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On commutative and non-commutative quantum stochastic diffusion flows

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Abstract

In this work we develop quantum stochastic solution flows of stochastic diffusion evolution equations of the form

(*SDE*) $$\begin{cases} Lx = F(x(t)),\ t > 0 \\ x(0) = x_0 \end{cases}$$

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on a suitable von Neumann (W^*-, Clifford) algebra C of operators with a finite (probability) regular trace. By $L := d/dt + A$ it is denoted a linear operator such that $-A$ (the Hamiltonian operator of a Quantum Mechanical or a Quantum Field System) is a non-negative and self-adjoint linear operator and the infinitesimal generator of the corresponding analytic semigroup acting on L^2-commutative (Bose-Einstein) of functions or on an L^2-non-commutative (Fermion-Dirac) of operators (possible unbounded operators) Hilbert space H. By F we mean a given H-valued quantum stochastic process. Our results apply on a Fock space generated by Hilbert space K with conjugation J, in a Quantum Mechanical or Quantum Field System, including interactions involving quantized Bose-Einstein and Fermion-Dirac fields (specifically spin ½ Dirac particles) with an external field via a cutoff Yukawa-type interaction.

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1 Introduction

This paper is devoted to quantum stochastic diffusion evolution equations of the form

$$(SDE) \qquad \begin{cases} Lx = F(x(t)),\ t > 0 \\ \quad x(0) = x_0 \end{cases}$$

on a suitable Hilbert space H defined by a suitable von Neumann (W^*-, Clifford) algebra C endowed with a probability regular trace.

The subject has roots in the interactions of elementary particles namely Bosons (photons, mesons, H^4, mesotrons, pions) and Fermions (neutrons, neutrinos, protons, electrons) have been studied from a variety of points of view (cf. [1], [2]).

In particular in their famous papers Carathéodory [3] and Einstein [4], investigated a foundation of Thermodynamics which has consequences for a better consideration of modern quantum fields models for the interactions of elementary particles.

Besides, Oppenheimer and Schwinger [5] examined an effort to take into account the relation of the source to the mesotron field than either Blabha's classical methods or the a priori postulation of isobars afforded.

Moreover, Yukawa, Sakata and Taketani in a series of papers [6], [7] and [8] following previous ideas of Heisenberg and Fermi studied the emission of light particles, i.e. a neutrino and an electron, after the transition of a "heavy" particle from neutron state to photon state. Years later, Glimm [9], Glimm and Jaffe [10] continue the investigations of Yukawa-type interacting coupling spaces.

On the other hand, Accardi, Anillesh and Volterra [11], Arnold and Sparber [12], Canizo, Lopez and Nieto [13], Lindsay [14], Lindsay and Wills [15], Lindsay and Parthasarathy [16], Sparber, Carrillo, Dolbeault and Markowich [17], considered a class of quantum evolution equations, quantum dynamical semigroups for diffusion models and studied a non-commutative generalization of a stochastic quantum differential equation (of Feynman-Kac type) deriving stochastic quantum flows.

In the present work we obtain quantum stochastic diffusion flows in a commutative case (Bose-Einstein interaction) and in a non-commutative case (Fermi-Dirac interaction).

We study (SDE) in the infinite dimensional case, where $L := d/dt + A$ denotes a linear operator such that $-A$ is a non-negative self-adjoint linear operator (the Hamiltonian operator) acting on a Hilbert space H such that $-A$ is the infinitesimal generator of an analytic semigroup e^{-tA}, $t \in \mathbf{R}^+$ and F is a given quantum stochastic process taking values in H.

2 Function spaces and flows

In what follows H will denote a general (complex) Hilbert space with norm $\|\cdot\|$. Let $-A$ be a non-negative self-adjoint operator acting on the Hilbert space H and let e^{-tA}, $t \in \mathbf{R}^+ := [0, \infty)$ be the analytic semigroup acting on H with infinitesimal generator $-A$.

