Theoretical Mathematics & Applications, vol.2, no.1, 2012, 23-34
ISSN: 1792- 9687 (print), 1792-9709 (online)
International Scientific Press, 2012

Differential equations of amplitudes and frequencies of equally-amplitudinal oscillations of the second order

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Abstract

Study of oscillations leads to a need to determine Sturm's two-amplitudinal but equally-frequent equations with positive coefficient (according to results from Prodi [1]). This is shown by Floké theorem [2] and monograph Babakov [3].

The problem of finding of amplitudes and frequencies hasn't been solved even in Sturm theory in Amrein et al. [4], because it is nonlinear problem even though originates from linear oscillations of the second order.

In this paper are set up differential equations for amplitude and frequency function of solution of equally-amplitudinal oscillations, and are given theorems on solution existence as well as the important examples.

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Article Info: *Received* : December 1, 2011. *Revised* : January 12, 2012
Published online : March 5, 2012

Mathematics Subject Classification : 34E05, 34K11

Keywords: Differential equation, Oscillation, Frequency function, Amplitude function, A series iteration method

1 Introduction and Preliminaries

Let linear homogenous differential equation of the second order

$$y''+a(x)y'+b(x)y=0 \tag{1.1}$$

where $a(x)$ and $b(x)$ are given continuous functions, satisfies conditions of Picard- Lindelöf theorem on existence and uniqueness of solution over half-open interval $[0,+\infty)$. Introducing the substitution

$$y=z\exp\left(-\frac{1}{2}\int a(x)dx\right) \tag{1.2}$$

equation (1.1) reduces to canonical equation

$$z''+\phi(z)z=0 \tag{1.3}$$

where

$$\phi(x)=b(x)-\frac{a'(x)}{2}+\frac{a^2(x)}{4}. \tag{1.4}$$

It is known, that for positive $\phi(x)$ on $[0,+\infty)$ its integral diverges, so equation (1.3) has oscillatory solutions z_1 and z_2 which are continuous and differentiable. In the papers [5,6,7,8,9] by series iteration method we have shown that the solutions of equation (1.3) have following form

$$\begin{aligned} z_1=\cos_{\phi(x)}x=1-\iint\phi(x)dx^2+\iint\phi(x)dx^2\iint\phi(x)dx^2-\\ -\iint\phi(x)dx^2\iint\phi(x)dx^2\iint\phi(x)dx^2+\cdots \end{aligned} \tag{1.5}$$

$$\begin{aligned} z_2=\sin_{\phi(x)}x=x-\iint x\phi(x)dx^2+\iint\phi(x)dx^2\iint x\phi(x)dx^2-\\ -\iint\phi(x)dx^2\iint\phi(x)dx^2\iint x\phi(x)dx^2+\cdots \end{aligned}\quad . \tag{1.6}$$

For $\phi(x)=k^2=const.$ solutions are analytical function which are equal to Euclidean sine and cosine, while for arbitrary and positive $\phi(x)$ have a lot of similarities and mutual characteristics with sine and cosine. That is why we named functions (1.5) and (1.6) general or Sturm's sine and cosine with basis $\phi(x)$. For them all analogies with standard trigonometry have been applied (zeros, inflections, Sturm's arrangements of zeros and extremes).

From approximate formulas, which have proven in [6], we have

$$\cos_{\phi(x)} x \approx \cos\left(x\sqrt{\phi(x)}\right), \tag{1.7}$$

$$\sin_{\phi(x)} x \approx \sin\frac{\left(x\sqrt{\phi(x)}\right)}{\sqrt{\phi(x)}}. \tag{1.8}$$

Here is obvious that

1° Frequency functions for both solutions are equal and approximately are $G(x)\approx x\sqrt{\phi(x)}$.

2° Amplitudes in general case are different. For $\cos_{\phi(x)} x$ is $A_1 \approx 1$, and for, $\sin_{\phi(x)} x$, $A_2 \approx \frac{1}{\sqrt{\phi(x)}}$. This verifies Prodi's theorem ([1]): if $\phi(x)\to+\infty$, when $x\to+\infty$, then one solution remains bounded and second tends to zero. In other words, if $\phi(x)\to+\infty$ solutions are two-amplitudinal but equally-frequent.

So, harmonic oscillations are equally-amplitudinal, but are they also only equally-amplitudinal or there are many types of them, we consider that these questions haven't been sufficiently elaborated and emphasized. Therefore, in this paper we shall try to determine simple equally-amplitudinal solutions

$$y_1(x)=A(x)\cos G(x)\,,\ y_2(x)=A(x)\sin G(x) \tag{1.9}$$

with amplitude $A(x)$ and unique common frequency $G(x)$.

