Theoretical Mathematics & Applications, vol.6, no.4, 2016, 87-101
ISSN: 1792-9687 (print), 1792-9709 (online)
Scienpress Ltd, 2016

Numerical solution of Fractional Integro-Differential Equations by Least Squares Method and Shifted Chebyshev Polynomials of the third kind method

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Abstract

In this paper, an implementation of an efficient numerical method of linear fractional integro-differential equations **(LFIDEs)** by least squares method with aid of shifted Chebyshev polynomials of the third kind method. The fractional derivative is described in the Caputo sense. The method is based upon shifted Chebyshev polynomials of the third kind approximations is introduced. Some numerical examples are presented to illustrate the theoretical results and compared with the results obtained by other numerical methods. We have computed the numerical results using Mathematica 9 programming.

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Article Info: *Received* : June 1, 2016. *Revised* : July 20, 2016.
Published online : October 30, 2016.

Mathematics Subject Classification: 65K10; 65G99; 35E99; 68U20
Keywords: Linear fractional integro-differential equations; Caputo fractional derivatives; Shifted Chebyshev polynomials of the third kind method; Least squares method

1 Introduction

Fractional derivatives have recently played a significant role in many areas of sciences, engineering, fluid mechanics, biology, physics and economies ([4], [15], [19]). Many real-world physical systems display fractional order dynamics, that is their behavior is governed by fractional order differential equations. Consequently, considerable attention has been given to the solutions of fractional differential equations (FDEs) and integral equations of physical interest ([1], [2], [10], [20], [22], [27] , [30]). Most non-linear FDEs do not have exact analytic solutions, so approximate and numerical techniques ([25]-[28]) must be used. Many mathematical problems in science and engineering are set in unbounded domains. There is a need to consider practical design and implementation issues in scientific computing for reliable and efficient solutions of these problems. Several numerical methods to solve the FDEs have been given such as variational iteration method [10], homotopy perturbation method ([22], [25]), Adomian's decomposition method ([11], [14]), homotopy analysis method [9] and collocation method ([12], [20], [30]).

Representation of a function in terms of a series expansion using orthogonal polynomials is a fundamental concept in approximation theory and forms the basis of spectral methods of solution of differential equations ([5], [8], [13]). In [12], Khader introduced an efficient numerical method for solving the fractional diffusion equation using the shifted Chebyshev polynomials. In [28] and et. al introduced an efficient numerical method for solving the fractional diffusion equation using the shifted Chebyshev polynomials of the third kind. Spectral collocation methods are efficient and highly accurate techniques for numerical solution of non-linear differential equations. The basic idea of the spectral collocation method is to assume that the unknown solution $v(x)$ can be approximated by a linear combination of some basis functions, called the trial functions, such as orthogonal polynomials. The orthogonal polynomials can be chosen according to their special properties, which make them particularly suitable for a problem under consideration ([13], [24]).

Our fundamental goal of this work is to develop a suitable way to approximate the fractional integro-differential equations using the shifted Chebyshev polynomials of the third kind with finite difference method together with Chebyshev collocation method [28].

In this paper, least squares method with aid of shifted Chebyshev polynomials of the third kind method is applied to solving fractional integro-differential equations [17]. Least squares method has been studied in ([3], [7], [16]-[18], [23], [31]).

In this paper, we are concerned with the numerical solution of the following linear fractional Integro-differential equation [16]:

$$D^{\nu}\varphi(x) = f(x) + \int_0^1 K(x,t)\varphi(t)dt, \qquad 0 \leq x, t \leq 1, \tag{1}$$

with the following supplementary conditions:

$$\varphi^{(i)}(0) = \delta_i, \qquad n-1 < \nu \leq n, \quad n \in N, \tag{2}$$

where $D^{\nu}\varphi(x)$ indicates the νth Caputo fractional derivative of $\varphi(x)$, $f(x)$, $K(x,t)$ are given functions, x and t are real variables varying in the interval [0, 1] and $\varphi(x)$ is the unknown function to be determined.

