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The optimal fuzzy portfolio strategy with option hedging

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Abstract

Owing to the fluctuation of financial market from time to time, the risk-free interest rate, volatility and stock price etc may occur imprecisely in the real world. Therefore, it is natural to consider the related parameters fuzzy. Portfolio selection concerns the allocation of wealth to assets such that return is maximized and risk is minimized. The institution hedges the asset's value using put options.This paper proposed the optimal fuzzy hedging portfolio strategy with options, maximizing the expected value of the portfolio and minimizing its Value-at-Risk. One problem with the model for multi-objective portfolio selection is that it is not easy to find a

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trade-off between the objectives due to a non-smooth multi-objective optimization problem. We propose a fuzzy programming algorithm to solve the optimization model. As an application, we calculate the optimal invest shares of stocks and put options for an investor seeking for maximum profit and meanwhile reducing risk.

Mathematics Subject Classification: O29

Keywords: Option portfolio, Fuzzy programming algorithm, Hedging, Multi-objective

1 Introduction

Portfolio selection concerns the allocation of wealth to assets such that return is maximized and risk is minimized. The best known mathematical model for portfolio selection is the Markowitz model [1]. The Markowitz model measures return by the expected value of the random portfolio return and risk by the variance of the portfolio return.There are a plenty of research on risk. A standard benchmark for firm-wide measures of risk is value at risk (VaR) [2]. Inspired by the concept of coherence introduced by Artzner et al. [3], very promising approaches have been published in the recent years using coherent risk measures for optimizing portfolios with asymmetric return distributions. Under these the conditional value-at-risk and the conditional drawdown have become the most popular ones. Rockafellar and Uryasev [4], Krokhmal et al. [5], Topaloglou et al. [6], and Chekhlov et al. [7] show how portfolio optimization problems based on these risk measures can be transformed and solved by linear programs using Monte Carlo simulations to approximate the return distribution.

In a portfolio with many instruments, this procedure may be computationally expensive or even infeasible. There is a related literature that investigates the optimal portfolio allocation with options. Liu and Pan [8] model stochastic volatility and jump processes and derive the optimal portfolio policy for a CRRA investor between one stock, a 5% OTM put option, and cash. Although they get an analytical closed-form for the optimal option allocation, they need to specify the parametric model and estimate its parameters. Jones [9] studies optimal portfolios to exploit the apparent put mispricing, in particular for short-term OTM put options. He estimates a general nonlinear latent factor model for put daily returns and maximizes a constrained mean-variance objective. Glasserman et al. [10] proposed an effective Monte Carlo simulation: using the information included in Delta- Gamma- Theta to select effective sample distribution.

The current papers of option portfolio are derived based on BS formula, and the input parameters of the Black–Scholes formula are usually regarded as the precise real-valued data; that is, the input data are considered as the real numbers. However, in the real world, some parameters in the Black–Scholes formula cannot always be expected in a precise sense. For instance, the volatility, stock price, and also the Greek parameters delta, gamma, theta, rho are fuzzy numbers. In fact, there are a plenty of literatures introducing fuzzy theory in the financial market, such as Wu and Sen [11] applied fuzzy sets theory to the Black–Scholes formula, making the financial analyst who can pick any European option price with an acceptable belief degree. Papadopoulos [12] presented an application of a new method of constructing fuzzy estimators for the parameters of a given probability distribution function, using statistical data. In the models of hedging, many papers suppose the Greek parameters are equal to zero, Gao [13] proposes a general linear programming model with risk bounds on all the Greek letters . In practice, the Greek letters of portfolio are difficult stable at the value of zero, and the investors are likely to keep them minimum.

As far as we know, there are no literatures concerning portfolio of options with return maximizing and risk minimizing, meanwhile. In our paper, we discuss the minimize risk of the asset over the next τ periods, maximize the return of the portfolio with option under fuzzy environment. We proposed a multi-objective model, one problem of whose is that it is not easy to find a trade-off between the objectives because this is a non-smooth multi-objective optimization problem. In order to solve this problem, we propose a fuzzy programming algorithm to solve the optimization model.

The rest of this paper is organized as follows. In Section 2, optimization model is described including objectives and constraints. In Section 3, explains the methodology employed to construct optimal portfolios. In Section 4, numerical examples and concluding remarks. We report on numerical tests in which we compare our model with the former model in aspect of profit. Finally, we present some concluding remarks in Section 5.

