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Approximate Solutions of the Brinkman-Forscheimer Model

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Abstract

This article discusses approximation methods for solving the strongly nonlinear Brinkman-Forscheimer equation describing the unidirectional flow of fluid through a horizontal channel filled with porous medium. Comparisons were made of approximations by semi-analytic methods and the finite difference method for this nonlinear problem. The results indicated that the approximations by the finite difference method compared well with solutions by semi-analytic methods in the literature.

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1 Introduction

The steady state and pressure driven unidirectional flow of fluid through a horizontal channel filled with porous media is often modeled by the Brinkman-

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Forscheimer momentum equations represented by the second-order nonlinear differential equation

$$\mu_{\text{eff}}\frac{du^*}{dy^{*2}} = \frac{\mu}{K}u^* + \frac{\rho C_f}{\sqrt{K}}u^{*2} + G \tag{1}$$

where the variables $C_f, G, K, \rho, \mu, \mu_{\text{eff}}$ denote respectively the drag coefficient, the pressure gradient, the permeability, the fluid density, the fluid viscosity and the effective fluid viscosity inside the porous medium [8]. The impermeable walls of the channel are assumed to be at the positions $y^* = \pm h$. By introducing the similarity variables

$$x = \frac{x^*}{P_e H}, \quad y = \frac{y^*}{h}, \quad u = \frac{Gh^2u^*}{\mu}$$

the dimensionless form of (1) is obtained as a

$$\frac{d^2u}{dy^2} - s^2u - Fsu^2 + \frac{1}{M} = 0 \tag{2}$$

subject to the boundary conditions

$$u(\pm 1) = 0.$$

In the above, the variables

$$M = \frac{\mu_{\text{eff}}}{\mu}, \quad F = \frac{C_f\rho GH^3}{\mu_{\text{eff}}\mu}, \quad Da = \frac{K}{H^2}, \quad s = \left(\frac{1}{MDa}\right)^{\frac{1}{2}}$$

where P_e, Da and H denote respectively the Peclet, the Darcy number and the total width of the channel.

Several numerical methods abound for approximating the solution of the above nonlinear two-point boundary value problem. These methods include, but not limited to, the shooting method, finite difference method, Green's function, etc. Over the years, semi-analytical methods and series solution methods like the Adomian decomposition method (ADM) [3] and Variational Iteration Method (VIM) [6, 7] have been employed to obtain semi-analytical solution of general two-point boundary value problem which covers problem (2). It is well known that ADM handles nonlinear boundary value problems quite very effectively [1, 4, 2]. The VIM has also been applied to solve both linear and nonlinear BVP over the years, and it has been reported that solutions obtained

via the VIM compare well with those of the ADM. However, due to ease of application, the ADM remains a preferred method over the VIM [6]. Research results have also indicated that the ADM is preferred over the shooting method [5].

In this article, we apply finite difference method to obtain approximations to the solution of the BVP (2). In addition, we discuss the variational iteration method as well as the Adomian decomposition method applied to (1). More, precisely, in Section 2.1, we discuss the variational iteration method applied to (2) and pointed out why the method is not attractive to solving this model problem. Since there is no closed form solution of problem (2) in general, numerical solutions are also computed via the Matlab routine BVP4c as reference solution. Our numerical results in Section 3 indicated that finite difference method handles Problem (2) better than the ADM for certain combination of the parameters in which case, solution by ADM is not available.

2 Numerical Methods for Problem (2)

In this section we shall consider popular methods for finding approximate solutions of (2), namely the semi-analytical methods: the Adomian Decomposition method and the Variational Iteration method, and the finite difference method as a numerical method among the three. We will provide brief description of the methods, and their applications to (2).

2.1 Variational Iteration Method

Here, we discuss the variational iteration method - a semi-analytical method that can be used to solve the model problem (2). More specifically, we highlight challenges in applying the method to the present problem.