The structure of this paper is arranged in the following way: In section 2, we introduce some basic definitions about Caputo fractional derivatives. In section 3, we give some properties of Chebyshev polynomials of the third kind which are of fundamental importance in what follows and we derive an approximate formula for fractional derivatives using Chebyshev polynomials of the third kind expansion. In section 4, the procedure of solution of linear fractional integro-differential equation. In section 5, numerical example is given to solve the **LFIDEs** and show the accuracy of the presented method. Finally, in section 6, the report ends with a brief conclusion and some remarks.

2 Preliminary and notations

In this section, we present some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

2.1 The Caputo fractional derivative

Definition 2.1. *The Caputo fractional derivative operator* D^{ν} *of order* ν *is defined in the following form:*

$$D^{\nu} f(x) = \frac{1}{\Gamma(m-\nu)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\nu-m+1}} dt, \quad \nu > 0,$$

where $m-1 < \nu \leq m, \ m \in \mathbb{N}, \ x > 0.$

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation:

$$D^{\nu}\left(\lambda f(x) + \mu g(x)\right) = \lambda D^{\nu} f(x) + \mu D^{\nu} g(x),$$

where λ and μ are constants. For the Caputo's derivative we have

$$D^{\nu} C = 0, \qquad \text{C is a constant,} \tag{3}$$

$$D^{\nu} x^n = \begin{cases} 0, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n < \lceil \nu \rceil; \\ \frac{\Gamma(n+1)}{\Gamma(n+1-\nu)} x^{n-\nu}, & \text{for} \quad n \in \mathbb{N}_0 \text{ and } n \geq \lceil \nu \rceil. \end{cases} \tag{4}$$

We use the ceiling function $\lceil \nu \rceil$ to denote the smallest integer greater than or equal to ν, and $\mathbb{N}_0 = \{0, 1, 2, ...\}$. Recall that for $\nu \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.
For more details on fractional derivatives definitions and its properties see ([6, 15, 19, 21]).

3 Some properties of Chebyshev polynomials of the third kind

3.1 Chebyshev polynomials of the third kind

The Chebyshev polynomials $V_n(x)$ of the third kind ([13], [28]) are orthogonal polynomials of degree n in x defined on the $[-1, 1]$

$$V_n = \frac{cos(n + \frac{1}{2})\Theta}{cos(\frac{\Theta}{2})},$$

where $x = cos\Theta$ and $\Theta \in [0, \pi]$.

They can be obtained explicitly using the Jacobi polynomials $P_k^{(\alpha,\beta)}(x)$, for the special case $\beta = -\alpha = 1/2$.
These are given by:

$$V_k(x) = \frac{2^{2k} P_k^{(-1/2,1/2)}(x)(\Gamma(k+1))^2}{\Gamma(2k+1)}. \tag{5}$$

Also, these polynomials $V_n(x)$ are orthogonal on $[-1, 1]$ with respect to the inner product:

$$< V_n(x), V_m(x) >= \int_{-1}^{1} \sqrt{\frac{1+x}{1-x}} V_n(x) V_m(x) dx = \begin{cases} \pi, & \text{for} \quad n = m; \\ 0, & \text{for} \quad n \neq m; \end{cases}. \tag{6}$$

where $\sqrt{\frac{1+x}{1-x}}$ is weight function corresponding to $V_n(x)$.
The polynomials $V_n(x)$ may be generated by using the recurrence relations

$$V_{n+1}(x) = 2xV_n(x) - V_{n-1}(x), \quad V_0(x) = 1, \quad V_1(x) = 2x - 1, \quad n = 1, 2, \ldots .$$

The analytical form of the Chebyshev polynomials of the third kind $V_n(x)$ of degree n, using Eq. (5) and properties of Jacobi polynomials to obtain they are given as:

$$V_n(x) = \sum_{k=0}^{[\frac{2n+1}{2}]} (-1)^k \, (2)^{n-k} \, \frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\,\Gamma(2n-2k+2)} \, (x+1)^{n-k}, \qquad n \in Z^+, \tag{7}$$

where $[\frac{2n+1}{2}]$ denotes the integer part of $(2n+1)/2$.

