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Some liminf results for the Increments of stable subordinators

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Abstract

Let $\{X(t), 0 \leq t < \infty\}$ be a stable subordinator defined on a probability space $(\Omega, \mathcal{F}, \mathcal{A})$. In this paper we establish some liminf for increments of stable subordinators and we obtain similar results for delayed sums.

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1 Introduction

Let $\{X(t), 0 \leq t < \infty\}$ be a stable subordinator with exponent α, $0 < \alpha < 1$, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{A})$. Let a_t, $t > 0$, be a non-negative

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valued function of t such that (i) $0 < a_t \leq 1$, (ii) $a_t \to \infty$ as $t \to \infty$ (iii) $a_t/t \to 0$ as $t \to \infty$. let $Y(t) = X(t + a_t) - X(t)$, $t > 0$ and $Y(0) = 0$. Define $\lambda_\beta(t) = \theta_\alpha a_t^{\frac{1}{\alpha}} (\log \frac{t}{a_t} (\log t)^\beta (\log a_t)^{1-\beta})^{\frac{\alpha-1}{\alpha}}$, where $\theta_\alpha = (B(\alpha))^{\frac{1-\alpha}{\alpha}}$, $B(\alpha) = (1-\alpha)\alpha^{\frac{\alpha-1}{\alpha}} (cos(\frac{\pi\alpha}{2}))^{\frac{1}{\alpha-1}}$, $0 < \alpha < 1$ and $0 \leq \beta \leq 1$. Observe that the process has the property that $t^{-\frac{1}{\alpha}} X(t)$ and $X(1)$ are identically distributed. A real valued increasing process $\{X(t), t > 0\}$ with stationary independent is called a subordinator. For any given t, the characteristic function of $X(t)$ is the form

$$E(e^{\{iuX(t)\}}) = exp\left\{-t|u|^\alpha \left(1 - \frac{ui}{|u|} tan\left(\frac{\pi\alpha}{2}\right)\right)\right\}, \quad 0 < \beta < 1.$$

Throughout the paper ε, c, δ and K (integer), with or without suffix, stand for positive constants; i.o. means infinitely often; we shall define for each $u \geq 0$ the functions $\log u = \log(max(u, 1))$, $\log\log u = \log\log(max(u, 3))$, $g(t) = (t \log t)/a_t$ and $g_\beta(t) = \frac{t}{a_t} (\log t)^\beta (\log a_t)^{1-\beta}$ with $0 \leq \beta \leq 1$, so that $\lambda_{(t,\beta)} = (2a_t \log g_\beta(t))^{-\frac{1}{2}}$.

Vasudeva and Divanji [6] have obtained the following limit inferior for the increments of stable subordinators. Under certain condition on a_t, it was shown that $\liminf_{t\to\infty} \frac{Y(t)}{\lambda_1(t)} = 1 a.s$

Hwang et al.[2] and Bahram and Shehawy [1] studied this subsequence principle for increments of Gaussian processes in obtaining limsup. In this paper we study an almost sure limit inferior behaviour for increments of stable subordinators for proper selection of subsequences and extended to delayed sums.

2 Main results

In order to prove Theorem 2.1, we need to give the following Lemma.

Lemma 2.1. *(see [5] or [6]) Let X_1 be a positive stable random variable with characteristic function*

$$E(exp\{iuX_1\}) = exp\left\{-|u|^\alpha \left(1 - \frac{iu}{|u|} tan\left(\frac{\pi\alpha}{2}\right)\right)\right\}, \quad 0 < \alpha < 1. \quad \textit{Then, as} \quad x \longrightarrow 0$$

$$P(X_1 \leq x) \simeq \frac{x^{\frac{\alpha}{2(1-\alpha)}}}{\sqrt{2\pi\alpha B(\alpha)}} exp\left\{-B(\alpha) x^{\frac{\alpha}{\alpha-1}}\right\}$$

where

$$B(\alpha) = (1-\alpha)\alpha^{\frac{\alpha-1}{\alpha}}(cos(\frac{\pi\alpha}{2}))^{\frac{1}{\alpha-1}}.$$

