

# Numerical solutions of integro-differential equation with purely integral condition by using Laplace transform method

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## Abstract

The aims of this paper are to prove the existence, uniqueness and continuous dependence upon the data of solution of following integro-differential hyperbolic equation with purely integral conditions.

$$\frac{\partial^2 u}{\partial t^2} - a \frac{\partial^2 u}{\partial x^2} - c \frac{\partial u}{\partial x} + \alpha u(x, t) = \int_0^t \vartheta(t-s)u(x, s) ds, \quad 0 < x < \ell, \quad 0 < t \leq T,$$

$$u(x, 0) = \phi(x), \quad 0 < x < \ell,$$

$$\frac{\partial u}{\partial t}(x, 0) = \chi(x), \quad 0 < x < \ell,$$

$$\int_0^\ell u(x, t) dx = E(t), \quad 0 < t \leq T,$$

$$\int_0^\ell xu(x, t) dx = G(t), \quad 0 < t \leq T.$$

The proofs are based on a priori estimates and Laplace transform method. We present a numerical approximate solution to integro-differential equation with integral conditions. A Laplace transform method is described

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for the solution of considered equation. Following Laplace transform of the original problem, an appropriate method of solving differential equations is used to solve the result at time-independent modified equation and solution is inverted numerically back into the time domain. Numerical results are provided to show the accuracy of the proposed method.

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## 1 Introduction

In this paper, we deal with a class of hyperbolic integro-differential equation with purely nonlocal conditions. The precise statement of the problem is as follows : let  $\ell > 0$ ,  $T > 0$ , and  $\Omega = \{(x, t) \in \mathbb{R}^2 : 0 < x < \ell, 0 < t < T\}$ . We shall determine a solution  $u$ , in  $\Omega$  of the differential equation

$$\frac{\partial^2 u}{\partial t^2} - a \frac{\partial^2 u}{\partial x^2} - c \frac{\partial u}{\partial x} + \alpha u(x, t) = \int_0^t \vartheta(t-s)u(x, s) ds, \quad 0 < x < \ell, \quad 0 < t \leq T \quad (1)$$

satisfying the initial conditions

$$u(x, 0) = \phi(x), \quad 0 < x < \ell, \quad (2)$$

$$\frac{\partial u}{\partial t}(x, 0) = \chi(x), \quad 0 < x < \ell, \quad (3)$$

and the integral conditions

$$\int_0^\ell u(x, t) dx = E(t), \quad 0 < t \leq T, \quad (4)$$

$$\int_0^\ell xu(x, t) dx = G(t), \quad 0 < t \leq T, \quad (5)$$

where  $\vartheta$ ,  $\xi$ ,  $\phi$ ,  $E$ ,  $G$  are known functions,  $\ell$ ,  $T$  and  $a$  are positive constants, and  $\alpha, c$  are the reals.

**Assumption 1.1.** For all  $x, t \in \overline{\Omega}$ , we assume that

$$\vartheta_0 \leq \vartheta(x, t) \leq \vartheta_1, \quad \vartheta_0 > 0, \quad \vartheta_1 > 0$$

$\phi$  is continuously derivate on  $(0, \ell)$ .

The notion of nonlocal condition has been introduced to extend the study of the classical initial value problems and it is more precise for describing nature phenomena than the classical condition since more information is taken into account, thereby decreasing the negative effects incurred by a possibly erroneous single measurement taken at the initial value. The inportance of nonlocal conditions in many applications is discussed in [10], [16].

Mathematical modelling by evolution problems with a nonlocal constraint of the form

$$\frac{1}{1 - \ell} \int_{\ell}^1 u(x, t) dx = \zeta(t)$$

is encountered in heat transmission theory, thermoelasticity, chemical engineering, underground water flow, and plasma physic.