Let us consider a general nonlinear problem

$$Lu(y) + Nu(y) = f(y)$$

where L, N, f denote a linear operator, nonlinear operator and a forcing term respectively. The basic idea of the variational iteration technique applied to

the above problem is to is to construct a correction functional

$$u_{n+1} = u_n + \int_0^y \lambda(Lu_n(\epsilon) + N\tilde{u}_n(\epsilon) - f(\epsilon))d\epsilon$$

for the nonlinear problem. In the above, λ represents a Lagrange multiplier which can be identified optimally through variation theory, u_n denotes the nth approximate solution and $\tilde{u}_n$ denotes a restricted variation in the sense that $\delta\tilde{u}_n = 0$ holds true. The series solution to the problem is then obtained from the approximate solutions $u_1, u_2, ...$ via

$$u(y) = \lim_{n\to\infty} u_n.$$

2.2 Application to Problem (2)

In order to apply the variational iteration technique described above to(2), the Lagrange multiplier λ has to be firstly identified. By defining the correction functional associated with (2) as

$$u_{n+1} = u_n + \int_0^y \lambda(y,\epsilon)\Big(u_n''(\epsilon) - s^2u_n(\epsilon) - Fs\,\tilde{u}_n^2(\epsilon) + \frac{1}{M}\Big)d\epsilon \tag{3}$$

the Lagrange multiplier λ can then be obtained as the solution of the following stationary conditions:

$$\delta u_n(\epsilon): \quad \lambda'' - s^2\lambda(y,\epsilon) = 0, \tag{4}$$

$$\delta u_n(y): \quad 1 - \lambda'(\epsilon)\Big|_{\epsilon=y} = 0, \tag{5}$$

$$\delta u_n'(y): \quad \lambda(\epsilon)\Big|_{\epsilon=y} = 0. \tag{6}$$

Now solving the stationary conditions (4)- (6) yields

$$\lambda(y,\epsilon) = \frac{1}{2s}\Big(e^{s(\epsilon-y)} - e^{-s(\epsilon-y)}\Big) = \frac{1}{2s}sinh(s(\epsilon-y)). \tag{7}$$

Substituting (7) into (3), we obtain the iterative scheme

$$u_{n+1} = u_n + \frac{1}{2s}\int_0^y sinh(s(\epsilon-y))\Big(u_n''(\epsilon) - s^2u_n(\epsilon) - Fs\,u_n^2(\epsilon) + \frac{1}{M}\Big)d\epsilon. \tag{8}$$

2.2.1 Slow convergence of VIM for problem (2)

As hinted in [7] the Lagrange multiplier for nonlinear problem is difficult to identify in general. In the immediate previous discussion, this parameter has been identified using the idea of restricted variation. It is well known that inexact identification of the Lagrange multiplier this way results in slow convergence of the iteration (8).

Typically, the first iteration of variational iteration method gives a very accurate approximation [7]. However, for this present problem, the second iteration of (8) with initial solution $u_0(y) = u_0 = \frac{y^2-1}{3}$ that obviously satisfies the boundary conditions $u(\pm 1) = 0$ yields an inaccurate approximation involving almost 160 terms! The iterative procedure therefore become unmanageable coupled with a very slow convergence. It is therefore not surprising (see [7] and references therein) that in spite of the popularity of the method for more than a decade, no attempt was made to apply variational iteration method to the Brinkman-Forscheimer model equation (2).

As we shall see in Section 3, the Adomian decomposition method, which otherwise is a method of choice for two-point boundary value problem is also sometimes too sensitive to the model parameters F, s and M.

2.3 Adomian Decomposition Method

To illustrate this method, let us consider a general nonlinear problem

$$Lu(y) + Ru(y) + Nu(y) = f \tag{9}$$

where L, R, N denote respectively the linear, linear remainder and non-linear operators, e.g. for (2) we could choose $Lu(y) = \frac{d^2u(y)}{dy^2}$, $Ru(y) = -s^2u(y)$, $Nu(y) = Fsu(y)^2$ and $f = \frac{1}{M}$. In the case $L = \frac{d^k}{dy^k}$ then the inverse operator L^{-1} represents a $k-$fold definite integration from y_0 to y such that $L^{-1}Lu = u - \Phi$ where Φ is a function incorporating the boundary conditions. Applying the inverse operator L^{-1} to both sides of (9) yields

$$u(y) = \gamma(y) - L^{-1}[Ru(y) + Nu(y)] \tag{10}$$

with $\gamma(y) = \Phi + L^{-1}f$.