As it is well-known we may assume that there exist positive real numbers M, δ such that

$$\| e^{-tA} \| \leq M e^{-\delta t}, \quad \text{for all } t \in \mathbf{R}^+.$$

Let $C_b(\mathbf{R}^+, H)$ the Banach space of bounded continuous functions $u : \mathbf{R}^+ \to H$ endowed with supremum norm

(2.1) $$|u| := \left\{ \| u(t) \| : t \in \mathbf{R}^+ \right\}$$

and let $C(\mathbf{R}^+, H)$ be the Fréchet space of continuous functions $u : \mathbf{R}^+ \to H$.

By a ***flow*** (***dynamical system***, ***nonlinear semigroup***) on a complete metric space X we mean a family $U = U(t)$, $t \in \mathbf{R}^+$ of functions $U(t) : X \to X$, enjoying the following properties;

(2.2) for every $t \in \mathbf{R}^+$, $U(t)$ is continuous from X into X

(2.3) for each $x \in X$ the function $t \mapsto U(t)x$ is continuous

(2.4) $U(0) = i$ (identity on X)

(2.5) $U(t+s)x = U(t)U(s)x$, whenever $t, s \in \mathbf{R}^+$ and $x \in X$

We recall that the function $t \mapsto U(t)x$ is called the ***trajectory*** of $x \in X$.

In practice flows arise from autonomous differential equations for which there are theorems concerning existence uniqueness and continuity of solutions.

3 Main results

3.1 The linear case

We start with the linear initial value problem

(3.1) $$\begin{cases} \left(\dfrac{d}{dt} + A\right)x(t) = f(t),\ t > 0 \\ x(0) = x_0 \end{cases}$$

where f is a given H-valued function on $\mathbf{R}^+$, $x_0 \in H$.

A function $u : \mathbf{R}^+ \to D(A)$ is called a ***classical solution*** on $\mathbf{R}^+$ of (3.1) if it is strongly differentiable for every $t \in \mathbf{R}^+$ and satisfies (3.1) for every t in $\mathbf{R}^+$.
On the other hand a function u in $C(\mathbf{R}^+, H)$ given by

(3.2) $$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} f(s)\,ds$$

is called the ***mild solution*** of (3.1) on $\mathbf{R}^+$, with initial data $u(0) = u_0$ in H.

Theorem 3.1. Let f be in the Fréchet space $C(\mathbf{R}^+, H)$. Then there exists exactly one mild solution u of (3.2) in $C(\mathbf{R}^+, H)$ and if $f \in C_b(\mathbf{R}^+, H)$ then also $u \in C_b(\mathbf{R}^+, H)$.

Proof. Let t in $\mathbf{R}^+$. By hypothesis the function $f : [0,t] \to H$ is bounded and continuous. Hence the Bochner integral

(3.3) $$\int_0^t e^{-(t-s)A} f(s)\,ds = \int_0^t e^{-sA} f(t-s)\,ds$$

is well-defined for every $t \geq 0$, since:

$$\int_0^t \| e^{-sA} f(t-s) \| \,ds \leq M_0 \int_0^t e^{-\delta s} \| f(t-s) \| \,ds \leq M_0 \mid f \mid_t \int_0^t e^{-\delta s}\,ds$$

(3.4) $$= M_0 \mid f \mid_t \delta^{-1}(1 - e^{-\delta t})$$

where $\mid f \mid_t := \sup\{ \| f(s) \|,\ s \in [0,t]\}$.

Then the function

$$t \mapsto u(t) := e^{-tA}u_0 + \int_0^t e^{-sA} f(t-s)ds$$

is the unique continuous mild solution of (3.1) (see also [18]).

Finally if $f \in C_b(\mathbf{R}^+, H)$ then also $u \in C_b(\mathbf{R}^+, H)$ since

$$\begin{aligned} \| u(t) \| &= \left\| e^{-tA}u_0 + \int_0^t e^{-sA} f(t-s)ds \right\| \\ &\le \left\| e^{-tA}u_0 \right\| + \left\| \int_0^t e^{-sA} f(t-s)ds \right\| \\ &\le M_0 \| u_0 \| + \int_0^t \left\| e^{-sA} f(t-s) \right\| ds \\ &\le M_0 \| u_0 \| + M_0 \mid f \mid \delta^{-1} \end{aligned} \tag{3.4}$$

□

3.2 The non-linear case

We consider the non-linear initial value problem

$$\begin{cases} \left(\dfrac{d}{dt} + A \right) x(t) = F(x(t)), \; t > 0 \\ x(0) = x_0 \end{cases} \tag{3.5}$$

where F is a given H-valued function on H, $x_0 \in H$.

