when $p \leq q$

$$\lim_{N\to\infty} p_i = 0 \text{ for } p \leq 0.5 \qquad 2.22$$

Clearly, if the gambler starts with an initial wealth $x_0 = i$, and playing against an adversary with infinite wealth and wishes to continue gambling for forever, then with $p > 0.5$ the gambler will never be brought to ruined, but when $p \leq q$ (negative expectation), the eminent ruin of the gambler become glaring.

With probability of win $p = 0.6$ and $q = 1 - p = 0.4$, if the gambler infinitely wagers a portion of bankroll on a played game, then the actuarial probability of getting rich denoted by $p_i{}^*$ is

$$p_i{}^* = 1 - \left(\frac{0.4}{0.6}\right) = 0.3333$$

Implying that $q_i{}^* = 0.6667$ (ie $1 - 0.3333$).

4. Results and Analysis

Kelly criterion, a much publicised work by the famed financial and investment analyst Kelly avers that, the gambler faced with a series of favourable placed bets should wager a fixed percentage of the bankroll at a time to maximize returns and ensure growth of the bankroll. The inherent sense of this to the investor is to avoid eventual ruin, while, growing the capital of the gambler.

In the light of the above, with an initial wealth of GH 80, and given levels of win and lose probabilities, the possibility of reaching an expected amount without necessarily getting broke or ruined can be accessed. Equations (2.6) and (2.7) provide useful response of these possibilities.

Table 1: Analysis of Ruin Probabilities

i **GH**	N GH	p	q	p_i	q_i
20	30	0.50	0.50	0.6667	0.3333
20	35	0.50	0.50	0.5714	0.4286
30	35	0.50	0.50	0.8571	0.1429
35	45	0.50	0.50	0.7778	0.2222
40	55	0.50	0.50	0.7272	0.2728
20	30	0.40	0.60	0.0173	0.9827
20	30	0.45	0.55	0.1323	0.8677
30	35	0.45	0.55	0.3661	0.6339
35	45	0.55	0.45	0.9992	0.0008
5	10	0.45	0.55	0.2683	0.7317
9	10	0.45	0.55	0.7899	0.2101
99	100	0.40	0.60	0.6667	0.3333

From earlier discussions, the labelled columns are explained as

i the initial capital of the gambler for placed bet

N the total expected fortune to reach

p the probability of win of a placed bet

q the probability of lose of a place bet
p_i probability of additional earnings
q_i probability of ruin.

A close examination of Table 1 indicates from the first two rows that, with equal probabilities of play win and lose, the probability of ruin in the case of seeking total fortune of $GH\,30$ from $GH\,20$ (i.e. $30 - 20 = 10$ units of income) is 33.3% compared with when the gambler seeks a total fortune of $GH\,35$ from $GH\,20$ (i.e. $35 - 20 = 15$ units of income) with same parameters, in which case, the ruin possibility of the gambler increased by 9.53 ($0.4286 - 0.3333$) percentage points. In essence, when the gambler's expected fortune to reach from initial capital is higher; the probability of ruin became higher. Also, a close look made from the first-half of the table for further observation confirms this remark. In fact, this is when the probability of win is equal to losing a particular game play. This makes much sense intuitively in risk analysis theory.
The instance where the probability of win is not equal to a lost for a placed bet (i.e. $p \neq q$), the gambler's risk of ruin (RoR) fluctuates amidst specified levels of probabilities, differential initial amounts and expected fortunes. We observe from the first row of the second-half of table (2.1) that, when $p = 0.4$ and $q = 0.6$ ($p \neq q$), the ruin probability of the gambler becomes very high (98.27%) with a negligible probability of 1.73% of attaining the goal of $N = 30$ from $i = 20$. In the second row, when the probability of win fell from 60% to 55%, the gambler RoR reduced by 11.5% percentage points from 98.27% to 86.77% cognizance of the same initial amount and expected fortune in both scenarios. Like our earlier observation, when the expected fortune to reach is GHC (from 30 to 35), with same dynamics of the market as second scenario above, the RoR fell further. This time twice what the probability of win for a placed bet is to 63.39%. Interesting, with a more positive possibility of winning a placed bet, the probability to reach the expected fortune is very much higher (almost to parity) than the ROR of the gambler. With same dynamics of initial amount i $GHC\,30$, expected fortune N, $GHC\,35$ and given probabilities are highlighted in the table, the ROR become very much negligible in which case the gambler expectation become realistic. This is in consonance with the inherent risk caution feature of the much publicised Kelly criterion. (*Read about this*). Again, in seeking additional GHC 1 to the wealth of the gambler which is minimal compared to the above described scenarios, when the probability of win is less than that of loses for a placed bet, the RoR of the gambler is low. However, with same conditions, but higher expected fortune to reach, the gambler RoR increased again affirming earlier conclusion and submission reached that, the higher expected fortune to reach correlates with the gambler's RoR.