The Adomian decomposition method now assumes a series solution

$$u = \sum_{n=0}^{\infty} u_n \tag{11}$$

while assuming a decomposition of the form

$$Nu = \sum_{n=0}^{\infty} A_n \tag{12}$$

for the nonlinear term. In the above, the coefficients A_n are called the Adomian polynomials which are given by

$$A_n = \frac{1}{n!}\frac{\partial d^n}{\partial \lambda^n}\left[N\left(\sum_{k=0}^{\infty} u_k \lambda^k\right)\right]_{\lambda=0}$$

for $n = 0, 1, 2, ...$ Now plugging in the decompositions (11) and (12) into (10), we obtain

$$u = \sum_{n=0}^{\infty} u_n = \gamma(y) - L^{-1}\left(R\sum_{n=0}^{\infty} u_n + \sum_{n=0}^{\infty} A_n\right). \tag{13}$$

By choosing $u_0(y) = \gamma(y)$ we obtain the solution components u_n through the classical Adomian recursive scheme

$$u_0(y) = \gamma(y), \tag{14}$$

$$u_{n+1}(y) = -L^{-1}(Ru_n + A_n), \quad n \geq 0. \tag{15}$$

Subsequently, the n-th order approximation of the solution of (9) can be obtained as

$$\psi(y) = \sum_{i=0}^{n-1} u_k(y). \tag{16}$$

2.3.1 Application to Problem (2)

To apply the decomposition method described above to (2), we choose

$$Lu = \frac{d^2u}{dy^2}, \quad Ru = -s^2u, \quad Nu = Fsu^2, \quad f = \frac{1}{M}$$

so that

$$L^{-1}(\cdot) = \int_{-1}^{y}\int_{-1}^{y}(\cdot)\, dydy.$$

The term Φ incorporating the boundary condition is obtained by employing the condition $u(-1) = 0$ as

$$\Phi = u(-1) + (y+1)u'(-1) = (y+1)A$$

where $A := u'(-1)$, a constant to be determined later on using the other boundary condition $u(1) = 0$. Consequently,

$$u_0 = \gamma(y) = A(y+1) - \int_{-1}^{y}\int_{-1}^{y}\left(\frac{1}{M}\right)dydy.$$

The solution components are then computed using the iteration (14), for example

$$u_1(y) = -\int_{-1}^{y}\int_{-1}^{y}\left(-s^2u_0 + Fsu_0^2\right)dydy$$

and n-th order approximation of the solution is obtained using (16). In Section 3, we will report on numerical results of the above implementation as computed in [10].

2.4 Finite Difference Method

The finite difference method is a popular method for approximating solutions of boundary value problems. The basic idea of the finite difference method for boundary value problems is to approximate the differential operator(s) occurring in the equation by appropriate difference schemes that are easily derived via Taylor expansion of the solution u.

Let us briefly describe the finite difference method for a general nonlinear two-point boundary value problem of the form

$$u'' = f(y, u, u'), \;\; a \leq y \leq b, \;\; u(a) = \alpha, u(b) = \beta. \tag{17}$$

In the above, it is assumed that the function u is smooth enough to allow a valid Taylor expansion. Firstly, we discretize the interval $[a, b]$ by inserting N equally spaced grid points or nodes $y_i = a + i\tau, \;\; i = 0, 1, .., N+1$ in the interval $[a, b]$ where the step size $\tau = \frac{b-a}{N+1}$. As a next step, we approximate the differential operator(s) appearing in the equation, namely here, the first and second derivatives u' and u'' respectively. If we assume that the solution

u possess a bounded fourth derivative, then simple algebraic manipulations of the Taylor expansions of $u(y \pm \tau)$