A function $u : \mathbf{R}^+ \to D(A)$ is called a ***classical solution*** on $\mathbf{R}^+$ of (3.5) if it is strongly differentiable for every $t \in \mathbf{R}^+$ and satisfies (3.5) for every t in $\mathbf{R}^+$. Moreover a solution u in $C(\mathbf{R}^+, H)$ of the integral equation

$$x(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} F(x(s))ds \tag{3.6}$$

will be called a ***mild solution*** of (3.5) on $\mathbf{R}^+$, with initial data $u(0) = u_0$ in H.

Let Φ be the corresponding ***Nemytskii operator*** of the non-linear operator $F: H \to H$ appearing in eq. (3.5), i.e. for every $y: \mathbf{R}^+ \to H$, Φy is defined by the formula:

$$\Phi y(t) := F(y(t)),\ t \in \mathbf{R}^+$$

Now we state the following condition concerning the Nemytskii operator Φ.

Condition (Φ): $\Phi y \in C_b(\mathbf{R}^+, H)$ provided that $y \in C_b(\mathbf{R}^+, H)$ and there exists a real-valued function $\gamma \in C_b(\mathbf{R}^+, \mathbf{R}^+)$ such that:

(3.7) $\quad \| \Phi y_1(t) - \Phi y_2(t) \| \le \gamma(t) \| y_1(t) - y_2(t) \|$, for all $y_1, y_2 \in C_b(\mathbf{R}^+, H)$ and $t \in \mathbf{R}^+$.

Theorem 3.2. Let condition (Φ) holds. Then for any given $u_0 \in H$ there exists exactly one mild solution $u := u(0, u_0)$ in $C_b(\mathbf{R}^+, H)$ of (3.5) satisfying $u(0) = u_0$. Moreover assuming that every mild solution is a classical solution of (3.5), there exists exactly one solution flow $U(t)$ on H with trajectories $t \mapsto U(t)x$ in $C_b(\mathbf{R}^+, H)$, $x \in H$.

Proof. Let $u_0 \in H$. Considering the Hamerstein-type operator

(3.8) $$\Pi : C_b(\mathbf{R}^+, H) \to C_b(\mathbf{R}^+, H)$$

which to any $y \in C_b(\mathbf{R}^+, H)$ associates (according to condition (Φ) and to Theorem 3.1) the unique mild solution

(3.9) $$\Pi y(t) := e^{-tA} u_0 + \int_0^t e^{-sA} \Phi y(t-s)\, ds,\ t \in \mathbf{R}^+$$

in $C_b(\mathbf{R}^+, H)$ of the linear initial value problem:

(3.10) $$\begin{cases} \left(\dfrac{d}{dt} + A \right) x(t) = \Phi y(t) \\ x(0) = u_0 \end{cases}$$

Now let $y_1, y_2 \in C_b(\mathbf{R}^+, H)$ and $t \in \mathbf{R}^+$.

Then applying (2.2) and condition (Φ) we see that:

$$\| \Pi y_2(t) - \Pi y_1(t) \| = \left\| \int_0^t e^{-sA} \Phi y_2(t-s) ds - \int_0^t e^{-sA} \Phi y_1(t-s) ds \right\|$$

$$\leq \int_0^t \left\| e^{-sA} (\Phi y_2(t-s) - \Phi y_1(t-s)) \right\| ds$$

$$\leq M_0 \int_0^t e^{-\delta s} \left\| \Phi y_2(t-s) - \Phi y_1(t-s) \right\| ds$$

$$\leq M_0 \, |\gamma| \int_0^t e^{-\delta s} \left\| y_2(t-s) - y_1(t-s) \right\| ds$$

$$\leq M_0 \, |\gamma| \int_0^{+\infty} e^{-\delta s} \, \| y_2(t-s) - y_1(t-s) \| \, ds$$