3.2 The shifted Chebyshev polynomials of the third kind

In order to use these polynomials on the interval $[0, 1]$, we define the so called shifted Chebyshev polynomials of the third kind [28] by introducing the change of variable $s = 2x - 1$. The shifted Chebyshev polynomials of the third kind are defined as $V_n^*(x) = V_n(2x - 1)$.
These polynomials are orthogonal on the support interval $[0, 1]$ as the following inner product:

$$< V_n^*(x), V_m^*(x) >= \int_{0}^{1} \sqrt{\frac{x}{1-x}} V_n^*(x) V_m^*(x) dx = \begin{cases} \frac{\pi}{2}, & \text{for} \quad n = m; \\ 0, & \text{for} \quad n \neq m; \end{cases}. \tag{8}$$

where $\sqrt{\frac{x}{1-x}}$ is weight function corresponding to $V_n^*(x)$. and normalized by the requirement that $V_n^*(1) = 1$.
The polynomials $V_n^*(x)$ may be generated by using the recurrence relations

$$V_{n+1}^*(x) = 2(2x-1)V_n^*(x) - V_{n-1}^*(x), \qquad V_0^*(x) = 1, \quad V_1^*(x) = 4x-3, \qquad n = 1, 2, \dots .$$

The analytical form of the shifted Chebyshev polynomials of the third kind $V_n^*(x)$ of degree n in x given by:

$$V_n^*(x) = \sum_{k=0}^{n} (-1)^k \, (2)^{2n-2k} \, \frac{(2n+1)\Gamma(2n-k+1)}{\Gamma(k+1)\,\Gamma(2n-2k+2)} \, (x)^{n-k}, \qquad n \in Z^+, \ (9)$$

In a spectral method, in contrast, the function $g(x)$, square integrable in $[0, 1]$, is represented by an infinite expansion of the shifted Chebyshev polynomials of the third kind as follows:

$$g(x) = \sum_{i=0}^{\infty} b_i \, V_i^*(x), \tag{10}$$

where b_i is a chosen sequence of prescribed basis functions. One then proceeds some how to estimate as many as possible of the coefficients bi, thus approximating $g(x)$ by a finite sum of $(m+1)$-terms such as:

$$g_m(x) = \sum_{i=0}^{m} b_i \, V_i^*(x), \tag{11}$$

where the coefficients b_i, $i = 0, 1, \dots$ are given by

$$b_i = \frac{1}{\pi} \int_{-1}^{1} g(\frac{x+1}{2}) \, V_i(x) \sqrt{\frac{1+x}{1-x}} \, dx, \tag{12}$$

where the coefficients b_i, $i = 0, 1, \dots$ are given by

$$b_i = \frac{2}{\pi} \int_{0}^{1} g(x) \, V_i^*(x) \sqrt{\frac{x}{1-x}} \, dx, \tag{13}$$

Theorem 3.1. *(Chebyshev truncation theorem) ([13], [24]) The error in approximating $g(x)$ by the sum of its first m terms is bounded by the sum of the absolute values of all the neglected coefficients. If*

$$g_m(x) = \sum_{i=0}^{m} b_i \, V_i(x), \tag{14}$$

then

$$E_T(m) \equiv |g(x) - g_m(x)| \leq \sum_{k=m+1}^{\infty} |b_i|, \tag{15}$$

for all $g(x)$, all m, and all $x \in [-1, 1]$.

The main approximate formula of the fractional derivative of $g_m(x)$ is given in the following theorem.