$$u(y+\tau) = u(y) + \tau u'(y) + \frac{\tau^2}{2}u''(y) + \frac{\tau^3}{3!}u'''(y) + \frac{\tau^4}{4!}u''''(\eta), \quad \eta \in (y, y+\tau),$$

$$u(y-\tau) = u(y) - \tau u'(y) + \frac{\tau^2}{2}u''(y) - \frac{\tau^3}{3!}u'''(y) + \frac{\tau^4}{4!}u''''(\eta), \quad \eta \in (y, y+\tau)$$

yield different approximations of the first and second derivatives of function u e.g. forward difference scheme, backward difference scheme and centered difference scheme.

Typically, the first and second derivatives are approximated by the centered difference formula. This common choice is mostly due to its second order convergence in the error. That is

$$u' = \frac{du}{dy} = \frac{u_{i+1} - u_{i-1}}{2\tau} - \frac{\tau^2}{6}y'''(\eta_i), \quad \eta_i \in (y_{i-1}, y_{i+1}) \tag{18}$$

$$u'' = \frac{d^2u}{dy^2} = \frac{u_{i+1} - 2u_i + u_{i-1}}{\tau^2} - \frac{\tau^2}{6}y''''(\varepsilon_i), \quad \varepsilon_i \in (y_{i-1}, y_{i+1}) \tag{19}$$

where we have adopted the notation $u_i := u(y_i)$ and the index $i = 1, 2, .., N+1$. Finally, by omitting the error terms, the relations (18)-(19) become approximations to the first and second derivatives of u and are inserted into the equation (17) to obtain a differencing scheme

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\tau^2} - f\Big(y_i, u_i, \frac{u_{i+1} - u_{i-1}}{2\tau}\Big) = 0, \quad i = 1, 2, .., N \tag{20}$$

with $u_0 = \alpha,\ u_{N+1} = \beta$. Equation (20) is a system of $N+1$ nonlinear equations which can be solved, for example by Newton's method, to obtain the unknown solutions $u_i, i = 1, 2, .., N$. The nonlinear finite difference scheme (20) has been established to be second order convergent, see [9, p. 433].

Let us write (20) as $F(\tilde{u}) = 0$ where $\tilde{u} = (u_1, u_2, ..., u_N)^T$. With a good choice of the initial value $\tilde{u}_0$, Newton's method solves (20) iteratively by the scheme

$$\tilde{u}_{k+1} = \tilde{u}_k + J(\tilde{u}_k)^{-1}F(\tilde{u}_k) \tag{21}$$

where J represents the Jacobian of the nonlinear system (20). It is a well known fact that the Newton's iterative scheme above converges to the zero of the nonlinear function F as stated below.

Proposition 2.1. *Let the initial condition be chosen close enough to the true solution and assume that $J(\tilde{u}_0)$, the Jacobian of the system at $\tilde{u}_0$ is non-singular, then the approximate solutions obtained via the iterative formula* (21) *converge to the true solution of* (17).

For the proof of the above proposition, we refer the reader to [9, Section 3.2].

2.5 Application to the Brinkman-Forscheimer BVP

Comparing the Brinkman-Forscheimer model (2) with (17) gives

$$f(y, u, u') = Fsu^2 + s^2u - \frac{1}{M},$$

with $\alpha = \beta = 0,\ a = -1, b = 1$. Therefore, the corresponding difference scheme based on (20) is obtained as

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{\tau^2} - Fsu_i^2 - s^2u_i + \frac{1}{M} = 0, \quad i = 1, 2, ..., N \tag{22}$$

with $u(-1) = u_0 = 0, u(1) = u_{N+1} = 0$. This translates to the system

$$\begin{aligned}
2u_1 - u_2 + \tau^2 Fsu_1^2 + \tau^2 s^2 u_1 - \frac{\tau^2}{M} &= 0,\\
-u_3 + 2u_2 - u_1 + \tau^2 Fsu_2^2 + \tau^2 s^2 u_2 - \frac{\tau^2}{M} &= 0,\\
&\vdots\\
-u_N + 2u_{N-1} - u_{N+2} + \tau^2 Fsu_{N-1}^2 + \tau^2 s^2 u_{N-1} - \frac{\tau^2}{M} &= 0,\\
-u_{N+1} + 2u_N - u_{N-1} + \tau^2 Fsu_N^2 + \tau^2 s^2 u_N - \frac{\tau^2}{M} &= 0
\end{aligned}$$