Many methods were used to investigate the existence and uniqueness of the solution of mixed problems which combine classical and integral conditions. J. R. Cannon [5] used the potentiel method, combining a Dirichlet and an intégral condition for a parabolic equation. L.A. Mouravey and V. Philinovoski [9] used the maximum principle, combining a Neumann and an integral condition for heat equation. M.Z. Djibibe and K. Tcharie [13], Ionkin [6] and L. Bougoffa [4] used the Fourier method for same purpose. Recently, mixed problems with integral conditions for generalization of equation (1) have been treated using the energy-integral method. See M.Z. Djibibe and K. Tcharie [11] , M.Z. Djibibe and K. Tcharie [12], M.Z. Djibibe el al. [14], [15], N.I. Yurchuk [16], [17], M. Mesloub, A. Bouziani and N. Kechkar [8]. Differently to these works, in the present paper we combine a priori estimate and Fourier's method to prove existence and uniqueness solution of the problem (1)- (5).

The results obtained in this paper generalize the results of [1], and constitute a new contribution to this emerging field of research.

It can be a part in the contribution of the development of a priori estimates an Laplace methods for solving such problems.

The questions related to these problems are so miscellaneous that the elaboration of a general theory is still premature. Therefore, the investigation of these problems requires at every time a separate study.

The remainder of the paper is organized as follows. After this introduction, in section 2, we present some preliminaries and basic lemmas. Then in Section 3, we establish a priori estimate. Finally, in section 4, we prove existence solution.

## 2 Preliminaries

We transform the problem (1)-(5) with nonhomogeneous boundary conditions (4) and (5) into a problem with homogeneous boundary conditions. For this, we introduce a new unknown function defined by  $u(x, y) = v(x, t) + w(x, t)$ , where

$$w(x, t) = \frac{2\ell - 4x}{\ell^4} E(t) + \frac{4x - 2\ell}{\ell^4} G(t) = \frac{4x - 2\ell}{\ell^4} (G(t) - E(t)).$$

Then, problem becomes :

$$\frac{\partial^2 v}{\partial t^2} - a \frac{\partial^2 v}{\partial x^2} - c \frac{\partial v}{\partial x} + \alpha v(x, t) = f(x, t), \quad 0 < x < \ell, \quad 0 < t \leq T, \quad (6)$$

$$v(x, 0) = \varphi(x), \quad 0 < x < \ell, \quad (7)$$

$$\frac{\partial v}{\partial t}(x, 0) = \psi(x), \quad 0 < x < \ell, \quad (8)$$

$$\int_0^\ell v(x, t) dx = 0, \quad 0 < t \leq T, \quad (9)$$

$$\int_0^\ell xv(x, t) dx = 0, \quad 0 < t \leq T, \quad (10)$$

where

$$\varphi(x) = \phi(x) - \frac{4x - 2\ell}{\ell^4} (G(0) - E(0)),$$

$$\psi(x) = \chi(x) - \frac{4x - 2\ell}{\ell^4} (G'(0) - E'(0)),$$

$$f(x, t) = \int_0^t \vartheta(t-s)u(x, s) ds - \frac{2(2x - \ell)}{\ell^4} (G''(t) - E''(t)) \\ - \frac{2(2\alpha x - \alpha\ell - 2c)}{\ell^4} (G(t) - E(t)).$$

Instead of searching for the function  $u$ , we search for the function  $v$ . So the solution of problem (1), (2), (3), (4) and (5) will be given by  $u(x, t) = v(x, t) + w(x, t)$ .

**Lemma 2.1 (Gronwall).** *Let  $x(t) \geq 0$ ,  $h(t)$ ,  $y(t)$  the integrables functions on  $[a; b]$ . If*

$$y(t) \leq h(t) + \int_a^t x(\tau)y(\tau) d\tau, \quad \forall t \in [a; b]$$

then

$$y(t) \leq h(t) + \int_a^t h(\tau)x(\tau) \exp\left(\int_\tau^t x(s) ds\right) d\tau, \quad \forall t \in [a; b].$$

