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An Alternative Mathematical Interpretation and Generalization of the Capital Growth Criterion

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Abstract

The Law of Large Numbers provides one basis for the capital growth criterion, which maximizes the expected value of log return per period. In this paper, an alternative derivation of this criterion is presented based on the first-order asymptotic expansion of the return per period. A second-order expansion gives rise to a more general growth criterion that approximates optimal compound growth for a finite number of periods.

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1 Introduction

> ”For him [Keynes] the short run was much more significant than the long run—that long run in which, as he used to say, ’we are all dead’.” A. C. Pigou, *Proc. Br. Acad.*, vol. 32, p. 13, 1946.

The capital growth criterion (CGC) of portfolio selection has been developed as the optimal long-term strategy for maximizing the cumulative return of a large number of sequential investments, as discussed in [1]–[14]. Briefly stated, the overall portfolio investment return (1 + yield) for any given investment period is written as

$$S = \mathbf{b}^T \mathbf{X} \tag{1}$$

where $\mathbf{X} \overset{\text{def}}{=} [\ X_1 \quad X_2 \quad \cdots \quad X_m\ ]^T$ is a vector of individual returns comprising the portfolio, and the vector $\mathbf{b} \overset{\text{def}}{=} [\ b_1 \quad b_2 \quad \cdots \quad b_m\ ]^T$ is the is the allocation of resources with the constraint

$$\sum_{i=1}^{m} b_i = 1. \tag{2}$$

The CGC then states that for repeated investments of the type (1), the resources are optimally allocated so as to maximize the expected value of $\log S$. In [3], [6], [7], [8], [9], [12], and [13], it has been argued that this criterion has both long term and short term optimality from many different perspectives. In this paper, the optimal asymptotic exponential growth property that results from this criterion will be examined. A recent approach with intuitive appeal uses the Law of Large Numbers to arrive at this result, e.g., [11] and [14]. Here, an alternative derivation of the CGC will be derived by appealing to a simple asymptotic expansion of the return per period, thus developing this result from still another viewpoint. Taking this expansion to second-order then develops a new more general growth criterion that optimizes compound growth for a finite number of periods.

2 Law of Large Numbers

If the proceeds from each investment period are reinvested, then the compounded return after n periods is

$$S_n = \prod_{i=1}^{n} \mathbf{b}^T \mathbf{X}_i \tag{3}$$

where $\mathbf{X}_i$ is the return vector of the ith period. It is assumed here that the $\mathbf{X}_i$'s are all independent and identically distributed. (The more general case is considered in [13].) The equivalent return per period is given by

$$R_n \stackrel{\text{def}}{=} S_n^{1/n} = \prod_{i=1}^{n} (\mathbf{b}^T \mathbf{X}_i)^{1/n}. \tag{4}$$

One motivation for the CGC is gleaned by writing the return per period as in [11] and [14]:

$$R_n = \exp\left[\frac{1}{n}\sum_{i=1}^{n} \ln(\mathbf{b}^T \mathbf{X}_i)\right] \tag{5}$$

where ln denotes natural log. Then, as noted in [3], [13], the Law of Large Numbers implies that

$$\frac{1}{n}\sum_{i=1}^{n} \ln(\mathbf{b}^T \mathbf{X}_i) \xrightarrow[n \to \infty]{} \Psi_\infty \stackrel{\text{def}}{=} E\{\ln(\mathbf{b}^T \mathbf{X})\} \tag{6}$$

where convergence is interpreted in the sense of probability. Thus, the asymptotic CGC maximizes $\Psi_\infty \stackrel{\text{def}}{=} E\{\ln(\mathbf{b}^T \mathbf{X})\}$ in order to maximize R_n for large n.

If n is not large, the more difficult problem of maximizing the mean return per period

$$\bar{R}_n \stackrel{\text{def}}{=} E\{R_n\} = [E\{(\mathbf{b}^T \mathbf{X})^{1/n}\}]^n \tag{7}$$

must be considered. This is the more realistic problem that one often faces when investing over a finite lifetime.