whose Jacobian is given by

$$J(\tilde{u}) = \begin{pmatrix}
j_{u_1} & -1 & 0 & \cdots & \cdots & 0\\
-1 & j_{u_2} & -1 & 0 & \cdots & \vdots\\
 & & & & & \\
0 & & \ddots & \ddots & & 0\\
\vdots & & & & j_{u_{N-1}} & -1\\
0 & & \cdots & 0 & -1 & j_{u_N}
\end{pmatrix}$$

where

$$j_{u_k} := 2 + \tau^2(2Fsu_k + s^2).$$

In view of Proposition 2.1, it is obvious that the choice of the initial condition for (22) is important for the convergence or otherwise of the approximate solutions u_i's. In general for the BVP (17), we choose $\tilde{u}_0 = (u_{i,0}),\ i = 1, 2, .., N$ where $u_{i,0} = \alpha + i\big(\frac{\beta-\alpha}{b-a}\big)\tau$. This translates to $\tilde{u}_0 = (0, 0, ..., 0)$. It is then clear that the Jacobian for the above system of equations is non-singular at $\tilde{u}_0$.

3 Numerical Results and Discussion

Here, we shall report on the numerical implementation of the methods described in the previous section. For the purpose of comparison and as a reference solution, problem (2) is also solved using the Matlab package bvp4c which is a collocation method based on the implementation of the three-stage Lobatto formula. The method produces a continuously differentiable approximate solution that is fourth-order accurate.

The first step in solving (2) by bvp4c is writing (2) as a system of first-order differential equations which is achieved by setting $u(y) = p_1,\ \frac{du}{dy} = p_2$ so that

$$\begin{aligned}\frac{dp_1}{dy} &= p_2,\\ \frac{dp_2}{dy} &= s^2 p_1 + Fsp_1^2 - \frac{1}{M}\end{aligned}$$

subject to the boundary conditions $p_1(\pm 1) = 0$. The resulting system of first-order ordinary differential equation is then solved with Matlab.

Let us now report on our numerical implementations.

Table 1 shows the result of numerical approximations of $u(0)$ by the finite difference method (FDM) and the reference solution by the Matlab routine bvp4c. As can be seen in the table and in Figure 1, the solutions obtained by FDM compare well with those of the reference solutions. On the other hand, Table 2 shows comparison of the approximations of $u(0)$ by the FDM and the Adomian decomposition method (ADM). The table depicts the over-sensitivity of the ADM to the parameter F, s and M. As can be seen in the table, for $F > 1$, ADM iteration failed to converge and therefore, no solution

Table 1: Comparison of numerical solutions obtained by finite difference method (FDM) for $u(0)$ with those obtained through bvp4c for different values of the parameters F, s, M

F	M	s	Approximation of $u(0)$ by FDM	Approximation of $u(0)$ by bvp4c
1	1	1	0.32380	0.32383
1.2	1	1	0.31911	0.31915
1.8	1	1	0.30640	0.30642
2.0	1	1	0.30255	0.30258
1	2	1	0.16837	0.16841
1	3	1	0.11384	0.11389
1	4	1	0.08600	0.08606
1	1	2	0.17438	0.17445
1	1	3	0.09754	0.09759
1	1	4	0.05941	0.05941

was returned. Furthermore, for $s >> 1$ ADM returned a completely misleading results. However, FDM always return fairly accurate solutions for various choices of the parameters as seen in Table 1. The solution profile for the case case $F = 2, M = 1$ and $s = 1$ is shown in Figure 2.

Therefore, in view of the failed implementation of the variational iteration method for (2) and oversensitivity of the preferred semi-analytical method Adomian decomposition method, it appears that finite difference based methods are method of choice for this nonlinear problem.

