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DrawDown Constraints and Portfolio Optimization

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Abstract

The seminal work by Markowitz in 1959 introduced portfolio theory to the world. The prevailing notion since then has been that portfolio risk is non linear i.e. you cannot use Linear Programming (LP) to optimize your portfolio. We will in this paper show that simple portfolio drawdown constraints are indeed linear and can be used to find for example maximum risk adjusted return portfolios. VaR for these portfolios can then be estimated directly instead of using computer intensive Monte Carlo methods.

JEL classification numbers: G10, G11

Keywords: drawdown, portfolio, risk, expected return

1 Introduction and Literature Review

Over the centuries a lot of people have been fascinated by the theory of

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investment. The main source of motivation has of course been money i.e. how can you make more money without taking too much risk. Markowitz (1959) explains that an investor needs to maximize his or her expected return while on the same time minimize the portfolio variance. Portfolio variance or portfolio risk has always been thought of as non-linear i.e. the variance of a diversified portfolio is lower than the variance of the individual assets in such a portfolio. We will in this paper show however that portfolio risk can successfully be expressed in linear terms which make the modeling much easier. Mean-variance optimization and value-at-risk (VaR) models represent a foundation in investment science. Liu (2005) argues that major investments banks have increasingly employed VaR models to measure the risk from their trading activities. Conditional distributions of portfolio returns are used to estimate VaR. The normality assumption is used to forecast VaR despite the fact that asset returns exhibits fat tails which leads to that that the true value at risk is being underestimated.

Palaro and Hotta (2006) explain that there are three main ways to estimate VaR; historical simulation, the variance-covariance and the Monte Carlo approach. The first approach does not assume any distribution assumptions. The last two approaches assume a given joint distribution. Copulas theory is essential when it comes to model joint multivariate distributions. The benefit of a Copula model is that the marginal distributions can be different. The authors further explain that conditional copulas can be a very powerful tool to estimate VaR. Berkowitz and O'Brien (2002) looked at the performance of VaR forecasts for six commercial banks. They concluded that despite the banks effort to present extensive and detailed information regarding VaR such risk measure did not outperform the forecasts of a simple ARMA/GARCH model of the banks profit/loss.

Alexandera and Baptistab (2002) explain another drawdown with VaR. If we for example have two mean-variance efficient portfolios then the portfolio with the highest variance might have a lower VaR. This has to do with the fact that VaR is also a function of expected return. The authors explain that a risk adverse investor that switches from using variance to VaR to measure risk might end up with a portfolio with larger standard deviation. Regulators should be aware of such VaR limitations. Beder (1995) explains that VaR is not a unified defined concept. The author investigated VaR calculations by dealers and end-users and found that VaR for the same portfolio can differ significantly. VaR is extremely dependent on parameters, data, assumptions and methodology. Rockafellar and Uryasevb (2002) explain that it is better to use conditional value-at-risk (CVaR) than VaR when it comes to measuring risk in financial markets. The reason is that CVaR is additive, can identify dangers beyond VaR and can solve larger scale portfolios optimization problems by using efficient and robust Linear Programming (LP) algorithms.

Grossman and Zhou (1993) create a model where the investor is exposed to a portfolio drawdown constraint. At each point in time the investor is not allowed to lose more than a fixed percentage of the maximum value of the portfolio achieved up to that point. The author shows that for a constant relative risk aversion utility function the investors optimal investment strategy is to invest a fraction of wealth that is proportional to the surplus W[t]-v*M[t] where W[t] is the wealth at time t, v drawdown parameter between 0 and 1 and M[t] is the maximum value that the portfolio was worth. Alexandera and Baptistab (2006) argue that drawdown constraints indeed can decrease portfolio variance however drawdown constraint can also lead to mean-variance inefficiency. In the presence of a benchmark a drawdown constraint can increase the portfolio variance and tracking error volatility hence reducing an investment manager’s ability to track a benchmark.

2 Theoretical Modeling

In order to illustrate the simple nature of drawdown constraints let assume that we have one weight vector **W**, one matrix with random returns **R** and one drawdown vector **DR** as seen in Table 1. We now multiply **R** with **W** which will give us a portfolio return vector called **Z** which can be seen on the second row in Table 1. Now the portfolio return vector is a time series that contains the portfolio return from t=1 to t=n. We assume that in each point in time the portfolio return has to be larger than the drawdown vector **DR**.