3 Asymptotic Expansion of Return Per Period

3.1 First Order

For any real number a,

$$a^{1/n} = 1 + \frac{\ln a}{n} + O\left(\frac{1}{n^2}\right) \tag{8}$$

and combining with (4) gives

$$R_n = \prod_{i=1}^{n} \left[1 + \frac{\ln(\mathbf{b}^T \mathbf{X}_i)}{n} + O\left(\frac{1}{n^2}\right) \right]. \tag{9}$$

Now, if the $\mathbf{X}_i$'s are independent stationary vectors, then the asymptotic expected return per period is given by

$$\bar{R}_n = [1 + \Psi_\infty/n + O(1/n^2)]^n \xrightarrow[n \to \infty]{} \exp(\Psi_\infty) \tag{10}$$

where $\Psi_\infty \stackrel{\text{def}}{=} E\{\ln(\mathbf{b}^T \mathbf{X})\}$ is the CGC objective function. Thus, the exponential growth property of the CGC follows directly from the first-order asymptotic expansion of the return per period. In the following, we explore the logical extension of this approach in order to generalize the CGC for a finite number of investment periods.

3.2 Second Order

The second-order version of the expansion (8) is given by

$$a^{1/n} = 1 + \frac{\ln a}{n} + \frac{(\ln a)^2}{2n^2} + O\left(\frac{1}{n^3}\right). \tag{11}$$

It is natural to formulate a more general capital growth criterion by once again combining with (4) and taking the expected value, resulting in the expression

$$\bar{R}_n = [1 + \Psi_\infty/n + E\{[\ln(\mathbf{b}^T \mathbf{X})]^2\}/(2n^2) + O(1/n^3)]^n. \tag{12}$$

Thus, the generalized CGC for a given number of periods n is to maximize the objective function

$$\begin{aligned} \Psi_n &\stackrel{\text{def}}{=} \Psi_\infty + E\{[\ln(\mathbf{b}^T \mathbf{X})]^2\}/(2n) \\ &= E\{\ln(\mathbf{b}^T \mathbf{X})\} + E\{[\ln(\mathbf{b}^T \mathbf{X})]^2\}/(2n). \end{aligned} \tag{13}$$

In the limit, $\Psi_n \xrightarrow[n \to \infty]{} \Psi_\infty$, and so the generalized CGC is consistent with the asymptotic CGC.

4 An Example

In order to illustrate the above results, the simple two-stock example of [10], [11] will be considered, whereby $X_1 = 1$ represents a constant stock (cash) and X_2 is a random variable that can take on the value 2 or 1/2 (double or half) with equal probability of 1/2. In this case, the portfolio allocation vector is written as $\mathbf{b} = (1-b, b)^T$, where the variable b is the fraction of capital risked on the volatile stock X_2. Applying the CGC shows that for this example,

$$\Psi_\infty = E\{\ln(\mathbf{b}^T\mathbf{X})\} = [\ln(1+b) + \ln(1-b/2)]/2 \tag{14}$$

is maximized for $b = 0.5$, i.e., risk one half of your capital at each investment period, and this will be the optimal policy for a large number of periods $n \to \infty$. For this example, it is easy to calculate from (7) the average equivalent return per period for an arbitrary number of investment periods, giving

$$\bar{R}_n = [(1+b)^{1/n}/2 + (1-b/2)^{1/n}/2]^n \xrightarrow[n\to\infty]{} \sqrt{9/8 - (b-1/2)^2/2}. \tag{15}$$

A plot of the average yield per period ($\bar{R}_n - 1$) as a function of the fraction of capital risked b is shown in Figure 1 for several values of n. For any value of n, $\bar{R}_n$ is maximized for $b = b_n^*$ which is easy to calculate for this simple example. As can be seen, for large n, the maximum average yield is attained for $b = b_\infty^* = 0.5$, in accordance with the CGC. However, for moderate values of n, the peak shifts to the right; in the extreme case for $n = 1$, one should risk everything on X_2.

