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Recent Progress in Chen's conjecture

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Abstract

In 1991, B-Y Chen states a very interesting conjecture that every biharmonic submanifold is minimal. From then, many researchers proved the conjecture in several cases. In this short note some old and new results concerning this conjecture are presented. Furthermore a generalization of this conjecture and the relation with harmonic maps is given.

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1 Introduction

Let $x : M \to E^n$ be an isometric immersion of an m-dimensional connected submanifold of a Euclidean space E^n. If we denote by $\vec{x}$, $\vec{H}$ and Δ the position vector field, the mean curvature vector field, and the Laplace operator respectively of M, with respect to the induced metric of M, then it is well

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known that

$$\Delta\vec{x} = -m\vec{H} \tag{1}$$

From this we conclude that M is a minimal submanifold of E^n if and only if its coordinate functions are harmonic. Easily we have that every minimal submanifold satisfies

$$\Delta\vec{H} = \vec{0}. \tag{2}$$

Submanifolds of E^n which satisfy condition (2) are said to have harmonic mean curvature vector field. Submanifolds which satisfy equation (2) are called biharmonic since, in view of (1), equation (2) is equivalent to $\Delta^2\vec{x} = 0$. The study of equation (1) turns to the study of the following fourth order elliptic semilinear partial differential system [8]

$$\begin{aligned} &\Delta H + \sum_{i=1}^{m} h(Se_i, e_i) = 0 \\ &m\nabla < H, H > +4trS^2 = 0, \end{aligned}$$

where $\{e_1, e_2, ..., e_m\}$ is an orthonormal frame of M, h the second fundamental form, S the shape operator and Δ is the Laplace operator. A conjecture of B.Y. Chen [9] states that:
C1: *the only biharmonic submanifolds of Euclidean spaces are the minimal submanifolds.*

In [9] B-Y Chen states another conjecture about biharmonic hypersurfaces. He states that:
C2: *Every biharmonic hypersurface of Euclidean spaces is minimal.*

The first conjecture received much attention from researchers. This fact made many geometers:

- to support the truth (or try to find counter examples),
- to develop and prove other conjectures

2 The progress of the conjecture

In [9] B.-Y. Chen proved that every biharmonic surface in E^3 is minimal. In [19], [20] I. Dimitric generalizing Chens result, proved that any biharmonic submanifold M of a Euclidean space E^n is minimal if it is one of the following:
(a) a curve,
(b) a submanifold with constant mean curvature,
(c) a hypersurface with at most two distinct principal curvatures,
(d) a pseudo-umbilical submanifold of dimension n 6 = 4, or
(e) a submanifold of finite type.

Th. Hasanis and Th. Vlachos in [25] proved that every biharmonic hypersurface in E^4 is minimal. For alternative proof of the same theorem one could see also [16]. Yu Fu in [31] proved that biharmonic hypersurfaces with three distinct principal curvatures in Euclidean 5-space are minimal. Biharmonic submanifolds which are complete and proper studied by Akutagawa and Maeta [1]. B.-Y. Chen and M. I. Munteanu in [15] studied $\delta(2)$-ideal and $\delta(3)$-ideal biharmonic hypersurfaces. Luo focused on weakly convex biharmonic submanifolds, [28]. In [30], Y.-L. Ou constructed examples to show that the original biharmonic conjecture is not valid in the case of biharmonic conformal submanifolds in Euclidean spaces. N. Nakauchi and H. Urakawa, studied biharmonic hypersurfaces in a Riemannian manifold with non-positive Ricci curvature, [29].

In contrast to the Euclidean cases, the conjecture generally fails for submanifolds in a pseudo-Euclidean space E^n_s. This is not unexpected issue since a problem formulated in Euclidean spaces may often appear considerably different when someone studied it in pseudo-Euclidean spaces.

B.Y. Chen and S.Ishikawa in [13] gave examples of non-minimal biharmonic space-like surfaces with constant mean curvature, in the pseudo-Euclidean spaces E^4_s $(s = 1, 2)$. For example, in E^4_1 $(+, +, +,)$ the surface $\vec{r}(u, v) = (u, v, f(u, v), f(u, v)$ with $\Delta f \neq 0$, $\Delta^2 f = 0$ is non-minimal but biharmonic. Furthermore, in [14] the same authors classified pseudo-Riemannian biharmonic surfaces of signature (1,1) with constant nonzero mean curvature and flat normal connection in E^4_s.

However, biharmonicity implies minimality in some special cases. In fact, in [Ch6] it was shown that any biharmonic surface in E^3_s $(s = 1, 2)$ is minimal, and in [18] it was proved that every biharmonic hypersurface M_r^3 of E_s^4 $(s = 1, 2, 3)$ whose shape operator is diagonal is minimal. Also in [3] proved the truth of the conjecture that any biharmonic Lorentz hypersurfaces in E_1^4 is minimal.