In particular, if  $x(t) \equiv c$  is invariable function and  $h(t)$  is an increasing function, then

$$y(t) \leq h(t)e^{c(t-a)}, \forall t \in [a; b].$$

## 3 Main Results

### 3.1 Uniqueness and Continuous Dependence of the solution

**Theorem 3.1.** *If  $v(x, t)$  is a solution of problem (1), (2), (3), (4) and (5), then we have a priori estimates*

$$\sup_{0 \leq t \leq T} \int_0^\ell \left( J_x^2 \left( \frac{\partial u}{\partial t} \right) + J_x^2 u dx + u^2 \right) dx \leq A \left( \int_0^\ell \varphi^2(x) dx + \int_0^\ell J_x^2 \psi(x) dx + \int_0^\ell J_x^2 \varphi(x) dx \right). \quad (11)$$

*Proof.* Appying  $J_x$  to (1), Multiplying with  $J_x \frac{\partial u}{\partial t}$  and integrating the results obtained over  $\Omega_\tau = (0, \ell) \times (0, \tau)$ . Observe that

$$\begin{aligned} & \int_0^\ell \int_0^\tau J_x \left( \frac{\partial^2 v}{\partial t^2} \right) J_x \left( \frac{\partial v}{\partial t} \right) dt dx - a \int_0^\ell \int_0^\tau J_x \left( \frac{\partial^2 v}{\partial x^2} \right) J_x \left( \frac{\partial v}{\partial t} \right) dt dx \\ & - c \int_0^\ell \int_0^\tau J_x \left( \frac{\partial v}{\partial x} \right) J_x \left( \frac{\partial v}{\partial t} \right) dt dx + \alpha \int_0^\ell \int_0^\tau J_x v(x, t) J_x \left( \frac{\partial v}{\partial t} \right) dt dx \\ & = \int_0^\ell \int_0^\tau J_x \left( \int_0^t \vartheta(t-s)u(x, s) ds \right) J_x \left( \frac{\partial v}{\partial t} \right) dt dx, \quad (12) \end{aligned}$$

where  $J_x(u(x, t)) = \int_0^x u(\xi, t) d\xi$ . Successive integration by parts of integrals on the left-hand of (12) are straight-forward but somewhat tedious. We give only their results

$$\begin{aligned}
& \frac{1}{2} \int_0^\ell J_x^2 \left( \frac{\partial u(x, \tau)}{\partial t} \right) dx - \frac{1}{2} \int_0^\ell J_x^2 \psi(x) dx + \frac{\alpha}{2} \int_0^\ell J_x^2 u dx - \frac{\alpha}{2} \int_0^\ell J_x^2 \varphi(x) dx \\
& + \frac{a}{2} \int_0^\ell u^2 dx - \frac{a}{2} \int_0^\ell \varphi^2(x) dx = a \int_{\Omega_\tau} u(\ell, t) \frac{\partial u}{\partial t} dx dt \\
& + c \int_{\Omega_\tau} u J_x \left( \frac{\partial u}{\partial t} \right) dx dt - \int_{\Omega_\tau} \left( a \frac{\partial u}{\partial x}(0, t) + u(0, t) \right) J_x \left( \frac{\partial u}{\partial t} \right) dx dt \\
& + \int_0^\ell \int_0^\tau J_x \left( \int_0^t \vartheta(t-s) u(x, s) ds \right) J_x \left( \frac{\partial v}{\partial t} \right) dt dx. \tag{13}
\end{aligned}$$

By the Cauchy inequality,

$$\begin{aligned}
\int_0^\ell \int_0^\tau J_x \left( \int_0^t \vartheta(t-s) u(x, s) ds \right) J_x \left( \frac{\partial v}{\partial t} \right) dt dx &\leq \frac{T\vartheta_0}{2} \int_{\Omega_T} J_x^2 u dx dt \\
&+ \frac{\vartheta_0 T}{2} \int_0^T \int_0^\ell J_x^2 \left( \frac{\partial u}{\partial t} \right) dx d\tau, \tag{14}
\end{aligned}$$

$$\int_{\Omega_\tau} u(\ell, t) \frac{\partial u}{\partial t} dx dt \leq \frac{1}{2} \int_{\Omega_T} u^2 dx dt + \frac{1}{2} \int_{\Omega_T} J_x^2 \left( \frac{\partial u}{\partial t} \right) dx dt, \tag{15}$$

$$\int_{\Omega_\tau} u J_x \left( \frac{\partial u}{\partial t} \right) dx dt \leq \frac{1}{2} \int_{\Omega_T} u^2 dx dt + \frac{1}{2} \int_{\Omega_T} J_x^2 \left( \frac{\partial u}{\partial t} \right) dx dt. \tag{16}$$