Table 2: The obtained approximations to $u(0)$ as computed by finite difference method, compared with solutions obtained through Adomian Decomposition method (ADM).

F	M	s	Approximation of $u(0)$ by FDM	Approximation of $u(0)$ by ADM [10]
1	1	1	0.32380	0.32385
1.2	1	1	0.31911	–
1.8	1	1	0.30640	–
2.0	1	1	0.30255	–
1	2	1	0.16837	0.16840
1	3	1	0.11384	0.11385
1	4	1	0.08600	0.08601
1	1	2	0.17438	0.17443
1	1	3	0.09754	−5.00495
1	1	4	0.05941	−4.87769

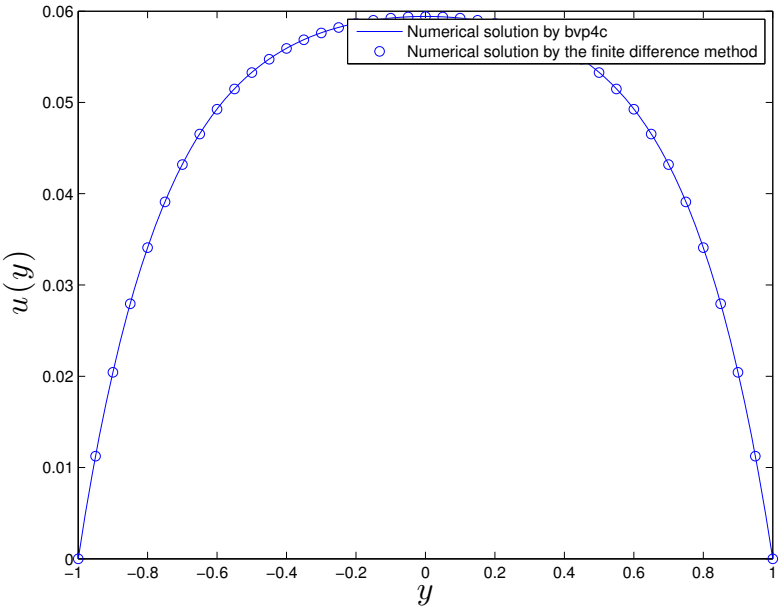

Figure 1: Velocity profile for the case $F = 1,\ M = 1,\ s = 4$

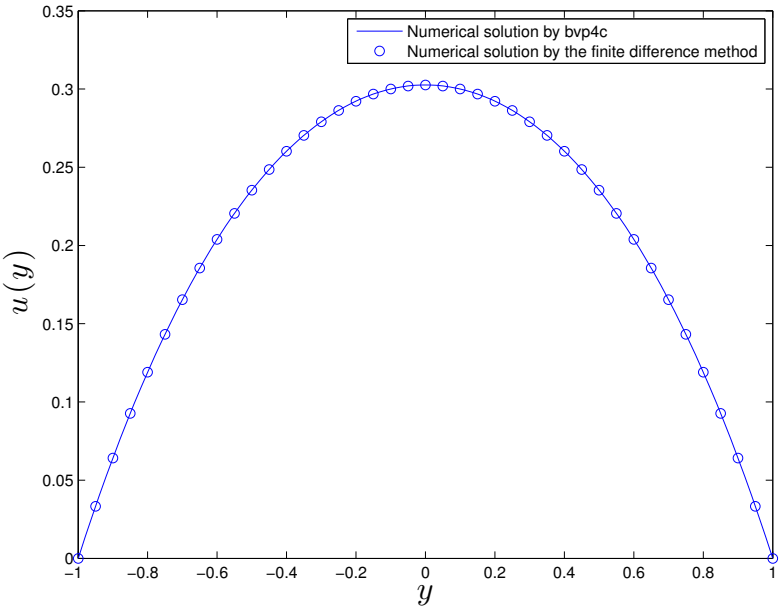

Figure 2: Velocity profile for the case $F = 2,\ M = 1,\ s = 1$

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