Table 1 : Basic DrawDown Constraint

$$W=\begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}, R=\begin{bmatrix} -5.00000 & -2.00000 & -14.00000 & 10.00000 \\ 3.00000 & 14.00000 & 11.00000 & 12.00000 \\ 11.00000 & -9.00000 & -3.00000 & 13.00000 \\ -4.00000 & 13.00000 & 2.00000 & 15.00000 \end{bmatrix}, DR=\begin{bmatrix} dr \\ dr \\ dr \\ dr \end{bmatrix}$$

$$Z = \begin{bmatrix} PR_{t=1} \\ PR_{t=2} \\ PR_{t=3} \\ PR_{t=4} \end{bmatrix} = \begin{bmatrix} -5.\,w_1 - 2.\,w_2 - 14.\,w_3 + 10.\,w_4 \\ 3.\,w_1 + 14.\,w_2 + 11.\,w_3 + 12.\,w_4 \\ 11.\,w_1 - 9.\,w_2 - 3.\,w_3 + 13.\,w_4 \\ -4.\,w_1 + 13.\,w_2 + 2.\,w_3 + 15.\,w_4 \end{bmatrix} > \begin{bmatrix} \boldsymbol{dr} \\ \boldsymbol{dr} \\ \boldsymbol{dr} \\ \boldsymbol{dr} \end{bmatrix}$$

Each element in the drawdown vector is the same and can be chosen by the investor. If the investor selects a too high value of **dr** i.e. +10 then the optimization will not find any feasible solution and if the investor selects a too low value of **dr** i.e. -10 then the constraint will not be binding. Due to these difficulties we assume that the investor does not directly select a value for **dr**. The optimization will give us an optimal drawdown constraint that will maximize the risk adjusted returns (RAR). Note also that if an investor is faced with a column dominate return matrix (as in our empirical section) then the least square solution to **R.W**=**ER** will give the investor the optimal portfolio allocation for any given expected return. Such decomposed method tends to be faster than the standard QPSolver that works on the square of the residuals. We will start by simulating some data. We will then test four optimized investment strategies as seen in Table

2. The first one minimizes portfolio variance, the second one maximizes risk adjusted returns, the third one maximizes expected return and the last one maximizes expected return and minimizes drawdown.

Table 2: Optimization: Objective Functions and Constraints

Objective and Solver	Objective Function	Constraints
Minimize Portfolio Variance QPSolve	Min [Transpose(W).Q.W]	Transpose(W).S=1 -1>w[i]<1
Maximize Risk Adjusted Return QPSolve	Min [Transpose(W).Q.W - Transpose(W).ER]	Transpose(W).S=1 -1>w[i]<1
Maximize Expected Return LPSolve	Max [Transpose(W).ER]	Transpose(W).S=1 -1>w[i]<1
Optimized DrawDown LPSolve	Max [ur +dr]	Transpose(W).S=1 -1>w[i]<1 Transpose(W).ER>ur Z>dr

We can see in Table 2 that the first two investment strategies use Quadratic Programming (QP) to find the optimal allocation. This has to do with the fact that such optimizations involve the covariance matrix **Q** which has quadratic terms i.e. squared weights. The two last investment strategies can be solved by using Linear Programming (LP) since both the objective functions and the constraints are linear. The portfolio expected return and the drawdown constraint are both linear. Note that a percentage drawdown constraint which refers to the portfolio value in the previous period would not work since it is not linear. It should also be said that **S** is simply a vector with 1's which means that Transpose(W).S=1 -> w[1] + w[2] + w[n] = 1. Note that **ur** is the threshold for the expected return and **dr** is the threshold for drawdown. The higher the value of **ur** is the larger the portfolio expected return will be. The higher the value of **dr** the less drawdown the portfolio will have hence we want to maximize such a value since we want as little drawdown as possible.