4.1 Optimal Betting Strategy

In the light of the above discussion, the question that easily comes to mind is: If the initial amount i to wager is $GHC\,10$, and the expected amount N to reach is

$GHC\ 100$; then between choosing betting $GHC\ 1$ repeatedly or $GHC\ 10$ from accrued revenue repeatedly, which should be adopted at specified level of probabilities of win or lose for a placed bet? These are important for the gambler for reasons of not betting too little to be denied of large funds when there is favourable result, and too big only to lose the funds to be brought to ruin.
With probabilities of $p = 0.49$ and $q = 0.51$ the probability of getting rich if $GHC\ 1$ is repeatedly wagered is determined to be

$$p_{10} = \frac{1-\left(\frac{0.51}{0.49}\right)^{10}}{1-\left(\frac{0.51}{0.49}\right)^{100}} = 0.0092 \qquad *$$

However, if the gambler bets $GHC\ 10$ at each game with same parameters of probabilities of win and lose as above, in which case $N = 10$ and $i = 1$ since we need to make a net total of 9 wins, then the probability to win $GHC\ 100$ in $GHC\ 10$ bets starting with $GHC\ 10$ funds is

$$p_1 = \frac{1-\left(\frac{0.51}{0.49}\right)}{1-\left(\frac{0.51}{0.49}\right)^{10}} = 0.0830 \qquad **$$

Clearly, a risk—averse gambler by every imagination will seek to absolve him/herself or minimize the correlation between RoR and the expected fortune to reach discussed earlier. We see from (*) that, the chance to win the 'expected fortune to reach' by repeatedly betting $GHC\ 1$ is very negligible as $0.0092 = 0.92\%$. Contrasting this with repeatedly betting $GHC\ 10$, the gambler's expected win is $0.0830 = 8.3\%$; that is, the chance to win is 8.3 in hundred which is 9 times better than playing $GHC\ 1$ repeatedly. Essentially, this indicates that, relative to the expected fortune to reach N from initial wealth i, the gambler should bet bold in a sub-fair ($p < q = 0.49 < 0.51$) situation, but tread cautiously in a super—fair ($p > q$) situation relative to the dynamics of the game of football scores.

4.2 Knowing When to Stop

From above discussions, subtle though, but, important questions arise for address and discussion relative to the point of exit for the gambler. These are

- How long does the gambler play the game of bets?
- At what revenue level does the gambler reach to warrant an exit?

Obviously, these are cerebral questions for reflection to ensure ruin free state of the gambler vis—a—vis the possibility of being brought to bankruptcy. In the light of this, the determination of the point of exit for the gambler largely known in stochastic and probability theory as stopping time becomes relevant for interrogation.

4.3 Stopping Time

Definition 4.1: Let $X = \{X_n : n \geq 0\}$ be a stochastic process. Then with respect to a random time X, we define a stopping time to be a time such that for each $n \geq 0$, the event $\{\tau = n\}$ can be determined to a large extent by the total information up to and including time n denoting $\{X_0, X_1, \dots, \dots, X_n\}$.

For the relevant of the discussion herein referred to, if $X_n{}^*$ is made to denote the total earnings after the n^{th} gamble, a stopping time τ is a rule that indicates what time to stop gambling. Heuristically, the point of exit for the gambler after a given number of plays is largely dependent on the information known up to and including time n and not on future information. That is, the decision depends on the history (the past) and the present, and not on the future of how much gains or losses have being made to warrant an exit or stop. In essence, for sample path w, such that $N(w) = n$, then the gambler stops playing the game based on information generated by the sample paths $X_1(w), X_2(w) \dots, \dots, X_3(w)$ which is independent of the future information of $X_{n+1}(w), X_{n+2}(w), \dots, \dots, \dots$. In essence, stopping time in gambling strives on current and past information, and not on future information.