3 A Generalization

A generalization of the equation $\Delta\vec{H} = \vec{0}$ is to study the condition $\Delta\vec{H} = \alpha\vec{H}$ for some non-zero constant α. This equation was first appeared in [11] where B.Y. Chen classified surfaces in E^3 which satisfying $\Delta\vec{H} = \alpha\vec{H}$. Also, in [12] it was shown that a submanifold M of a Euclidean space satisfies $\Delta\vec{H} = \alpha\vec{H}$ if and only if M is biharmonic or of 1-type or of null 2-type.

Hypersurfaces in E^4 satisfying $\Delta\vec{H} = \alpha\vec{H}$ with the additional condition of conformal flatness were classified in [24] by O. Garay, and in [17] F. Defever proved that every hypersurface of E^4 satisfying $\Delta\vec{H} = \alpha\vec{H}$ has constant mean curvature. A. Ferrandez and P. Lucas in [22] classified surfaces M_r^2 $(r = 0, 1)$ in the Lorenz-Minkowski space E_1^3. The same authors studied [23] the case of hypersurfaces M_r^{m-1} $(r = 0, 1)$ in E_1^n satisfying $\Delta\vec{H} = \alpha\vec{H}$ and such that the minimal polynomial of the shape operator is at most of degree two, and show that M_r^{m-1} has constant mean curvature. B.-Y. Chen in [10] classified submanifolds in De Sitter space-time satisfying $\Delta\vec{H} = \alpha\vec{H}$.

In [2] the authors proved that a hypersurface of the pseudo-Euclidean space E_s^4 satisfying $\Delta H = \lambda H$ and with shape operator which is diagonalizable, has constant mean curvature.

4 The conjecture and Harmonic maps

In this section the relation of equation $\Delta\vec{H} = \alpha\vec{H}$ to the theory of harmonic and biharmonic maps is described. R. Caddeo, S. Montaldo and C. Oniciuc in [4] classified the nonharmonic biharmonic submanifolds of the unit three dimensional sphere S^3.

Definition 4.1 *Let (M^m, g) and (N^n, h) be Riemannian manifolds. A smooth map $\varphi : M \to N$ is said to be* ***harmonic*** *if it is a critical point of the energy* *functional:*

$$E_1 = \frac{1}{2}\int_M |d\varphi|^2 dv_g.$$

If we denote by ∇^{φ} the connection of the vector bundle φ^*TN induced from the Levi-Civita connection ∇^h of (N,h). Then the second fundamental form $\nabla D\varphi$ is defined by

$$\nabla d\varphi(X,Y) = \nabla^{\varphi}_X d\varphi(Y) - d\varphi(\nabla_X Y), \quad X,Y \in \Gamma(TM),$$

where ∇ is the Levi-Civita connection of (M,g). The tension field $\tau(\varphi)$ is a section of $\varphi^*(TN)$ defined by

$$\tau(\varphi) = tr(\nabla d\varphi).$$

It is well known that the map φ is harmonic if and only if its tension field vanishes.

Now assume that $\varphi : M \to N$ is an isometric immersion with mean curvature vector field $\vec{H}$. Then $m\vec{H} = \tau(\varphi)$ (cf. [21]). Therefore the immersion φ is a harmonic map is and only if M is a minimal submanifold of N.

Definition 4.2 *A smooth map $\varphi : M \to N$ is called **biharmonic** if it is a critical point of the bienergy functional:*

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 dv_g.$$

In [26] and [27] G.Y. Jiang proved that the Euler-Lagrange equation for E_2 is given by

$$\tau(\varphi) = -J_{\varphi}(\tau(\varphi)) = 0.$$

Here J_{φ} is the Jacobi operator of φ acting on sections $V \in \Gamma(\varphi^*TN)$.

If $x : (M^m, g) \to (E^n, canonical)$ is an isometric immersion, then we get:

$$\tau_2(x) = \Delta\tau(x) = \Delta(m\bar{H}) = m\Delta\bar{H}.$$

Therefore, M^m is a biharmonic submanifold of the Euclidean space E^n with the canonical metric if and only if the immersion $x : M^m \to E^n$ is a biharmonic map.

Using the above Caddeo, Montaldo and Oniciuc [5] proved that every biharmonic surface in the hyperbolic 3-space $H^3(-1)$ of constant curvature -1 is minimal. They also proved that biharmonic hypersurfaces of $H^n(-1)$ with at most two distinct principal curvatures are minimal [5]. Based on these, Caddeo, Montaldo and Oniciuc state the following conjecture in [6], [7]: C3:*Any biharmonic submanifold of a Riemannian manifold with non-positive sectional curvature is minimal.* (generalized conjecture)

References

[1] K. Akutagawa and S. Maeta, Biharmonic properly immersed submanifolds in Euclidean spaces, *Geom. Dedicata*, **164**, (2013), 351-355.

[2] A. Arvanitoyeorgos, F. Defever and G. Kaimakamis, Hypersurfaces of E_s^4 with proper mean curvature vector, *Journal of the Math. Soc. of Japan*, to appear.

[3] A. Arvanitoyeorgos, F. Defever, G. Kaimakamis and B.J. Papantoniou, Biharmonic Lorentz hypersurfacesin E_1^4 , *Pacific J. Math.*, **229**(2), (2007), 293-306.