"Minimize Portfolio Variance" = [0.0072578, 0.059975, 8.2636]

"Maximize Risk Adjusted Returns" = [0.051710, 0.20079, 3.8830]

"Maximize Expected Returns" = [0.91597, 0.36311, 0.39642]

"Optimized DrawDown" = [0.041751, 0.16248, 3.8917]

[$dr = 0.010148$, $ur = 0.16248$]

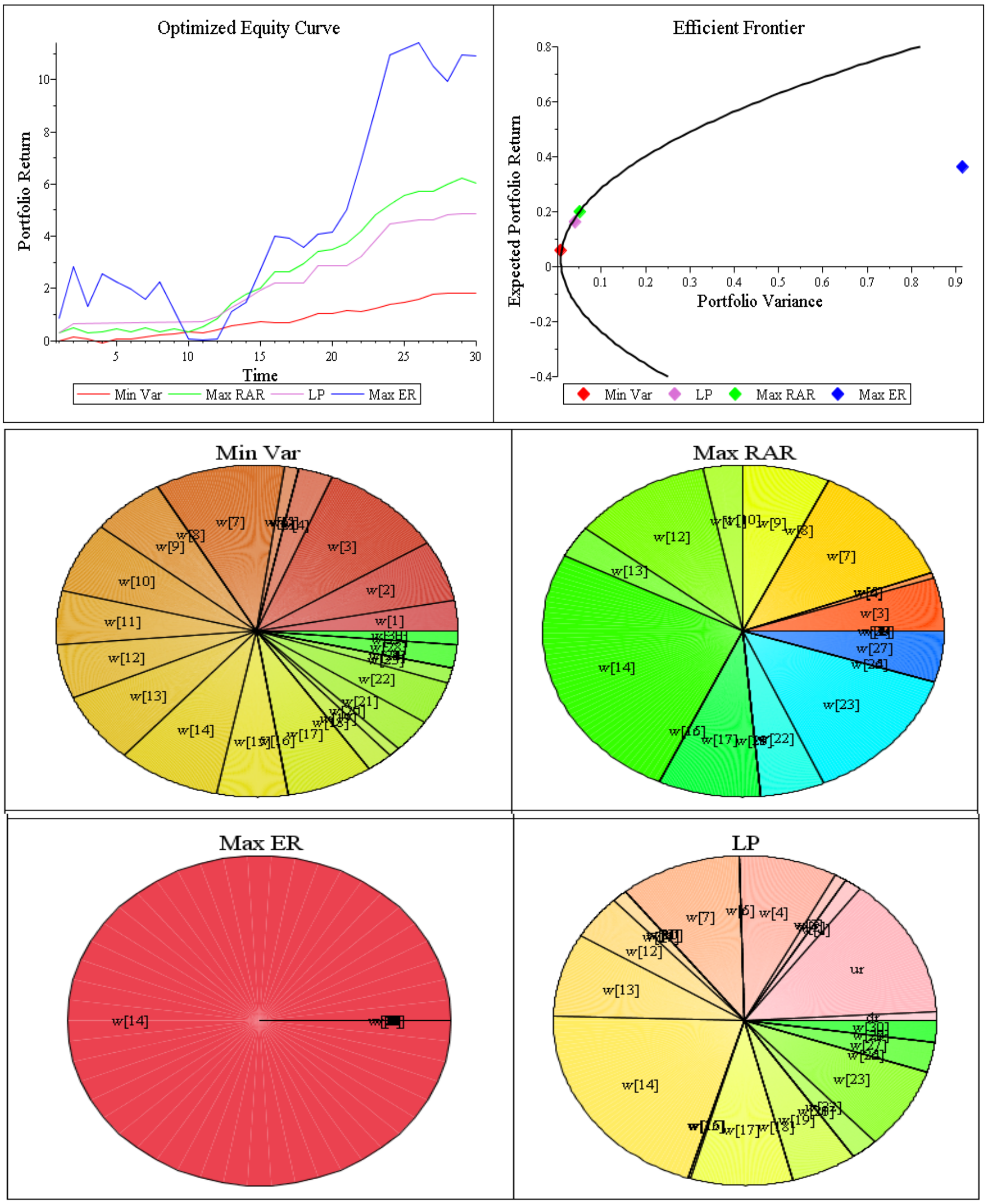


Figure 1: Simulated Data and Investment Strategies

In Figure 1 we can see the performance data [portfolio variance, portfolio expected return and portfolio risk adjusted returns] and the optimized equity curves for the four strategies. We can also see the efficient frontier and the allocations for our strategies.