Theorem 2.1. *Let a_t, $t > 0$ be a nondecreasing function of t such i) $0 < a_t \leq t$, ii) $a_t \to \infty$, as $t \to \infty$ and iii) $a_t/t \to 0$ as $t \to \infty$. Let (t_k) be an increasing sequence of positive integers such that*

$$\limsup_{k \longrightarrow \infty} \frac{t_{k+1} - t_k}{a_{t_k}} < 1. \tag{1}$$

Then

$$\liminf_{k \longrightarrow \infty} \frac{Y(t_k)}{\lambda_\beta(t_k)} = \varepsilon^* \quad a.s.,$$

where

$$\varepsilon^* = inf\{\varepsilon > 0 : \sum_k (g_\beta(t_k))^{-\varepsilon^{-\gamma}} < \infty, \quad 0 \leq \beta \leq 1\} \quad and \quad \gamma = \frac{\alpha}{\alpha - 1}, \quad 0 < \alpha < 1.$$

Proof Equivalently, we show that for any given $\varepsilon_1 > 0$, as $k \longrightarrow \infty$,

$$P(Y(t_k) \leq (\varepsilon^* + \varepsilon_1)\lambda_\beta(t_k) \quad i.o.) = 1 \tag{2}$$

and

$$P(Y(t_k) \leq (\varepsilon^* - \varepsilon_1)\lambda_\beta(t_k) \quad i.o.) = 0. \tag{3}$$

The condition (1) implies that $t_{k+1} < t_k + a_{t_k}$, for large k and by Mijnheer [5], we have

$$P\left(Y(t_k) \leq (\varepsilon^* + \varepsilon_1)\lambda_\beta(t_k)\right) = P\left(X_1 \leq \frac{(\varepsilon^* + \varepsilon_1)\lambda_\beta(t_k)}{a_{tk}^{1/\alpha}}\right) \tag{4}$$

Observe that

$$\frac{(\varepsilon^* + \varepsilon_1)\lambda_\beta(t_k)}{a_{tk}^{1/\alpha}} = (\varepsilon^* + \varepsilon_1)\theta_\alpha \left(\log g_\beta(t_k)\right)^{\frac{\alpha-1}{\alpha}}$$

taken as x, in the above lemma, one can find a k_1 and some conctant C_1, such that for all $k \geq k_1$,

$$P(X_1 \leq \frac{(\varepsilon^* + \varepsilon_1)\lambda_\beta(t_k)}{a_{tk}^{1/\alpha}}) \geq c_1(\log g_\beta(t_k))^{\frac{-1}{2}} exp\{-(\varepsilon^* + \varepsilon_1)^{\frac{\alpha}{\alpha-1}} \log g_\beta(t_k)\}$$

where $g_\beta(t) = \frac{t}{a_t}(\log t)^\beta (\log a_t)^{1-\beta}$ and $0 \leq \beta \leq 1$.
Notice that from the definition of ε_*, we have $\varepsilon_* \geq 1$ implies that there exists

$\varepsilon_2 > 0$ such that $(\varepsilon^* + \varepsilon_1)^{\frac{\alpha}{\alpha-1}} < (1-\varepsilon_2) < 1$. Hence $P(X_1 \leq \frac{(\varepsilon^*+\varepsilon_1)\lambda_\beta(t_k)}{a_{tk}^{1/\alpha}}) \geq \frac{c_1}{(\log g_\beta(t_k))^{\frac{1}{2}}(g_\beta(t_k))^{1-\varepsilon_2}}$. Let $l_k = \frac{t_k}{a_{t_k}}$ and $m_k = (\log t_k)^\beta (\log a_{t_k})^{1-\beta}$. Since $\frac{a_{tk}}{tk} \to 0$, as $k \to \infty$, l_k is non decreasing and $m_k \to \infty$, as $k \to \infty$, one can find a constant $k_2 \geq k_1$ such that $\frac{l_k^{\varepsilon_2} m_k^{\varepsilon_2}}{(\log l_k m_k)^{\frac{1}{2}}} \geq 1$, whenever $k \geq k_2$. By condition (1), for all $k \geq k_2$, we therefore have,