Substituting (16) into (15), yields

$$\begin{aligned}
& \frac{1}{2} \int_0^\ell J_x^2 \left( \frac{\partial u(x, \tau)}{\partial t} \right) dx + \frac{\alpha}{2} \int_0^\ell J_x^2 u dx + \frac{a}{2} \int_0^\ell u^2 dx \leq \frac{a}{2} \int_0^\ell \varphi^2(x) dx \\
& + \frac{1}{2} \int_0^\ell J_x^2 \psi(x) dx + \frac{\alpha}{2} \int_0^\ell J_x^2 \varphi(x) dx + \frac{|a| + T\vartheta_0 + |c|}{2} \int_{\Omega_T} u^2 dx dt \\
& + \frac{\vartheta_0 T + |c| + 1}{2} \int_{\Omega_T} J_x^2 \left( \frac{\partial u}{\partial t} \right) dx d\tau + \frac{T\vartheta_0}{2} \int_{\Omega_T} J_x^2 u dx dt. \tag{17}
\end{aligned}$$

The right-hand side of (17) is independent of  $\tau$ , hence, replacing the left-hand side by the upper bound with respect to  $\tau$  and by Gronwall Lemma, we get

$$\begin{aligned}
\sup_{0 \leq \tau \leq T} \int_0^\ell \left( J_x^2 \left( \frac{\partial u(x, \tau)}{\partial t} \right) + J_x^2 u dx + u^2 \right) dx &\leq e^{\lambda T} \left( \int_0^\ell \varphi^2(x) dx \right. \\
& \left. + \int_0^\ell J_x^2 \psi(x) dx + \int_0^\ell J_x^2 \varphi(x) dx \right), \tag{18}
\end{aligned}$$

where

$$\lambda = \frac{\max\left(\frac{1}{2}, \frac{\alpha}{2}, \frac{T\vartheta_0 + |c| + 1}{2}, \frac{T\vartheta_0 + |a| + |c|}{2}\right)}{\min\left(\frac{1}{2}, \frac{\alpha}{2}, \frac{a}{2}\right)}.$$

From (18), we obtain the priori estimates (11). This complete the proof of Theoreme (3.1), with  $A = e^{\lambda T}$ .  $\square$

**Corollary 3.2.** *If problem (1), (2), (3), (4) and (5) has a solution, then this solution is unique and depends continuously on  $(\varphi, \psi)$ .*

### 3.2 Existence of the Solution

Laplace transform is widely used in the area of engineering technology and mathematical science. There are many problems whose solution may be found in terms of the Laplace. In fact, it is an efficient method for solving many diffential equations and partial differential equation. The mains difficult of the method of Laplace domain into the real domain.

Hence in this section, we apply the technique of the Laplace transform to find solutions of the problem (1)-(5).

Suppose that  $v(x, t)$  is défined and is of the exponential order for  $t \geq 0$ , there exists  $\lambda > 0$ ,  $\beta > 0$  and  $t_0 > 0$  such that  $|v(x, t)| \leq \lambda e^{\beta t}$  for  $t \geq t_0$ . Then the Laplace transform  $V(x, s)$  includind the function  $v(x, t)$  is defined by

$$V(x, s) = \int_0^{+\infty} v(x, t) e^{-st} dt, \quad (19)$$

where  $s$  is know as a Laplace variable and  $V$  is a function in the Laplace domain.