We can see that the optimal value for **dr** was 0.010 and the optimal value for the expected return **ur** was 0.16. We can see that our optimized drawdown strategy (LP) actually had a higher risk adjusted return than the traditionally "maximize risk adjusted return" strategy but despite such a fact, due to the lower expected return, it was located slightly below the "maximize risk adjusted return" strategy on the efficient frontier. The interesting thing with our optimized investment strategy is that it is flexible. The investor can maximize expected return for a given drawdown or minimize the drawdown for a given expected return or both simultaneously. The location on the efficient frontier will depend on such selection. The efficient frontier was mapped out by using the two-fund separating theorem.

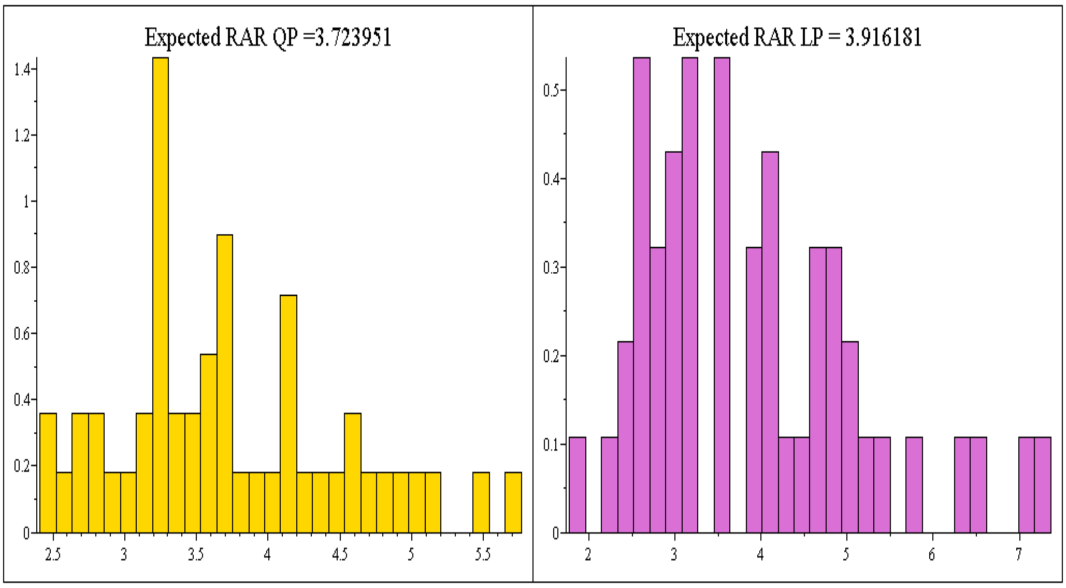


Figure 2: Simulated Data and Risk Adjusted Returns (RAR)

We can now compare the two investment strategies; maximize risk adjusted returns and the optimized drawdown strategy. We can see in Figure 2 the distribution of risk adjusted returns for our two strategies after 50 simulations. For each simulation 50 return series where created. We can see that our optimized drawdown strategy (LP) has actually a higher expected risk adjusted return than the "maximize risk adjusted return" strategy.

3 Empirical Modeling

We can now test our four investment strategies on some empirical data as seen in Figures 3 and 4. The dataset consists of monthly data (in total 73 return observations for each stock) for approximately 469 SP500 stocks for the period 2003-2009. In Figure 3 we simply ran the optimization over the entire sample period to produce the optimized equity curve. In exhibit-6 we split the dataset into two. The return for all stocks from t=1 to t=34 was used to optimized the portfolio while the return from t=35 to t=73 was used for forward testing. Both exhibits contain two figures each. In the first figure the "Maximize Expected Return" strategy is included and in the second figure it is excluded. This has to do with the fact that such strategy masks a lot of the volatility in the other strategies due to its high return. After each optimization we then used the optimal weight to evaluate **Z** which contains the portfolio returns to get the portfolio return time series. The expected portfolio variance and the portfolio expected return was simply found by evaluating W'.Q.W and W'.ER with the optimal portfolio weights and the expected risk adjusted return was simply found by dividing the previous two with each other.