Theorem 3.2. *Let $g(x)$ be approximated by shifted Chebyshev polynomials of the third kind as (11) and also suppose $\alpha > 0$, then*

$$D^{\alpha}(g_m(x)) = \sum_{i=\lceil\alpha\rceil}^{m} \sum_{k=0}^{i-\lceil\alpha\rceil} b_i \, N_{i,k}^{(\alpha)} \, x^{i-k-\alpha}, \tag{16}$$

where $N_{i,k}^{(\alpha)}$ is given by

$$N_{i,k}^{(\alpha)} = (-1)^k \frac{2^{2i-2k}\,(2n+1)\Gamma(2i-k+1)\Gamma(i-k+1)}{\Gamma(k+1)\Gamma(2i-2k+2)\Gamma(i-k+1-\alpha)}. \tag{17}$$

Proof. ([28]). □

4 Procedure solution using shifted Chebyshev polynomials of the third kind collocation method

In this section, the least squares method with aid of shifted Chebyshev polynomials of the third kind collocation method is applied to study the numerical solution of the linear fractional Integro-differential equation (1).

The procedure of the implementation is given by the following steps:

1. Substitute by Eq.(11) into Eq.(1) we obtain [16]:

$$D^{\nu}\left(\sum_{i=0}^{m} c_i V_i^*(x)\right) = f(x) + \int_0^1 K(x,t)\left(\sum_{i=0}^{m} c_i V_i^*(x)\right) dt. \tag{18}$$

2. Hence the residual equation is defined as

$$R(x, c_0, c_1, ..., c_n) = \sum_{i=0}^{m} c_i D^{\nu} V_i^*(x) - f(x) - \int_0^1 K(x,t)\left(\sum_{i=0}^{m} c_i V_i^*(x)\right) dt. \tag{19}$$

3. Let
$$S(c_0, c_1, ..., c_n) = \int_0^1 (R(x, c_0, c_1, ..., c_n))^2 \cdot \omega(x)dx. \tag{20}$$
where $\omega(x)$ is the positive weight function defined on the interval [0, 1]. In this work we take $\omega(x) = \sqrt{\frac{x}{1-x}}$.

4. Thus
$$S(c_0, c_1, ..., c_n) =$$
$$= \int_0^1 \left(\sum_{i=0}^{m} c_i D^{\nu} V_i^*(x) - f(x) - \int_0^1 K(x,t) \left(\sum_{i=0}^{m} c_i V_i^*(x) \right) dt \right)^2 \omega(x)dx. \tag{21}$$

5. So, finding the values of $c_i, i = 0, 1, ..., n$, which minimize S is equivalent to finding the best approximation for the solution of the LFIDEs (1).

6. The minimum value of S is obtained by setting
$$\frac{\partial S}{\partial c_i} = 0 \qquad i = 0, 1, ..., m. \tag{22}$$

7. Applying (22) to (21) we obtain
$$\int_0^1 \left(\sum_{i=0}^{m} c_i D^{\nu} V_i^*(x) - f(x) - \int_0^1 K(x,t) \left(\sum_{i=0}^{m} c_i V_i^*(x) \right) dt \right) \times$$
$$\times \left(D^{\nu} V_i^* - \int_0^1 K(x,t) V_i^*(x) \right) \omega(x)dx. \tag{23}$$

By evaluating the above equation for $i = 0, 1, ..., n$ we can obtain a system of $(m + 1)$ linear equations with $(m + 1)$ unknown coefficients c_i. This system can be formed by using matrices form as follows:
$$A = \begin{pmatrix} \int_0^1 R(x, c_0)h_0 dx & \int_0^1 R(x, c_1)h_0 dx & ... & \int_0^1 R(x, c_m)h_0 dx \\ \int_0^1 R(x, c_0)h_1 dx & \int_0^1 R(x, c_1)h_1 dx & ... & \int_0^1 R(x, c_m)h_1 dx \\ ... & ... & ... & ... \\ \int_0^1 R(x, c_0)h_m dx & \int_0^1 R(x, c_1)h_m dx & ... & \int_0^1 R(x, c_m)h_m dx \end{pmatrix}, \tag{24}$$
$$B = \begin{pmatrix} \int_0^1 f(x)h_0 dx \\ \int_0^1 f(x)h_1 dx \\ \vdots \\ \int_0^1 f(x)h_m dx \end{pmatrix}, \tag{25}$$