2 The optimization model

We first define some terminology. The price of an underlying asset S_t, whose process is governed by the following stochastic differential equation:

$$dS_t = S_t \mu dt + \sigma S_t dB_t$$

where μ and σ are the drift and the diffusion of the asset value, and B_t is a standard Brownian motion. The institution hedges the asset's value using put options. Define the market price today (i.e., time t) of a τ -period put as $P_t = P(S_t, K, r, \tau, \sigma)$, where the strike price of the option equals K and the interest rate is r. For simplicity, we assume that all options are priced according to the Black-Scholes option pricing model [14]:

$$P_t = Ke^{-r\tau}\Phi(d_1) - S_t\Phi(d_2)$$

where $d_1 = \dfrac{\log\dfrac{K}{S} - \left(r - \dfrac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$, $d_2 = \dfrac{\log\dfrac{K}{S} - \left(r + \dfrac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}$.

2.1 Objectives of the model

(1) The total return of options portfolio is $E\left(\sum_{i=1}^{n} S_{T,i}w_i + \sum_{j=1}^{m} p_{i,j}x_j\right)$ which the investor wishes to maximize, where $P_{i,j} = K_{i,j}e^{-rT}\Phi(d_{i,j,1}) - S_{i,0}\Phi(d_{i,j,2})$ means the put option price based on the $i-$ th stock and $j-$ th strike price.

$$d_{i,j,1} = \frac{\ln\left(K_{i,j} / S_{i,0}\right) - \left(r - \frac{1}{2}\sigma_i^2\right)T}{\sigma_i\sqrt{T}}, d_{i,j,2} = \frac{\ln\left(K_{i,j} / S_{i,0}\right) - \left(r + \frac{1}{2}\sigma_i^2\right)T}{\sigma_i\sqrt{T}}$$

$\Phi(\cdot)$ is the cumulative normal distribution. A put options strategy consist of long positions, $h_{i,j}$ in m options of the i stocks with strike prices $K_{i,j}$.

(2) The institution is concerned about its exposure to the asset over the next τ periods, and the relevant measure of risk is the position's VaR. Define $VarR_{t+\tau}$ as the dollar loss at the $\alpha\%$ level of the distribution on the institution's exposure

relative to investing the time t value of the portfolio in the risk-free asset. This future value provides the natural benchmark since a riskless portfolio will thereby yield a VaR of zero. The VaR of a position translates to the statistical statement: "With $(1-\alpha)$% confidence, the dollar loss in the future value of the cash flow in τ periods will not exceed $\$VaR_{t+\tau}$". To calculate this VaR, note first that the conditional distribution of the future value of the unhedged asset is lognormal. The VaR of the unhedged position is

$$VarR_T = \sum_{i=1}^{n} \alpha_i S_{i,0} \exp(r\tau) - \left[\sum_{i=1}^{n} \left(1 - \sum_{j=1}^{m} h_{i,j} \right) S_{i,0} \exp(\theta_i(\alpha)) + \sum_{i=1}^{n} \sum_{j=1}^{m} h_{i,j} K_{i,j} - \sum_{i=1}^{n} \sum_{j=1}^{m} h_{i,j} P_{i,j} \exp(r\tau) \right]$$

$$\theta_i(\alpha) = \left(\mu_i - \frac{1}{2}\sigma_i^2 \right) T + c(\alpha)\sigma_i \sqrt{T}$$

$c(\cdot)$ is the cut-off point of the cumulative distribution of a standard normal.

2.2 The constraints

$(1')$ The first constraint is the investment budget:

$$\sum_{i=1}^{n} s_{0,i}\alpha_i + \sum_{j=1}^{m} p_{0,j} x_j \le a_0$$

where $c_{0,i}$, $p_{0,j}$ are respectively the cost of buying an call or put option and a_0 means the initial amount in which the investor wish to invest.

$(2')$ We assume that the exposure is never fully hedged, i.e., $\sum_{i=1}^{n} \sum_{j=1}^{m} h_{i,j} < 1$. In general, this constraint will not bind for reasonable levels of expenditures on hedging.

$(3')$ $\alpha_i \geq 0, w_i \geq 0, x_j \geq 0$.

Notice that all variances are fuzzy numbers related to real finance market except a_i, w_j which are investment shares.

3 Introduction to solutions to fuzzy multi-objective problems

In this section we solve a general fuzzy multi-objective problem, and we will follow the steps to find the optimal solution in the next section.