We can see that our optimized drawdown strategy had lower risk adjusted returns that then "Maximize Risk Adjusted Return" investment strategy for the forward testing. This had to do with the fact that the expected return for our

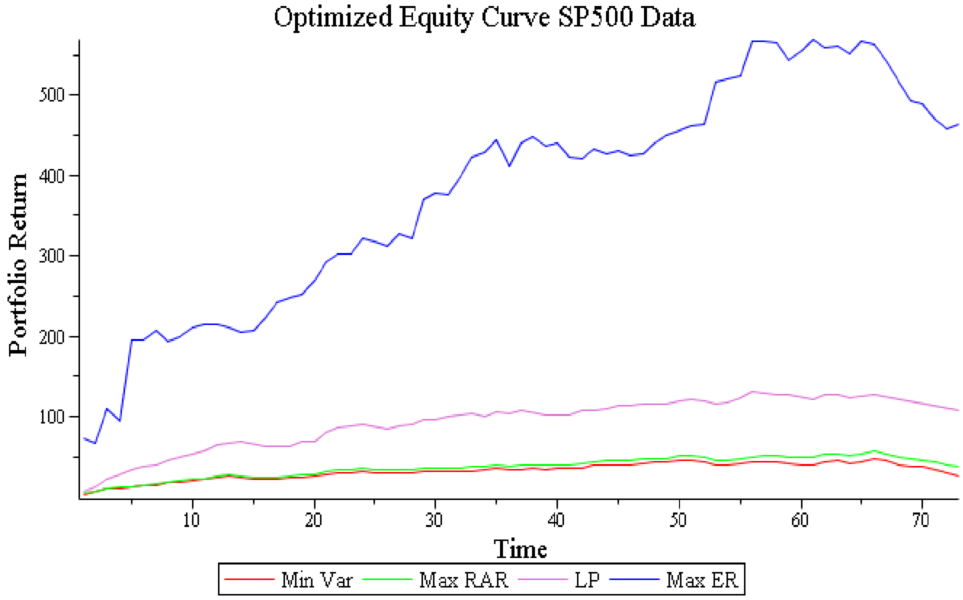
"Minimize Portfolio Variance" = [3.6015, 0.36250, 0.10065]

"Maximize Risk Adjusted Returns" = [3.6706, 0.50505, 0.13759]

"Maximize Expected Returns" = [441.89, 6.3637, 0.014401]

"Optimized DrawDown" = [12.023, 1.4844, 0.12347]

$[dr = -2.5739, ur = 1.4844]$



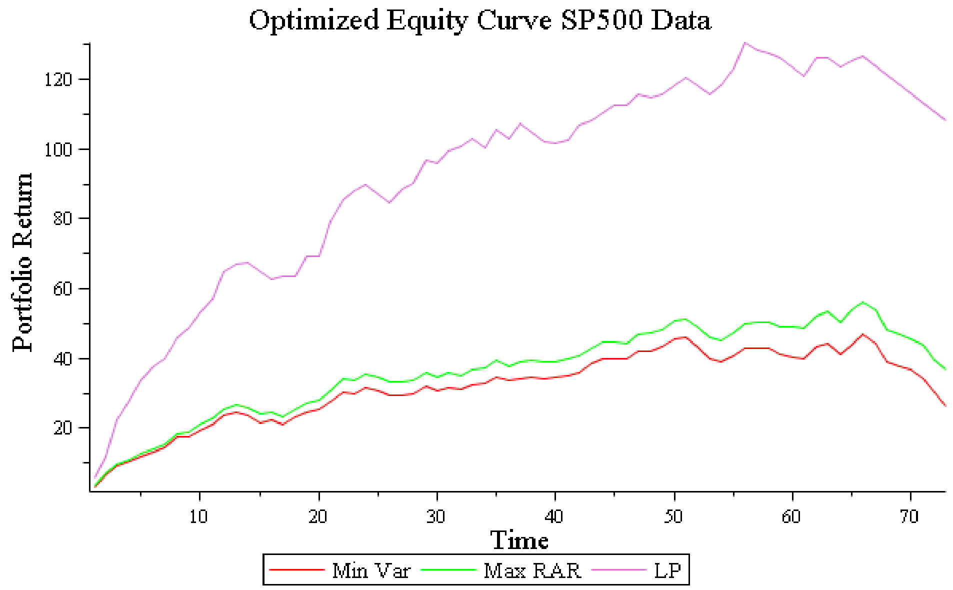


Figure 3: Optimized Equity Curve Empirical Data

"Minimize Portfolio Variance" = [0.10943, 1.1194, 10.229]