where

$$h_i = D^{\nu} V_i^*(x) - \int_0^1 K(x,t) \sum_{i=0}^{m} c_i V_i^*(x)\omega(x) dt, \; i = 1, 2, ..., m,$$

$$R(x,t) = \sum_{i=0}^{m} c_i D^{\nu} V_i^*(x) - \int_0^1 K(x,t) \left(\sum_{i=0}^{m} c_i V_i^*(x) \right) dt, \; i = 0, 1, ..., m$$

By solving the above system we obtain the values of the unknown coefficients and the approximate solution of 1.

5 Applications and numerical results

In this section, we numerical examples of linear fractional integro-differential equation are presented to illustrate the above results. All results are obtained by using Mathematica 9 programming.

Example 1:

Consider the following linear fractional integro-differential equation [16]

$$D^{1/2}\varphi(x) = \frac{(8/3)x^{3/2} - 2x^{1/2}}{\sqrt{\pi}} + \frac{x}{12} + \int_0^1 xt\varphi(t)dt, \qquad 0 \le x, t \le 1, \quad (26)$$

subject to $\varphi(0) = 0$ with the exact solution $\varphi(x) = x^2 - x$.

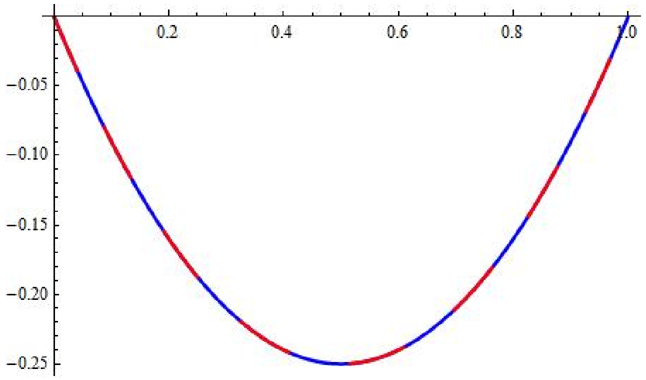

Figure 1: Comparison between the numerical solution and exact solution.

Applying the least squares method with aid of shifted Chebyshev polynomials collocation of third kind $V_i^*(x),\ i = 0, 1, ..., m$ at $m = 5$, to the linear fractional integro-differential equation (26), we obtain a system of (24) linear equations with (25) unknown coefficients $c_i,\ i = 0, 1, ..., 5$.

The solution obtained using the suggested method is in excellent agreement with the already exact solution and show that this approach can be solved the problem effectively. It is evident that the overall errors can be made smaller by adding new terms from the series (11). Comparisons are made between approximate solutions and exact solutions to illustrate the validity and the great potential of the proposed technique. Also, from our numerical results we can see that these solutions are in more accuracy of those obtained in [16].

Example 2:

Consider the following linear fractional integro-differential equation [16]

$$D^{5/6}\varphi(x) = f(x) + \int_0^1 xe^t\varphi(t)dt, \qquad 0 \le x, t \le 1, \tag{27}$$

where $f(x) = \frac{-3x^{1/6}\Gamma(5/6)(-91+216x^2)}{91\pi} + (2-2e)x$, subject to $\varphi(0) = 0$ with the exact solution $\varphi(x) = x - x^3$.