Most formally, let $(\mathcal{F}_t)_{t \in \mathbb{T}}$ be a filtration on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ where $\mathcal{F}$ is a $\alpha -$field and $\mathbb{P}$ is a probability measure. A random variable τ taking values in $\mathbb{T}\ U\{\infty\}$ is a stopping time for the filtration $(\mathcal{F}_t)$ if the event $\{\tau \leq t\} \in \mathcal{F}_t$ for every $t \in \mathbb{T}$.

4.4 Stopping Time Model

Let $X_1, X_2, \dots, \dots$ be a sequence of random variables of observations for the gambler from placed bets as far as possible. For each $n = 1,2,3 \dots, \dots$ after observing $X_1 = x_1, X_2 = x_2, \dots, X_n = x_n$ the gambler may either stop or continue relative to the received reward or losses $X^*(x_1, x_2, \dots, \dots)$ if the desire is for an unlimited period. The problem that stopping time addresses for the gambler is to choose which time to stop to maximise the expected reward (odds) which could be made from decisions about X^*. In essence, at time n (having observed $X^*(x_1, x_2, \dots, \dots)$, the gambler is obliged to choose a probability of stopping from these observations to indicate the point of exit. With associated probabilities $\mathbb{P}(p_1, p_2 \dots, \dots p_n)$, the stopping time is a sequence of random variables with the function $X^*(\mathbb{P}) = (x_1(p_1), x_2(p_2), \dots, \dots, x_n(p_n))$.

Obviously, the random variables determine the random time N at which stopping time occurs, $0 \leq \tau \leq 1$

The probability mass function (pmf) of $x_1, x_2, \dots, \dots$, of N given $x = (x_1, x_2 \dots, \dots)$ is

$$\psi_n(X_1, X_2, \dots, \dots, X_n) = P(N = n | X = x) for\ x = 0,1,2,3 \dots \dots$$

4.5 Gambler's Ruin and Stopping Time

From the above discussion of the gambler's ruin problem under "Optimal Betting Strategy" in section (2.5), if the gambler has $GHC\ 10$ with an expected amount to reach of $GHC\ 100$, and play $GHC\ 1$ per bet repeatedly, then representing n_1, n_2 as the numbers of plays for reaching an expected amount and point of ruin respectively, the following stopping time scenarios arise:

- The number of plays $\tau = n_1$ for the gambler to achieve the desire expected amount to reach for exit is a stopping time.
- The number of plays $\tau = n_2$ for the gambler to be brought to ruin (deplete bankroll) for exit is a stopping time
- The gambler stops (point of exit) either when the expected amount to reach is achieved or the capital is depleted, whichever comes first in which case the number $\tau = \min\{n_1, n_2)$ indicates the stopping time.

That is, the gambler's points of exit is either $x_n = N \text{ or } x_n = 0$ whichever happens first. The first passage time is thus represented as set $A = \{0, N\}$. By first passage time, we mean that, if X is a discrete state space and i is a fixed state, then

$$\tau = \min\{n \geq 0 : X_n = i\}$$

This is also referred to as the hitting time of the process to state i. And can be describe as the best point of exit for the gambler. For example, if $C = \{5,6,8,11\}$ is a set of states and τ is the first hitting time into the set C, then we have

$$\tau = \min\{n \geq 0, X_n \in C\}.$$

If $\{X_n\}$ is a process adapted to $\{\mathcal{F}_n\}$ on a Borel set B. Then defining a random variable

$$\tau(w) = \inf\{n \geq 0 : X_n(w) \in B\} \text{ for } \tau(w) = \infty$$

if $X_n \notin B$ for all n. In this regard, τ is a stopping time for $\{X_n\}$. It is fair to conclude easily that, hitting time is the first time the process takes a value in set B, if it ever takes a value. This leads to the following theorem.

Theorem 4.1: *Hitting times are stopping times*

Proof:

$$\text{Let } \{X_0 \in A\} = \{\tau = 0\} \text{ for } X_0$$
$$\text{for } n \geq 1, \{\tau = n\} = \{X_0 \notin A, X_1 \notin A, \dots, \dots X_{n-1} \notin A, X_n \in A\},$$

and this depends only on $\{X_0, X_1, \dots, \dots . X_n\}$ which is required.