"Maximize Risk Adjusted Returns" = [0.95637, 2.8737, 3.0048]

"Maximize Expected Returns" = [535.54, 12.584, 0.023497]

"Optimized DrawDown" = [58.243, 7.9481, 0.13647]

[$dr = 2.0047$, $ur = 7.9481$]

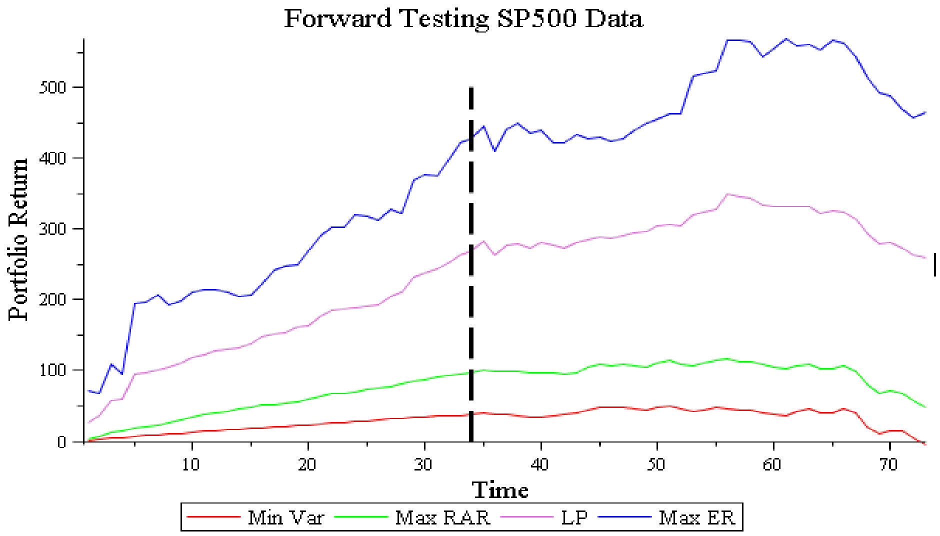


Figure 4: Forward Testing Empirical Data

strategy was 6 times as high as the expected return for the “Maximize Risk Adjusted Return” which resulted in that the return volatility increased as well. We can also see that none of the strategies did exceptional well during the forward testing. The only strategy that performed reasonable was the “Maximize Expected Returns” strategy. Both the “Minimize Portfolio Variance” and the “Maximize Risk Adjusted Returns” strategies had similar performance during the forward testing period. Note that we have not included any short positions in our investment benchmarks. Introducing short positions will generally lead to lower portfolio return variance.

4 Conclusions

We have in this paper discussed portfolio drawdown constraints and its relationship to traditional portfolio optimization. We have shown that simple drawdown constraints are indeed linear which means that portfolio risk can be model with linear optimization tools such as Linear Programming (LP). We have used both simulated and empirical data to compare the performance of such investment model with the performance of the traditionally Quadratic Programming (QP) investment models. The conclusion was that the performance was on par with the traditional portfolio optimization framework. Once the optimal allocation is found VaR for these portfolios can then be estimated directly instead of using computer intensive Monte Carlo methods. The maximize expected return and minimize portfolio drawdown investment strategy proposed in this paper is also very flexible. An investor can augment such strategy by for example maximize expected return for a given portfolio drawdown or minimize portfolio drawdown for a given expected return or do both which is the investment strategy proposed in this paper. All of these optimizations are solved with Linear Programming (LP) which has been proved to be very fast and stable.

Another thing to note is that drawdown constraints that are defined as the percentage change between the portfolio value in this period and the portfolio value in the previous period are not linear hence LP cannot be used. Instead the investor has to start with an initial portfolio value of let say 10 000 USD and then specify how much capital he is willing to risk on each trading day i.e. if a portfolio value of 9500 after the first trading day is acceptable then his drawdown constraint is -500. Now over time the percentage loss will not be constant. During the first trading day his potential percentage loss will be (9500-10000)/10000 = -5%. During the second trading day his potential percentage loss will be (9000-9500)/9500 = -5.26%. During the third trading day his potential percentage loss will be (8500-9000)/9000 = - 5.5%. This is not a very attractive feature.

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Appendix 1