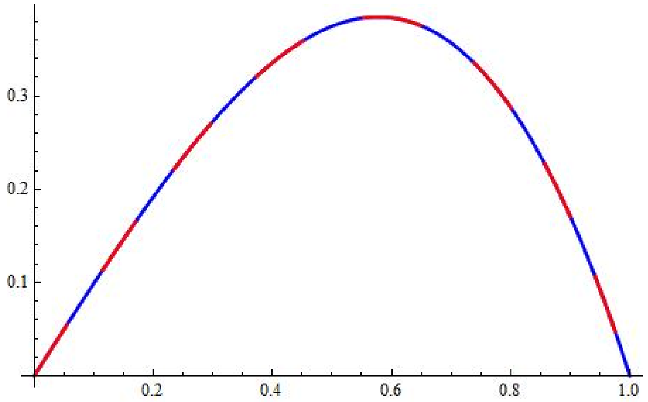

Figure 2: Comparison between the numerical solution and exact solution.

Similarly as in Example 1 applying the least squares method with aid of shifted Chebyshev polynomials collocation of third kind $V_i^*(x),\ i = 0, 1, ..., m$ at $m = 5$, to the fractional integro-differential equation (27) the numerical results are shown in Figure 2 and we obtain the approximate solution which is the same as the exact solution.

Example 3:

Consider the following fractional integro-differential equation [16]

$$D^{5/3}\varphi(x) = f(x) + \int_0^1 (xt + x^2t^2)\varphi(t)dt, \qquad 0 \le x, t \le 1, \tag{28}$$

where $f(x) = \frac{3\sqrt{3}\Gamma(2/3)x^{1/3}}{\pi} - x^2/5 - x/4$, subject to $\varphi(0) = 0$ with the exact solution $\varphi(x) = x^2$.

Similarly as in Examples 1 and 2 applying the least squares method with aid of shifted Chebyshev polynomials collocation of third kind $V_i^*(x)$, $i = 0, 1, ..., m$ at $m = 5$, to the fractional integro-differential equation (28) the numerical results are shown in Figure 3 and we obtain the approximate solution which is the same as the exact solution.

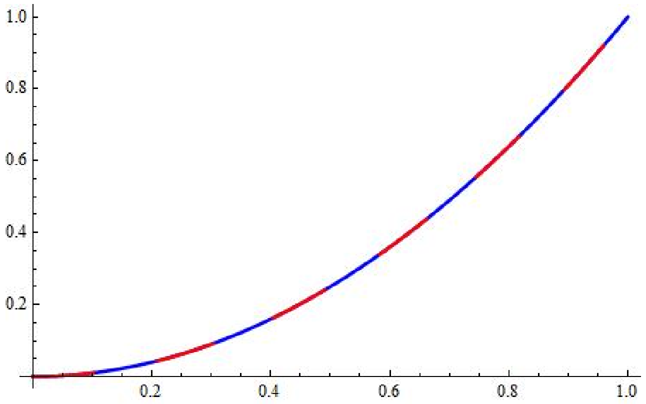

Figure 3: Comparison between the numerical solution and exact solution.

6 Conclusion

In this article, we introduced an accurate numerical technique for solving linear fractional integro-differential equation. We have introduced an approximate formula for the Caputo fractional derivative of the shifted Chebyshev polynomials of the third kind method in terms of classical shifted Chebyshev polynomials of the third kind method. The results show that the algorithm converges as the number of m terms is increased. The solution is expressed as a truncated shifted Chebyshev polynomials series and so it can be easily evaluated for arbitrary values of time using any computer program without any computational effort. Some numerical examples are presented to illustrate the theoretical results and compared with the results obtained by other numerical methods. We have computed the numerical results using Mathematica 9 programming.

ACKNOWLEDGEMENTS. Thank you for the referees their efforts. The authors would like to thank Prof. Dr. K. R. Mohamed, Department of Mathematics, Faculty of Science, Alazhar University, Cairo, Egypt, which provided support.

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