$$\begin{aligned} P(X_1 \leq \frac{(\varepsilon^*+\varepsilon_1)\lambda_\beta(t_k)}{a_{tk}^{1/\alpha}}) &\geq c_1(g_\beta(t_k))^{-1} \\ &= c_1(\frac{t_k(\log t_k)^\beta(\log a_{t_k})^{1-\beta}}{a_{t_k}})^{-1} \\ &= c_1(\frac{a_{t_k}}{t_k}(\frac{\log a_{t_k}}{\log t_k})^\beta \frac{1}{\log a_{t_k}}) \\ &\geq c_1(\frac{a_{t_k}}{t_k}(\frac{\log a_{t_k}}{\log t_k})\frac{1}{\log a_{t_k}}) \\ &= c_1(g(t_k))^{-1} \\ &= c_1\frac{t_{k+1}-t_k}{t_k \log t_k}. \end{aligned} \tag{5}$$

Observing that

$$\sum_{k=k_2}^{\infty} \frac{t_{k+1}-t_k}{t_k \log t_k} \geq \int_c^\infty \frac{dt}{t \log t}$$

for some $c > 0$ and that $\int_c^\infty \frac{dt}{t\log t} = \infty$. Hence from (4) and (5), we get

$$\sum_{k=k_2}^{\infty} P(Y(t_k) \leq (\varepsilon^*+\varepsilon_1)\lambda_{(t_k,\beta)}) = \infty.$$

The Condition (1) implies that $t_{k+1} \leq t_k + a_{t_k}$, for large k one can observe that $Y(t_k)^{'s}$ are mutually independent and hence by Borel-Cantelli Lemma, we have,

$$P(Y(t_k) \leq (\varepsilon^*+\varepsilon_1)\lambda_{(t_k,\beta)} \quad i.o) = 1,$$

which establishes (2).

Now we complete the proof by showing that, for any $\varepsilon_1 \in (0,1)$,

$$P(Y(t_k) \leq (\varepsilon^*-\varepsilon_1)\lambda_\beta(t_k) i.o) = 1.$$

From condition (1), we have $t_{k+1} \leq t_k + a_{t_k}$, for large k and from Mijnheer [5], one can find a k_2 such that for all $k \geq k_2$,

$$P(Y(t_k) \leq (\varepsilon^*-\varepsilon_1)\lambda_\beta(t_k) \quad i.o) = P(X(t_k+a_{t_k})-X(t_k)) \leq (\varepsilon^*-\varepsilon_1)\lambda_\beta(t_k) \quad i.o)$$

Hence in order to prove (3), it is enough to show that

$$P(X(t_k + a_{t_k}) - X(t_k) \leq (\varepsilon^* - \varepsilon_1)\lambda_\beta(t_k) \quad i.o) = 0. \tag{6}$$

We know that $t^{1/\alpha}X(t) = X(1)$, which implies

$$P(X(t_k + a_{t_k}) - X(t_k) \leq (\varepsilon^* - \varepsilon_1)\lambda_\beta(t_k)) = P\left(X(1) \leq \frac{(\varepsilon^* - \varepsilon_1)\lambda_\beta(t_k)}{a_{t_k}^{1/\alpha}}\right)$$

and

$$\frac{(\varepsilon^* - \varepsilon_1)\lambda_\beta(t_k))}{a_{t_k}^{1/\alpha}} = (\varepsilon^* - \varepsilon_1)\theta_\alpha \left(\log(g_\beta(t_k))^{(\alpha-1)/\alpha}\right. .$$

By taking $x = (\varepsilon^* - \varepsilon_1)\theta_\alpha \left(\log(g_\beta(t_k))^{(\alpha-1)/\alpha}\right.$, where $g_\beta(t) = \frac{t}{a_t}(\log t)^\beta(\log a_t)^{1-\beta}$, in the above lemma, one can find a k_4 and c_1 such that for all $k \geq k_4$,

$$P\left(X(1) \leq \frac{(\varepsilon^* - \varepsilon_1)\lambda_\beta(t_k)}{a_{t_k}^{1/\alpha}}\right) \leq \frac{c_2}{(\log(g_\beta(t_k)))^{1/2}} exp\{-(\varepsilon^* - \varepsilon_1)^{\frac{\alpha}{(\alpha-1)}} \log g_\beta(t_k)\}.$$