(3.11) $$\leq M_0 \, |\gamma| \, \delta^{-1} \, | y_2 - y_1 |$$

Applying (3.11) and induction we deduce

(3.12) $$\left\| \Pi^n y_2(t) - \Pi^n y_1(t) \right\| \leq \frac{\left(M_0 \, |\gamma| \, \delta^{-1} \right)^n}{n!} \, | y_2 - y_1 |, \text{ for all } n \in \mathbf{N}.$$

From (3.12) and for n large enough we conclude that Π is a contraction operator on $C_b(\mathbf{R}^+, H)$ and has a unique fixed point $u := u(0, u_0)$ satisfying

(3.13) $$u(t) := e^{-tA} u_0 + \int_0^t e^{-sA} \Phi u(t-s) ds, \quad t \in \mathbf{R}^+$$

Therefore the function $u : \mathbf{R}^+ \to H$ is the unique mild solution of (3.5) in $C_b(\mathbf{R}^+, H)$ with $u(0) = u_0$ (see also [18]).

Then setting

(3.14) $$U(t) u_0 := u(t)$$

whenever $t \in \mathbf{R}^+$ and $u_0 \in H$ and assuming that u is a classical solution of (3.5) we must infer that $U(t)$, $t \in \mathbf{R}^+$, is the unique solution flow on H, with trajectories $t \mapsto U(t) u_0$ in $C_b(\mathbf{R}^+, H)$.

We have first to justify that $U(t)$ satisfies conditions (2.10) and (2.11).

Let $t \in \mathbf{R}^+$.

Let also a sequence $(u_0^{(n)})$ in H such that:

(3.15) $$\ell im_{n\to\infty} u_0^{(n)} = u_0$$

Moreover we consider the corresponding solutions

$$u_n(0, u_0^{(n)}) := u_n, \quad \text{for every } n \in \mathbf{N}, \quad \text{and} \quad u(0, u_0) := u,$$

such that:

(3.16) $$u_n(t) = e^{-tA}u_0^{(n)} + \int_0^t e^{-sA}\Phi u_n(t-s)ds, \; t \in \mathbf{R}^+$$

(3.17) $$u(t) = e^{-tA}u_0 + \int_0^t e^{-sA}\Phi u(t-s)ds, \; t \in \mathbf{R}^+$$

Then combining condition (Φ), (3.16) and (3.17) we have:

$$\| U(t)u_0^{(n)} - U(t)u_0 \| = \| u_n(t) - u(t) \|$$

$$= \left\| e^{-tA}(u_0^{(n)} - u_0) + \int_0^t e^{-sA}(\Phi u_n(t-s) - \Phi u(t-s))\, ds \right\|$$

$$\leq \left\| e^{-tA}(u_0^{(n)} - u_0) \right\| + \int_0^t \left\| e^{-sA}(\Phi u_n(t-s) - \Phi u(t-s)) \right\| ds$$

$$\leq M_0 \left\| u_0^{(n)} - u_0 \right\| + M_0 \int_0^t e^{-\delta s} \left\| \Phi u_n(t-s) - \Phi u(t-s) \right\| ds$$

$$= M_0 \left\| u_0^{(n)} - u_0 \right\| + M_0 \int_0^t e^{-\delta(t-s)} \left\| \Phi u_n(s) - \Phi u(s) \right\| ds$$

(3.18) $$\leq M_0 \left\| u_0^{(n)} - u_0 \right\| + M_0 \,|\gamma|\, \int_0^t \left\| u_n(s) - u(s) \right\| ds$$

Thus from (3.18) and making use of Gronwall inequality we get:

$$\| U(t)u_0^{(n)} - U(t)u_0 \| = \| u_n(t) - u(t) \|$$

$$\leq M_0 \left\| u_0^{(n)} - u_0 \right\| e^{\int_0^t M_0|\gamma| ds}$$

(3.19) $$\leq M_0 \left\| u_0^{(n)} - u_0 \right\| e^{tM_0|\gamma|}$$

Consequently by (3.15) and (3.19) it follows

(3.20) $$\lim_{n\to\infty} U(t)u_0^{(n)} = U(t)u_0 .$$

Next let $u_0 \in H$. Consider also a sequence (t_n) and $t \in \mathbf{R}^+$ such that