[4] R. Caddeo and S. Montaldo, C. Oniciuc Biharmonic submanifolds of S^3, *Intern. J. Math.*, **12**(8), (2001), 867-876.

[5] R. Caddeo, S. Montaldo and C. Oniciuc, Biharmonic submanifolds in spheres, *Israel J. Math.*, **130**, (2002), 109-123.

[6] A. Balmus, S. Montaldo and C. Oniciuc, Biharmonic hypersurfaces in 4-dimensional space forms, *Math. Nachr.*, **283**, (2010), 1696-1705.

[7] A. Balmus, S. Montaldo and C. Oniciuc, New results toward the classification of biharmonic submanifolds in Sn, *An. St. Univ. Ovidius Constanta*, **20**(2), (2012), 89-114.

[8] B.-Y. Chen, *Total Mean Curvature and submanifolds of finite type*, Singapore, 1984.

[9] B.-Y. Chen, Some open problems and conjectures on submanifolds of finite type, *Soochow J. Math.*, **19**, (1991), 169-188.

[10] B.-Y. Chen, Submanifolds in De Sitter space-time satisfying $\Delta H = \lambda H$, *Israel J. of Math.*, **91**, (1995), 373-391.

[11] B.-Y. Chen, Null two-type surfaces in E^3 are circular cylinders, *Kodai Math. J.*, **11**, (1988), 295-299.

[12] B.-Y. Chen, Null two-type surfaces in Euclidean space, *Proc. Symp. in honor of Cheng-Sung Hsuand Kung-Sing Shih: Algebra, Analysis, and Geometry*, World Scientific, Teaneck, NJ, (1988), 1-18.

[13] B.-Y. Chen, S. Ishikawa Biharmonic surfaces in pseudo-Euclidean spaces, *Memoirs of the Fac. of Science, Kyushu University*, Series A, **45**, (1991), 323-347.

[14] B.-Y. Chen, S. Ishikawa Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces, *Kyushu Journal of Mathematics*, **52**, (1998), 167-185.

[15] B.-Y. Chen and M. I. Munteanu, Biharmonic ideal hypersurfaces in Euclidean spaces, Differential Geom. Appl., **31**, (2013), 1-16.

[16] F. Defever, Hypersurfaces of E^4 with harmonic mean curvature vector, *Math. Nachr.*, **196**, (1998), 61-69.

[17] F. Defever, Hypersurfaces of $\bar{\mathbb{E}}^4$ satisfying $\Delta\vec{H} = \lambda\vec{H}$, *Michigan Math. J.*, **44**, (1997), 355-363.

[18] F. Defever, G. Kaimakamis, B.J. Papantoniou, Biharmonic hypersurfaces of the 4-dimensional semi-Euclidean space E^4_s, *J. Math. Anal. Appl.*, **315**(1), (2006), 276-286.

[19] I. Dimitric, *Quadratic Representation and Submanifolds of Finite Type*, Doctoral thesis Michigan State University, 1989.

[20] I. Dimitric, Submanifolds of Em with harmonic mean curvature vector, *Bull Inst. Math. Acad. Sinica*, **20**, (1992), 53-65.

[21] J. Eells, J.H. Sampson, Harmonic mappings of Riemannian manifolds, *Amer. J. Math.*, **86**, (1964), 109-160.

[22] A. Ferrandez, P. Lucas, On surfaces in the 3-dimensional Lorentz-Minkowski space, *Pacific J. Math.*, **152**(1), (1992), 93-100.

[23] A. Ferrandez, P. Lucas, Classifying hypersurfaces in the Lorentz-Minkowski space with a characteristic eigenvector, *Tokyo J. Math.*, **15**(2), (1992), 451-459.

[24] O.J. Garray, A classification of certain 3-dimensional conformally flat Euclidean hypersurfaces, *Pacific J. Math.*, (1994), 16213-25.

[25] Th. Hasanis, Th. Vlachos Hypersurfaces in E^4 with harmonic mean curvature vector field, *Math. Nachr.*, **172**, (1995), 145-169.

[26] G.Y. Jiang 2-harmonic isometric immersions between Riemannian manifolds, *Chinese Ann. Math.*, Ser A, **7**(2), (1986), 130-144.

[27] G.Y. Jiang 2-harmonic maps and their first and second variational formulas, *Chinese Ann. Math.*, Ser A, **7**(4), (1986), 389-402.

[28] Y. Luo, Weakly convex biharmonic hypersurfaces in Euclidean spaces are minimal, arXiv:1305.71981v1, 2013.

[29] N. Nakauchi and H. Urakawa, Biharmonic hypersurfaces in a Riemannian manifold with non-positive Ricci curvature, *Ann. Glob. Anal. Geom.*, **40**, (2011), 125-131.

[30] Y.-L. Ou, On conformal biharmonic immersions, *Ann. Global Anal. Geom.*, **36**(2), (2009), 133-142.

[31] Yu Fu, Biharmonic hypersurfaces with three distinct principal curvatures in Euclidean 5-space, *Journal of Geometry and Physics*, **75**, (January, 2014), 113-119.