Now, the generalized CGC (13) will be applied, which for this example maximizes

$$\Psi_n = \Psi_\infty + \{[\ln(1+b)]^2 + [\ln(1-b/2)]^2)\}/(4n) \tag{16}$$

where Ψ_∞ is given by (14). Maximizing (16) over b then results in an estimate of the optimal value b_n^*, which will be denoted as $\hat{b}_n^*$. Table 1 lists the estimated values $\hat{b}_n^*$ in the third column for various values of n in the first column.

For comparison, the exact values b_n^* determined by maximization of (15) are included in the second column. As can be seen, the agreement is quite close for this example, with a maximum deviation of less than 0.3% over all values of n. This example shows that the generalized CGC can well approximate optimal return per period, whereas the conventional CGC does not capture the dependency on the number of periods n.

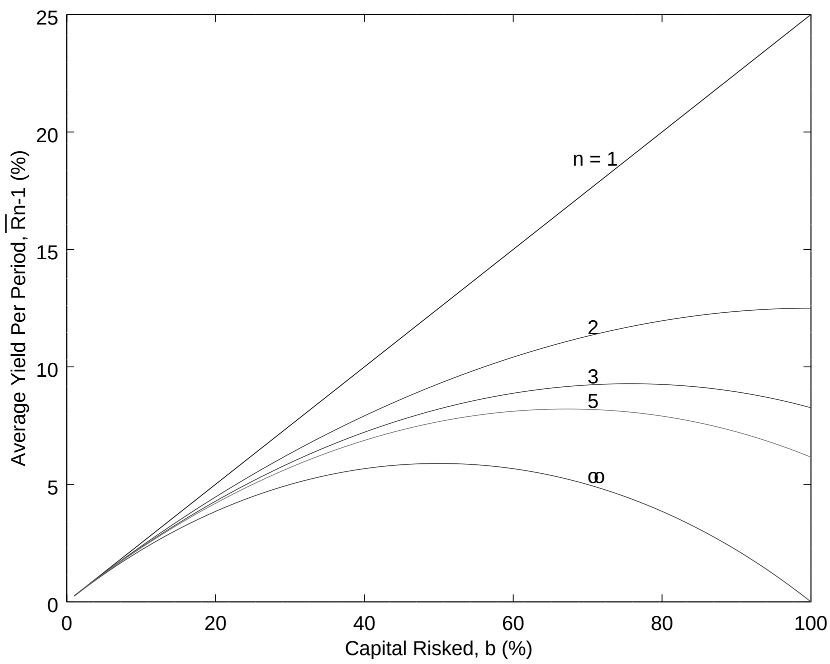

Figure 1: Average percent yield per investment period $(\bar{R}_n - 1)$ as a function of percentage capital risked b for example in text.

Table 1: Optimal Fraction of Capital Risked b_n^* and Approximation $\hat{b}_n^*$ Determined From Generalized Capital Growth Criterion for Example in Text, as a Function of Number of Investment Periods n.

n	b_n^*	$\hat{b}_n^*$
1	1.0000	1.0000
2	1.0000	1.0000
3	0.7574	0.7555
5	0.6296	0.6283
10	0.5577	0.5574
20	0.5274	0.5273
50	0.5106	0.5106
100	0.5053	0.5052
∞	0.5000	0.5000

5 Conclusions

The capital growth criterion can be developed from a number of viewpoints. In this paper, still another perspective on the problem is developed by using an asymptotic expansion of the return per period. This approach also results in a natural extension of the CGC for a finite number of investment periods n. A simple example showed that applying this generalized CGC results in a very close approximation to the optimal portfolio allocation for all n.

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