(3.21) $$\lim_{n\to\infty} t_n = t$$

and let $t_0 \in \mathbf{R}^+$ with

(3.22) $$|t_n| = t_n \le t_0 ,\ \forall n \in \mathbf{N} .$$

We also put

(3.23) $$t_1 := \max\{t, t_0\} .$$

Then by (3.12), (3.15) and (3.23) we deduce

$$\| U(t_n)u_0 - U(t)u_0 \| = \| u(t_n) - u(t) \|$$

$$= \left\| e^{-t_n A}u_0 + \int_0^{t_n} e^{-sA}\Phi u(t_n - s)\,ds - e^{-tA}u_0 - \int_0^{t} e^{-sA}\Phi u(t-s)\,ds \right\|$$

$$\le \left\| e^{-t_n A}u_0 - e^{-tA}u_0 + \int_0^{t_1} e^{-sA}(\Phi u(t_n - s) - \Phi u(t-s))\,ds \right\|$$

$$\le \left\| e^{-t_n A}u_0 - e^{-tA}u_0 \right\| + \int_0^{t_1} \left\| e^{-sA}(\Phi u(t_n - s) - \Phi u(t-s)) \right\| ds$$

$$\le \left\| e^{-t_n A}u_0 - e^{-tA}u_0 \right\| + M_0 \int_0^{t_1} \left\| \Phi u(t_n - s) - \Phi u(t-s) \right\| ds$$

(3.24) $$\le \left\| e^{-t_n A}u_0 - e^{-tA}u_0 \right\| + M_0\, |\gamma| \int_0^{t_1} \left\| u(t_n - s) - u(t-s) \right\| ds$$

for every $n \in \mathbf{N}$.

Thus by (3.21), (3.24) and the Lebesgue Dominated Convergence Theorem it follows that:

(3.25) $$\lim_{n\to\infty} U(t_n)u_0 = U(t)u_0$$

Finally, by standard arguments, we have $U(0)u_0 = u_0$ and

$$U(t_1)U(t_2)u_0 = U(t_1 + t_2)u_0 ,\ \text{for all } t_1, t_2 \in \mathbf{R}^+ ,$$

and the proof of the theorem is complete. □

4 Applications

4.1 Bose-Einstein case

Let E be the complexification Hilbert space of a real Hilbert space E' and let $\wedge_s(E)$ denote the Hilbert space of symmetric tensors over E.

Then there exists an isomorphism of $\wedge_s(E)$ (via a unitary operator) onto the Hilbert space $L^2(E', B(E'), d_{2c})$, with

(4.1) $$d_{2c}(\Gamma) = (2\pi t)^{-\frac{k}{2}} \int_{\Theta} e^{-\frac{\|x\|^2}{4c}} d\lambda^k(x)$$

where $\Gamma = P^{-1}(\Theta)$, Θ is a Borel set in the image PE' of a k-dimensional orthogonal projection P on E' and $(\mathbf{R}^k, B(\mathbf{R}^k), \lambda^k)$ is the Borel-Lebesgue measure in PE' (cf. [19]).

Therefore we can take the case

(4.2) $$H := L^2(E', B(E'), d_{2c}) = \wedge_s(E) .$$

4.2 Fermion (Fermion-Dirac) case

It is well-known that the Banach lattices $L^p(X, S, \mu)$, $1 \le p \le \infty$ when (X, S, μ) is a measure space can be extended in a non-commutative algebraic context.

We start recalling briefly some well-known facts concerning a non-commutative integration theory in which, instead of integrating functions on a measurable space with respect to a given measure, one integrates (possibly unbounded) operators “affiliated” with a von Neumann algebra V with respect to a “gage” (or a “trace”) on V. We shall restrict on “probability gages” since these gages are relevant for the study of Fermions.

Let E be a complex Hilbert space (the ***Fermion one-particle*** space) and let $\wedge_a^n(E)$ denote the Hilbert space of antisymmetric tensors of rank n over E, whenever $n = 1,2,\ldots$ and let $\wedge_a^0(E)$ be the complex numbers $\mathbf{C}$.

We shall denote by $\wedge_a(E)$ the (***Fermion-Dirac***) ***Fock space***, that is the Hilbert space direct sum

(5.1) $$\oplus_{n=0}^{\infty} \wedge_a^n(E)$$

and ω will denote the complex number ("bare vacuum" or no-particle state) $1 \in \wedge_a^0(E)$.