3.1 A general fuzzy multi-objective model

Let $A = (a_{ij})_{m\times n}$, $B = (c_{ij})_{r\times n}$, $b = (b_1, b_2 \cdots b_m)^T$, $x = (x_1, x_2, ..., x_m)^T$, $Z = (Z_1, Z_2, ..., Z_r)^T$, a multi-objective fuzzy programming is described as:

$$\max Z = Cx$$
$$s.t \begin{cases} Ax \leq b \\ x \geq 0 \end{cases} \quad \text{(P1)}$$

Because the objective functions of multi-objective programming are more than one, it is difficult to reach a certain point for all of the objective functions, to whose maximum, that is the optimal solution is usually does not exist. Therefore, it needs to make a compromise plan making each target function as large as possible in a specific problem. And fuzzy mathematical programming method can deal with the problem, which will turn the multi-target model to a single one.

3.2. One solution of multi-objective linear programming model with fuzzy mathematics

Step1: to solve every single maximum objective $Z_i, i = 1, 2, ..., r$ under the constrains $(1')\,(2')$,

$$Z_i^* = \max\left(Z_i \middle| Z_i = \sum_{j=1}^{n} c_{ij}, Ax \le b, x \ge 0 \right), i = 1, 2 \cdots r$$

To choose $d_i, (d_i > 0)$ as the corresponding fuzzy telescopic factor for each target $Z_i, i = 1, 2, ..., r$. Generally, fuzzy telescopic factors are choosen according to various sub-targets importance, that is the more important goal, the smaller the flexible index should be. In this paper, let $d_i = Z_i^* - Z_i^-$, where $Z_i^- = \min Z_i$, which is solved following the same method as Z_i^*.

Step 2: constructing fuzzy objective $\tilde{M}_i$ of target Z_i, whose membership function is:

$$\tilde{M}_i(x) = g_i\left(\sum_{j=1}^{n} c_{ij} x_j\right) = \begin{cases} 0, & \sum_{j=1}^{n} c_{ij} x_j < Z_i^* - d_i \\ 1 - \dfrac{1}{d_i}\left(Z_i^* - \sum_{j=1}^{n} c_{ij} x_j\right), & Z_i^* - d_i \le \sum_{j=1}^{n} c_{ij} x_j < Z_i^* \\ 1, & Z_i^* \le \sum_{j=1}^{n} c_{ij} x_j \end{cases}$$

Let $\lambda = \tilde{M}(x) = \bigcap_{i=1}^{r} \tilde{M}_i(x)$, the multi-objective problem P1 is transformed as:

$$\begin{cases} \max Z = \lambda \\ 1 - \dfrac{1}{d_i}\left(Z_i^* - \sum_{j=1}^{n} c_{ij} x_j\right) \ge \lambda, \qquad i = 1, 2, \ldots, r \\ \sum_{j=1}^{n} a_{kj} x_j \le b_k, \qquad k = 1, 2, \ldots, m \\ \lambda \ge 0, \qquad x_1, x_2, \ldots, x_n \ge 0 \end{cases} \tag{P2}$$

This is a single objective linear programming solved with LINGO easily.

4 Numerical examples and concluding remarks

In this section, we calculate the maximum profit minimizing risk under constrains with fuzzy programming algorithm. We choose the parameters as Table 1:

Table 1 : Stocks and option data for portfolio compositions used in the example

Securities	S_1	S_2	P_1	P_2
Risk return characteristics				
$\mu(\%)$	11.00	8.05	-34.43	-33.61
$\sigma(\%)$	26.86	16.3	97.07	83.89

Option data based on simulations.

Example assumes a time horizon of one year. The annual interest rate for the riskless investment is given by $r = 5\%$.

Table 2: Characteristics of available put and call options

option	underlying	market price	option price	exercise
Put P_1	Stock S_1	$S_1(0)$=15.59	$P_1(0)$=1.87	K_{12}=16.50
Put P_2	Stock S_2	$S_2(0)$=13.71	$P_2(0)$=1.48	K_{22}=15.00

Taking these base levers of the parameters, we calculated the optimal invest strategy by LINGO. The result is with believe degree 1 to buy one share stock S_1, one share stock S_2, one share put P_1, zero share put P_2, and get the max profit 34.469 with the minimum risk.

5 Conclusions

This paper contributes to four directions. First, it considers the derivative instruments as the investment instrument, not just as hedging instruments. Second, we construct multi-objective optimal portfolio strategy with options, considering profit and risk meanwhile. Thirdly, introducing fuzzy mathematics to the option portfolio, closed to real finance market. Additionally, proposing a fuzzy programming algorithm to solve the optimization model.

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