Applying the Laplace transform on both sides of (14), we obtain

$$\begin{aligned} \int_0^{+\infty} \frac{\partial^2 v}{\partial t^2} e^{-st} dt - a \int_0^{+\infty} \frac{\partial^2 v}{\partial x^2} e^{-st} dt - c \int_0^{+\infty} \frac{\partial v}{\partial x} e^{-st} dt + \\ + \alpha \int_0^{+\infty} v(x, t) e^{-st} dt = \int_0^{+\infty} \int_0^t \vartheta(t-s) v(x, s) ds dt. \end{aligned} \quad (20)$$

The standard integration by parts of the terms on the left-hand of (20), leads

$$\int_0^{+\infty} \frac{\partial^2 v}{\partial t^2} e^{-st} dt = -\psi(x) - s\varphi(x) + s^2 V(x, s), \quad (21)$$

$$\int_0^{+\infty} \frac{\partial^2 v}{\partial x^2} e^{-st} dt = \frac{d^2 V(x, s)}{dx^2}, \quad (22)$$

$$\int_0^{+\infty} \frac{\partial v}{\partial x} e^{-st} dt = \frac{dV(x, s)}{dx}. \quad (23)$$

Substituting (21), (22) and (23) into (20), we get

$$-a \frac{d^2 V(x, s)}{dx^2} - c \frac{dV(x, s)}{dx} + (\alpha + s^2)V(x, s) = f(x, s) + \psi(x) + s\varphi(x), \quad (24)$$

where  $f(x, s) = \int_0^{+\infty} \int_0^t \vartheta(t-s)v(x, s) ds dt$ .

Similarly, we have

$$\int_0^\ell V(x, s) dx = 0, \quad (25)$$

$$\int_0^\ell xV(x, s) dx = 0. \quad (26)$$

Now, we distinguish the following cas :

► Case 1 : If  $c^2 + 4a\alpha + 4as^2 = 0$ .

► Case 2 : If  $c^2 + 4a\alpha + 4as^2 > 0$ .

► Case 3 : If  $c^2 + 4a\alpha + 4as^2 < 0$ .

In this article, we only deal Case 1 and Case 2

For Case 1, that is  $c^2 + 4a\alpha + 4as^2 = 0$ , the solution general of (26) is given by

$$V(x, s) = (C_1(s)x + C_2(s))e^{-\frac{c}{2a}x} - \frac{1}{a} \int_0^x (x-\tau)(f(\tau, s) + \psi(\tau) + s\varphi(\tau))e^{\frac{c}{2a}\tau} d\tau. \quad (27)$$



Putting the intagral conditions (25) and (26) in (27), we get

$$\begin{aligned} & \left( 2 - \left( 1 + \frac{c\ell}{2a} e^{-\frac{c\ell}{2a}} \right) \right) C_1(s) + \frac{c}{2a} (1 - e^{-\frac{c}{2a}}) C_2(s) \\ &= 2 \int_0^\ell \int_0^x (x-y)(f(\tau, s) + \psi(\tau) + s\varphi(\tau)) e^{\frac{c}{2a}\tau} d\tau dx, \end{aligned}$$

$$\begin{aligned} & 4a^2 + (c^2\ell^2 - 4acl - 4a^1)C_1(s) + (2a - (2a + \ell c))C_2(s) \\ &= 4a^2 \int_0^\ell \int_0^x x(x-\tau)(f(\tau, s) + \psi(\tau) + s\varphi(\tau)) e^{\frac{c}{2a}\tau} d\tau dx. \end{aligned}$$

For Case 1, that is  $c^2 + 4a\alpha + 4as^2 > 0$ , Using the method of variation of parameter, to solve (26), we have the general solution as

$$\begin{aligned} V(x, s) &= C_1(s) \exp\left(-\frac{c - \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) \\ &+ C_2(s) \exp\left(-\frac{c + \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) - \frac{2}{\sqrt{c^2 + 4a(\alpha + s^2)}} \\ &\times \int_0^x (f(\tau, s) + \psi(\tau) + s\varphi(\tau)) e^{-\frac{c(x-\tau)}{2a}} \sinh\left(\frac{\sqrt{c^2 + 4a(\alpha + s^2)}}{2a}(x-\tau)\right) d\tau, \end{aligned} \quad (28)$$

where  $C_1$  et  $C_2$  are arbitrary functions of  $s$ .