For every x in E, the ***creation operator*** C_x is the bounded linear operator on $\wedge_a(E)$ with norm $\| C_x \| = \| x \|$ such that:

(5.2) $$C_x(u) = (n+1)^{\frac{1}{2}} P_a(x \otimes u)$$

whenever $u \in \wedge_a^n(E)$, where P_a denotes the ***antisymmetrization projection***.

The ***annihilation operator***, A_x, $x \in E$ is defined to be the adjoint of C_x, that is $A_x := C_x^*$.

Now let J be a conjugation on E. We recall that a function $J : E \to E$ is said to be a ***conjugation*** on E if J is ***antilinear*** ($J(ax+by) = \overline{a}J(x) + \overline{b}J(y)$, whenever $x, y \in E$ and for all complex numbers a and b), J is ***antiunitary*** ($< J(x), J(y) > = < y, x >$, whenever $x, y \in E$, where $< , >$ denotes the inner product on E) and J has ***period*** two ($J^2 = I$).

We also denote by C the von Neumann algebra generated by all operators (the "Fermion-Dirac fields") B_x, $x \in E$ on $\wedge_a(E)$ defined by the formula:

(5.3) $$B_x = C_x + A_{J(x)}$$

We note that C is the ***weakly closed Clifford algebra*** over E relative to the conjugation J.

A ***regular probability gage space*** is a triple (K,V,τ), where K is a complex Hilbert space, V is a von Neumann algebra of linear operators on K and τ is a faithful, central, normal ***trace*** (***state***) on V, i.e. τ is a linear functional from V into $\mathbf{C}$ such that:

(τ_1) τ is a ***state***, i.e. $\tau(I)=1$, $T\in V$, $T\geq 0$ implies $\tau(T)\geq 0$

(τ_2) τ is ***completely additive***, namely, if O is any set of mutually orthogonal projections in V with upper bound Y then $\tau(Y)=\sum_{P\in O}\tau(P)$

(τ_3) τ is ***regular*** or ***faithful***, i.e. if $T\in V$, $T\geq 0$, $\tau(T)=0$ implies $T=0$

(τ_4) τ is ***central***, i.e. $\tau(TS)=\tau(ST)$, whenever $T,S\in V$.

$(\wedge_a(E),C,\tau)$ is a regular probability gage space, where $\tau: C\to\mathbf{C}$, and

(5.4) $$\tau(u):=<u\omega,\omega> \text{ for every } \omega\in C$$

(cf. Segal [20])

For any closed linear operator T on E we put

(5.5) $$|T|:=\left(T^*T\right)^{\frac{1}{2}}$$

For $1\leq p<\infty$, $L^p(E,C,\tau)$ is defined to be the completion of C with respect to the norm $T\mapsto\|T\|_p=\tau\left(|T|^p\right)^{\frac{1}{p}}$. $L^\infty(E,C,\tau)$ is defined to be the Banach space C with respect to its operator norm. It has been shown that the Banach space $L^p(E,C,\tau)$, $1\leq p\leq\infty$ are spaces of linear (possible unbounded) operators on E (cf. Segal [20]).

In particular the function $u\mapsto u\omega$ extends to a unitary operator from $L^2(E,C,\tau)$ onto $\wedge_a(E)$ (cf. [21]).

Now we can take the case

(5.6) $$H:=L^2(E,C,\tau)=\wedge_a(E)$$

since $L^2(E,C,\tau)$ can be regarded as an ordered Hilbert space of operators on E.

Next let S be a four-dimensional complex spin space with positive definite inner product $(\,,\,)$ and let K be the Hilbert space of S-valued functions on $\mathbf{R}^3$ with

(5.7) $$\|\psi\|_K^2 = \int_{\mathbf{R}^3} (\psi(x),\psi(x))\, d\lambda^3(x) < \infty.$$

Then we can also take H the Hilbert state space $\wedge_a(Z)$ over the Hilbert space Z ***of a free spin ½ Dirac particle with an external field via a cutoff Yukawa-type interaction*** such that

(5.8) $$Z = K_+ \oplus K_+$$

where K_+ is the irreducible part of K when the infinitesimal generator of time translation is positive on K_+.

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