Observe that using properties of $\{a_t\}$, one can find some constant C_3 and k_4 such that for all $k \geq k_4$,

$$\begin{aligned} P(X(1) &\leq (\varepsilon^* - \varepsilon_1)\theta_\alpha \left(\log(g_\beta(t_k))^{(\alpha-1)/\alpha}\right)) \\ &\leq \frac{c_3}{(g_\beta(t_k))^{(\varepsilon^* - \varepsilon_1)^{\frac{\alpha}{(\alpha-1)}}}}. \end{aligned}$$

Notice that $\varepsilon^* = inf\{\varepsilon > 0 : \sum_k (g_\beta(t_k))^{-\varepsilon^{-\gamma}} < \infty, \quad 0 \leq \beta \leq 1\}$ and $\gamma = \frac{\alpha}{\alpha-1} < 0$, $0 < \alpha < 1$ which yields $\varepsilon^* \geq 1$.
Since $\varepsilon_1 \in (0,1)$, choose ε_1 sufficiently small one can find k_5 such that for all $k \geq k_5$,

$$\sum_{k=k_5}^{\infty} P\left(X(1) \leq \frac{(\varepsilon^* - \varepsilon_1)\lambda_\beta(t_k)}{a_{t_k}^{\frac{1}{\alpha}}}\right) \leq \sum_{k=k_5}^{\infty} \frac{C_3}{(g_\beta(t_k))^{(\varepsilon^* - \varepsilon_1)^\gamma}} < \infty,$$

where $\gamma = \frac{\alpha}{\alpha-1}, \quad 0 < \beta < 1$.
By Borel-Cantelli Lemma, (3) holds which implies (6) holds and proof of the theorem is completed. □

Theorem 2.2. *Let a_t, $t > 0$ be a nondecreasing function of t such i) $0 < a_t < t$, ii) $a_t \longrightarrow \infty$, as $t \longrightarrow \infty$ and iii) $a_t/t \to 0$ as $t \to \infty$. Let (t_k) be an increasing sequence of positive integers such that*

$$\liminf_{k \longrightarrow \infty} \frac{t_{k+1} - t_k}{a_{t_k}} > 1. \tag{7}$$

Then

$$\liminf_{k\longrightarrow\infty}\frac{Y(t_k)}{\lambda_\beta(t_k)}=1 \quad a.s.,$$

where $0\leq\beta\leq 1$.

Proof To prove the Theorem, it is enough to show that for any $\varepsilon\in(0,1)$,

$$P(Y(t_k)\leq(1+\varepsilon)\lambda_\beta(t_k)\quad i.o.)=1 \tag{8}$$

and

$$P(Y(t_k)\leq(1-\varepsilon)\lambda_\beta(t_k)\quad i.o.)=0 \tag{9}$$

By the Theorem of Vasudeva and Divanji [6], we claim that

$$\liminf_{k\longrightarrow\infty}\frac{Y(t_k)}{\lambda_\beta(t_k)}\geq\liminf_{k\longrightarrow\infty}\frac{Y(t_k)}{\lambda_1(t_k)}\geq\liminf_{t\longrightarrow\infty}\frac{Y(t)}{\lambda_1(t)}=1 \quad a.s.,$$

which establishes (9).

The condition (7) implies that there exists a k_1 such that $t_{k+1}>t_k+a_{t_k}$, for all $k\geq k_1$. This in turn implies that $\{Y(t_k),k\geq 1\}$ is a sequence of mutually independent r.v.s. We can observe that with a minor modification, the proof of (8) follows on similar lines of (2). That is using Lemma 2.1, one can find C_1 and k_2 such that for all $k\geq k_2$.

$$P(X_1\leq\frac{(1+\varepsilon)\lambda_\beta(t_k)}{a_{tk}^{1/\alpha}}\geq c_1(g(t_k))^{-(1+\varepsilon)^{\frac{\alpha}{\alpha-1}}}.$$

Choose $\varepsilon^{'}>0$ such that $(1+\varepsilon)^{a_{tk}^{1/\alpha}}<(1-\varepsilon^{'})<1$ and hence we have,

$$P(X_1\leq\frac{(1+\varepsilon)\lambda_\beta(t_k)}{a_{tk}^{1/\alpha}})\geq c_1(g(t_k))^{-(1-\varepsilon^{'})}.$$