Substuting (28) into (25) and (26), we have

$$\begin{aligned} & C_1(s) \int_0^\ell \exp\left(-\frac{c - \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx \\ &+ C_2(s) \int_0^\ell \exp\left(-\frac{c + \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx = \frac{2}{\sqrt{c^2 + 4a(\alpha + s^2)}} \\ &\times \int_0^\ell \int_0^x (f(\tau, s) + \psi(\tau) + s\varphi(\tau)) e^{-\frac{c(x-\tau)}{2a}} \sinh\left(\frac{\sqrt{c^2 + 4a(\alpha + s^2)}}{2a}(x-\tau)\right) d\tau dx, \end{aligned} \quad (29)$$

$$\begin{aligned}
& C_1(s) \int_0^\ell x \exp\left(-\frac{c - \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx \\
& + C_2(s) \int_0^\ell x \exp\left(-\frac{c + \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx = \frac{2}{\sqrt{c^2 + 4a(\alpha + s^2)}} \\
& \times \int_0^\ell \int_0^x x (f(\tau, s) + \psi(\tau) + s\varphi(\tau)) e^{-\frac{c(x-\tau)}{2a}} \sinh\left(\frac{\sqrt{c^2 + 4a(\alpha + s^2)}}{2a}(x - \tau)\right) d\tau dx,
\end{aligned} \tag{30}$$

where

$$\begin{pmatrix} C_1(s) \\ C_2(s) \end{pmatrix} = \begin{pmatrix} a_{11}(s) & a_{12}(s) \\ a_{21}(s) & a_{22}(s) \end{pmatrix}^{-1} \times \begin{pmatrix} b_1(s) \\ b_2(s) \end{pmatrix},$$

and

$$a_{11}(s) = \int_0^\ell \exp\left(-\frac{c - \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx, \tag{31}$$

$$a_{12}(s) = \int_0^\ell \exp\left(-\frac{c + \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx, \tag{32}$$

$$a_{21}(s) = \int_0^\ell x \exp\left(-\frac{c - \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx, \tag{33}$$

$$a_{22}(s) = \int_0^\ell x \exp\left(-\frac{c + \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx, \tag{34}$$

$$b_1(s) = -\frac{a}{\sqrt{c^2 + 4a(\alpha + s^2)}} \int_0^\ell \int_0^x (f(\tau, s) + \psi(\tau) + s\varphi(\tau)) e^{-\frac{c(x-\tau)}{2a}} \tag{35}$$

$$\times \sinh\left(\frac{\sqrt{c^2 + 4a(\alpha + s^2)}}{2a}(x - \tau)\right) d\tau dx, \tag{36}$$

$$b_2(s) = -\frac{a}{\sqrt{c^2 + 4a(\alpha + s^2)}} \int_0^\ell \int_0^x x (f(\tau, s) + \psi(\tau) + s\varphi(\tau)) e^{-\frac{c(x-\tau)}{2a}} \tag{37}$$

$$\times \sinh\left(\frac{\sqrt{c^2 + 4a(\alpha + s^2)}}{2a}(x - \tau)\right) d\tau dx. \tag{38}$$

If it is not possible to calculate the integral directly, then we calculate them numerically. If the Laplace inversion is possibly computed directly for (28) and

(38), we obtain our solution explicitly. Otherwise, we use the suitable approximate method, then we use the numerical inversion of the Laplace transform. We have

$$\begin{aligned}
a_{11}(s) &= \int_0^\ell \exp\left(-\frac{c - \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx \\
&= \frac{1}{2} \sum_{i=1}^n k_i \exp\left(-\frac{c - \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}(x_i + 1)\right), \\
a_{12}(s) &= \int_0^\ell \exp\left(-\frac{c + \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx \\
&= \frac{1}{2} \sum_{i=1}^n k_i \exp\left(-\frac{c + \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}(x_i + 1)\right), \\
a_{21}(s) &= \int_0^\ell x \exp\left(-\frac{c - \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx \tag{39} \\
&= \frac{1}{4} \sum_{i=1}^n k_i (x_i + 1) \exp\left(-\frac{c - \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}(x_i + 1)\right), \\
a_{22}(s) &= \int_0^\ell x \exp\left(-\frac{c + \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}x\right) dx \\
&= \frac{1}{4} \sum_{i=1}^n k_i (x_i + 1) \exp\left(-\frac{c + \sqrt{c^2 + 4a(\alpha + s^2)}}{2a}(x_i + 1)\right), \tag{40}
\end{aligned}$$