When $A = \{i\}$, then it reduces the hitting time to state i and

$$\{\tau = n\} = \{X_0 \neq i, \dots, \dots, X_{n-1} \neq i, X_n = i\}.$$

In the gamblers' ruin derivation above, the wealth of the gambler represented a simple random walk (symmetric in nature) as

$$X_n = i + \text{ɣ}_1 + \text{ɣ}_2 + \cdots + \text{ɣ}_n \text{ with } X_0 = i \in \{1,2, \dots, N-1\}$$

That is $\tau = \min\{n \geq 0 : X_n \in A\}$ is a stopping time relative to both $\{X_n\}$ and $\{\text{ɣ}_n\}$.
Thus for the information flow (paths) $\{X_0, X_1, \dots., X_n\}$ and $\{\text{ɣ}_1, \text{ɣ}_2, \dots., \text{ɣ}_n\}$ for $n \geq 2$, then we have

$$\{\tau = n\} = \{X_0 \notin A, \dots., X_{n-1} \notin A, X_n \notin A\},$$
$$= \{i + \text{ɣ}_1 \notin A, i + \text{ɣ}_1 \notin A, + \cdots, \text{ɣ}_{n-1} \notin A, i + \text{ɣ}_1 + \cdots + \text{ɣ}_n \in A\}$$

The point is that, by knowing that $X_0 = i$(the initial condition), $\{X_n\}$ and $\{\text{ɣ}_n\}$ generate the same information to the gambler.

4.6 Gambler's Ruin, Stopping Time and Wald's Equation

The Gambler's ruin analysis and stopping point are amenable to past and current flow of information for decision making purposes. Leaning the above discussions to stochastic theory, reminiscent of the Markov property easily comes to mind. The Markov property states that, given the present state X_n at any time n, the future $\{X_{n+1}, X_{n+2} \dots..\}$ is independent of the past $\{X_0, X_1 \dots., \dots., X_{n-1}\}$. By replacing the stopping time τ in the place of the deterministic time n {i.e. $\tau = n$}, the Markov property is retained, but in a strong form.
For independent and identically distributed (i.i.d) random variables as in section (3.12), for $\{X_n : n \geq 1\}$, and with a common mean value function $E(X)$, the sum of the r.v.s up to time τ can be

$$E\left\{\sum_{n=1}^{\tau} X_n\right\} = E(\tau)E(X) \qquad 3.1$$

Provided $E(\tau) < \infty$ and $E(|X|) = \infty$. And this is the statement of the Wald's equation.

Thus, the Wald's equation states that the expectation value of the sum $x_1 + x_2 + \dots\dots x_\tau$ is equal to the expectation of x times the expectation value of τ

Proof:

$$\sum_{n=1}^{\tau} X_n = X_1 + \cdots + \cdots + X_\tau$$

$$\sum_{n=1}^{\tau} X_n = \sum_{n=1}^{\infty} X_n I\{\tau > n-1\}$$

Where $I\{\tau > n-1\}$ denotes the random variable for the event $\{\tau > n-1\}$. By the definition of stopping time, $\{\tau > n-1\}$ can only depend (**at most**) on $\{X_1, \dots, X_{n-1}\}$. Analogously, X_n is independent of the event $\{\tau > n-1\}$.

$$E\left\{\sum_{n=1}^{\tau} X_n\right\} = E(X) \sum_{n=1}^{\infty} P(\tau > n-1)$$

$$= E(X) \sum_{n=0}^{\infty} P(\tau > n)$$

$$=E(X)E(\tau) \qquad 3.2$$

For any fixed integer $k \geq 1$, we can replace the deterministic time k by the expected value of a random time τ when τ is a stopping time.

$$E(X_1 + \cdots + X_k) = kE(X)$$

5. Conclusion

The ruin of the gambler at any point in time remains eminent vis-a-vis the randomness of the game of betting and gambling. However, optimal betting strategy exists for the adoption of bettors relative to the amount for wagering and best point of exit for reason of ruin avoidance.

This is made possible by observing the current and past events for purpose of projecting appropriately into future.

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