Following similar arguments of proof of (4) and (5), we get

$$P(Y(t_k)\leq(1+\varepsilon)\lambda_\beta(t_k)=\infty,$$

which in turn implies the proof of (8). Hence the proof of the Theorem is completed. □

3 Similar result for delayed sums

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d strictly positive stable r.v.s with index α, $0 < \alpha < 1$. Let $\{a_n, n \geq 0\}$ be a sequence of non-decreasing functions of positive integers of n such that $0 < a_n < n$, for all n and we assume that $a_n/n \downarrow 0$ as $n \to \infty$. Define $\lambda_\beta(n) = \theta_\alpha a_n^{\frac{1}{\alpha}}(\log \frac{n}{a_n} + \beta \log\log n + (1-\beta)\log\log a_n)^{\frac{\alpha-1}{\alpha}}$, where $\theta_\alpha = (B(\alpha))^{\frac{1-\alpha}{\alpha}}$, $B(\beta) = (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(cos(\frac{\pi\alpha}{2}))^{\frac{1}{\alpha-1}}$, $0 \leq \beta \leq 1$ and $0 < \alpha < 1$. Observe that the process has the property that $n^{-1/\alpha}X(n)$ and $X(1)$ are identically distributed. Let $S_n = \sum_{k=1}^{n} X_k$ and set $M_n = S_{n+a_n} - S_n$, where $\{M_n, n \geq 1\}$ is called a (forward) delayed sum (See Lai [3]). Define the $X_n = X(n) - X(n-1)$, $n = 1, 2, ...$; $X(0) = 0$, then $S_n = \sum_{k=1}^{n} X_k$ with $S_0 = 0$, which yields $M_n = S_{n+a_n} - S_n = X(n+a_n) + X(n) = Y(n)$.

Theorem 3.1. *Let $\{a_n, n > 0\}$ be a sequence of non-decreasing functions of positive integers of n such that $0 < a_n < n$ non-decreasing and a_n/n non-increasing. Let $(n_k, k \geq 1)$ be any increasing sequence of positive integers such that*

$$\limsup \frac{n_{k+1} - n_k}{a_{n_k}} < 1. \tag{10}$$

Then

$$\liminf_{k \longrightarrow \infty} \frac{M_{n_k}}{\lambda_\beta(n_k)} = \varepsilon^* \quad a.s.,$$

where

$$\varepsilon^* = inf\{\varepsilon > 0 : \sum_k (g_\beta(n_k))^{-\varepsilon^\gamma} < \infty, \quad 0 \leq \alpha \leq 1\},$$

and

$$\gamma = \frac{\alpha}{\alpha - 1}, \quad 0 < \alpha < 1.$$

Proof To prove the theorem it is sufficient to show that for any given $\varepsilon_1 \in (0, 1)$

$$P(M_{n_k} \leq (\varepsilon_* + \varepsilon)\lambda_\beta(n_k) \quad i.o.) = 1, \tag{11}$$

and

$$P(M_{n_k} \leq (\varepsilon_* - \varepsilon_2)\lambda_\beta(n_k) \quad i.o.) = 0. \tag{12}$$

The proof of (11) is an immediate consequence of (2) and the proof of (12) follows on the similar lines of Vasudeva and Divanji [6]. Hence the details are omitted. □

Theorem 3.2. *Let $\{a_n, n > 0\}$ be a sequence of non-decreasing functions of positive integers of n such that 1) $0 < a_n \leq n$, $n > 0$, 2) $a_n \to \infty$, as $n \to \infty$, and $a_n/n \to 0$, as $n \to \infty$. Let $(n_k, k \geq 1)$ be any increasing sequence of positive integers such that $\liminf_{k\to\infty} \frac{n_{k+1} - n_k}{a_{n_k}} > 1$. Then*

$$\liminf_{k\longrightarrow\infty} \frac{M_{n_k}}{\lambda_\beta(n_k)} = 1 \quad a.s.$$

Proof The proof of Theorem 3.2 is a direct consequence of above Theorem 2.2 and hence the details are omitted. □

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