$$\begin{aligned}
b_1(s) &= -\frac{a}{\sqrt{c^2 + 4a(\alpha + s^2)}} \int_0^\ell \int_0^x (f(\tau, s) + \psi(\tau) + s\varphi(\tau)) e^{-\frac{c(x-\tau)}{2a}} \\
&\quad \times \sinh\left(\frac{\sqrt{c^2 + 4a(\alpha + s^2)}}{2a}(x - \tau)\right) d\tau dx
\end{aligned}$$

$$\begin{aligned}
&= -\frac{a}{4\sqrt{c^2+4a(\alpha+s^2)}} \sum_{i=1}^n k_i \left[ f\left(\frac{1}{2}(x_i+1), s\right) + \psi\left(\frac{1}{2}(x_i+1)\right) \right. \\
&\quad + \left(1 - \frac{1}{2}(x_i+1)\right) \exp\left(-\frac{c}{2a} \left[\left(1 - \frac{1}{2}(x_i+1)\right) x_j \right. \right. \\
&\quad \left. \left. + \left(1 + \frac{1}{2}(x_i+1)\right)\right]\right) \sum_{j=1}^n \sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}\right) \\
&\quad \left. \times \left[\frac{1}{2} \left[\left(1 - \frac{1}{2}(x_i+1)\right) x_j + \left(1 + \frac{1}{2}(x_i+1)\right)\right] - \frac{1}{2}(x_i+1)\right] \right) \\
b_2(s) &= -\frac{a}{\sqrt{c^2+4a(\alpha+s^2)}} \int_0^\ell \int_0^x x [f(\tau, s) + \psi(\tau) + s\varphi(\tau)] e^{-\frac{c(x-\tau)}{2a}} \\
&\quad \times \sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}(x-\tau)\right) d\tau dx
\end{aligned}$$

$$\begin{aligned}
b_2(s) &= -\frac{a}{4\sqrt{c^2+4a(\alpha+s^2)}} \sum_{i=1}^n k_i \left[ f\left(\frac{1}{2}(x_i+1), s\right) + \psi\left(\frac{1}{2}(x_i+1)\right) \right. \\
&\quad + s\varphi\left(\frac{1}{2}(x_i+1)\right) \left. \left(1 - \frac{1}{2}(x_i+1)\right) \left(\frac{1}{2} \left[\left(1 - \frac{1}{2}(x_i+1)\right) x_j \right. \right. \right. \right. \\
&\quad \left. \left. + \left(1 + \frac{1}{2}(x_i+1)\right)\right]\right) \exp\left(-\frac{c}{2a} \left[\left(1 - \frac{1}{2}(x_i+1)\right) x_j \right. \right. \\
&\quad \left. \left. + \left(1 + \frac{1}{2}(x_i+1)\right)\right]\right) \sum_{j=1}^n \sinh\left(\frac{\sqrt{c^2+4a(\alpha+s^2)}}{2a}\right) \\
&\quad \left. \times \left[\frac{1}{2} \left[\left(1 - \frac{1}{2}(x_i+1)\right) x_j + \left(1 + \frac{1}{2}(x_i+1)\right)\right] - \frac{1}{2}(x_i+1)\right] \right).
\end{aligned}$$

where  $x_i$  and  $k_i$  are the abscissa and weights, defined as

$$x_i : i^{th} \text{ zero of } P_n(x), \quad k_i = \frac{2}{(1-x_i^2)(P'(x))^2}$$

Their tabulated values can be found in [7] for different values of  $N$ .

Using Stehfest's algorithm, the time domain solution is approximated as

$$v(x, t) \simeq \frac{\ln 2}{t} \sum_{k=1}^{2n} \sum_{i=\left[\frac{k+1}{2}\right]}^{\min(k,n)} \frac{i^n (2i)! (-1)^{k+n}}{(n-i)! i! (i-1)! (k-i)! (2i-k)!} V\left(x, \frac{k \ln 2}{t}\right). \quad (41)$$

## References

- [1] A. Merad, On solvability of the integrodifferential hyperbolic equation with purely nonlocal conditions, *Science Direct, Acta Mathematica Science*.
- [2] A. Merad and al., Existence and Uniqueness for a Solution of pseudohyperbolic equation with Nonlocal Boundary Condition, *Applied Mathematics & Information Science*, **9**(2), (2015), 1-7.
- [3] A. Merad and al., Numerical Solutions of the hyperbolic equation with purely integral condition by using Laplace Transform method, *Palestine Journal of Mathematics*, **4**(1), (2015), 30-36.
- [4] L. Bougoffa, Parabolic equations with nonlocal conditions, *Applied Mathematical Science*, **1**(21), (2007), 1041-1048.
- [5] J.R. Cannon, The solution of heat equation to the specification of energy, *Quart. Appl. Math.*, **21**(2), (1963), 155-160.
- [6] N.I. Ionkin, Solution of boundary value problem in heat conduction theory with non local boundary condition, *Differentsial'nye Uravneniya*(Russian), **13**, (1977), 294-304.
- [7] M. Abramowitz and I.A. Stegun, *Hand book of Mathematical Functions*, New York, Dover, 1972.
- [8] M. Mesloub, A. Bouziani and N. Kechkar, A strong solution of a problem with integral conditions, *Georgian Mathematical*, **9**(1), (2002), 149-159.
- [9] L.A. Mouravey and V. Philipovski, Sur un problème avec une condition aux limites non locale pour une équation parabolique, *Mat. Sb.*(Russian), **182**(10), (1991), 1479.
- [10] A.A. Samarkii, Some problems in differential equations theory, *Differents. Uravn.*, **16**(11), (1980), 1925-1935.
- [11] Moussa Zakari Djibibe, K. Kokou Tcharie and N. Iossifovich Yurchuk, Existence, Uniqueness and Continuous Dependence of Solution of Nonlocal Boundary Conditions of Mixed Problem for Singular Parabolic Equation

- in Nonclassical Function Spaces, *Pioneer Journal of Advances in Applied Mathematics*, **7**(1), (2013), 7-16.
- [12] Moussa Zakari Djibibe and K. Kokou Tcharie, A Nonlocal Boundary Value Mixed Problem for Second-order Hyperbolic-parabolic Equations in Nonclassical Function Spaces, *British Journal of Mathematics & Computer Science*, SCIENCEDOMAIN international, **4**(5), (2014), 597-611.
- [13] Moussa Zakari Djibibe and K. Kokou Tcharie, On the Solvability of an Evolution Problem with Weighted Integral Boundary Conditions in Sobolev Function Spaces with a Priori Estimate and Fourier's Method, *British Journal of Mathematics & Computer Science*, SCIENCEDOMAIN international, **3**(4), (2013), 801-810.
- [14] Moussa Zakari Djibibe, K. Kokou Tcharie and N. Iossifovich, An energy inequality and its applications of Nonlocal boundary conditions of mixed problem for singular parabolic equations in nonclassical function spaces, *International Scholarly Research Network ISRN Applied Mathematics*, **2012**, Article ID 138308, (2012), 9 pages, doi:10.5402/2012/138308.
- [15] M.Z. Moussa Zakari Djibibe, K. Kokou Tcharie and N. Iossifovich, Continuous dependence of solutions to mixed boundary value problems for a parabolic equation, *Electronic Journal of Differential Equations*, **2008**(17), (2008), 1-10.
- [16] N.I. Yurchuk, Mixed problem with an integral condition for certain parabolic equations, *Differents. Uravn.*, **22**(12), (1986), 2117-2126.
- [17] N.I. Yurchuk, Solvability of boundary value problems for certain operator-differential equation, *Differentsial'nye Uravney*, **13**(4), (1977), 626-636.