From general differential equation of the second order

$$\begin{vmatrix} y'' & y' & y \\ y_1'' & y_1' & y_1 \\ y_2'' & y_2' & y_2 \end{vmatrix} = 0$$

expanding by first row and dividing by Wronskian

$$W = W\left(y_1, y_2\right) = y_1' y_2 - y_1 y_2' = -A^2\left(x\right) G'\left(x\right)$$

we have

$$y'' - \frac{y_1'' y_2 - y_1 y_2''}{W} y' - \frac{y_1'' y_2' - y_1' y_2''}{W} y = 0.$$

After elementary calculation we obtain equation

$$y'' - \left(2\frac{A'(x)}{A(x)} + \frac{G''(x)}{G'(x)}\right) y' + \left(2\left(\frac{A'(x)}{A(x)}\right)^2 + \frac{A'(x)G''(x)}{A(x)G'(x)} - \frac{A''(x)}{A(x)} + \left(G'(x)\right)^2\right) y = 0 \quad (1.10)$$

Equation (1.10) has sense if $A(x) \neq 0$ and $G'(x) \neq 0$, i.e., if $A(x)$ has no zeros on $[0, +\infty)$ and if $G(x)$ is monotonous. If from (1.4) we calculate $\phi(x)$, then we obtain

$$\phi(x) = \left(G'(x)\right)^2 + \frac{1}{2}\frac{G'''(x)}{G'(x)} - \frac{3}{4}\left(\frac{G''(x)}{G'(x)}\right)^2. \quad (1.11)$$

Comparing (1.10) and (1.1) we have

$$\begin{aligned} a(x) &= -\frac{y_1'' y_2 - y_1 y_2''}{W} = -\left(2\frac{A'(x)}{A(x)} + \frac{G''(x)}{G'(x)}\right), \\ b(x) &= -\frac{y_1'' y_2' - y_1' y_2''}{W} = 2\left(\frac{A'(x)}{A(x)}\right)^2 + \frac{A'(x)}{A(x)}\frac{G''(x)}{G'(x)} - \frac{A''(x)}{A(x)} + \left(G'(x)\right)^2. \end{aligned} \quad (1.12)$$

Thus, frequency function $\phi(x)$ is the same for equations (1.10) i (1.1).

Let's notice that because of (1.2) the same cannot be applied for amplitudes because $\exp\left(-\frac{1}{2}\int a(x)dx\right) = A(x)\sqrt{G'(x)}$. If with $A(y)$ we mark amplitude of equation (1.10), and with $A(z)$ amplitude of canonical equation (1.3) then from obvious equality $\max y = \max\left(A(x)\sqrt{G'(x)}\right)\max z$ follows

$$A(y) = \max\left(A(x)\sqrt{G'(x)}\right)A(z). \tag{1.13}$$

Equation (1.10) shall be identical with its canonical if applies

$$a(x) = -\left(2\frac{A'(x)}{A(x)} + \frac{G''(x)}{G'(x)}\right) = 0. \tag{1.14}$$

This differential equation gives the first integral of form

$$A^2(x)G'(x) = c_1^2 = const., G'(x) > 0. \tag{1.15}$$

Regarding the linearity of solution (1.9), equation (1.3), it sufficient in equation (1.15) to take $c_1 = 1$.

So, problem of differential equations of frequency and amplitudes we first observe for narrower classes of functions. First, for the most general case $a(x) \neq 0$ solution $G(x)$ is not simple. Second, more elementary, when condition (1.14) applies, i.e., $a(x) = 0$, connection of amplitude and frequency is given by equation (1.15).

2 Main Results

In this Section in the form of theorem we present differential equations for amplitudes and frequencies.

Theorem 2.1. *General unambiguous amplitude* $A(x)$, *of differential equations* (1.10), *that is* (1.4), *which solutions are given by equation* (1.9), *with successive approximations of Picard is obtained from differential equation*

$$A''(x) + b(x)A(x) = \frac{c_1^4}{A^3(x)}. \tag{2.1}$$

Proof. Let a general equation (1.1) is given, where $a(x)$ and $b(x)$ are given continuous functions which satisfies Lipschitz condition. Comparing (1.1) and (1.10) we obtain nonlinear system of differential equations of the second order

$$\begin{aligned} a(x) &= -\left(2\frac{A'(x)}{A(x)}+\frac{G''(x)}{G'(x)}\right), \\ b(x) &= 2\left(\frac{A'(x)}{A(x)}\right)^2+\frac{A'(x)}{A(x)}\frac{G''(x)}{G'(x)}-\frac{A''(x)}{A(x)}+\left(G'(x)\right)^2. \end{aligned} \tag{2.2}$$

From here, we obtain the equation of the fourth order for $A(x)$ or for $G(x)$. Out of the first equation of the system we found

$$G(x)=c_1^2\int\frac{\exp\left(-\int a(x)dx\right)}{A^2(x)}dx+c_2. \tag{2.3}$$

Substituting (2.3) in second equation of the system (2.2) we obtain differential equation of amplitude

$$A''(x)+a(x)A'(x)+b(x)A(x)=c_1^4\frac{\exp\left(-2\int a(x)dx\right)}{A^3(x)}+c_2\ . \tag{2.4}$$

For $a(x)\neq 0$, (2.4) is nonlinear differential equation of the second order. Evaluating this we find two more constants c_3 and c_4.

Let $a(x)=0$. Since equation (2.4) transforms into equation (2.1), for it is important not just solution existence but also possible quadratures. From here $b(x)=\phi(x)$, where $\phi(x)$ is given with (1.12). Equation (2.1) has solutions for $A(x)\neq 0$. Substituting $A'(x)=u$ it can be presented as system of equations

$$A'(x) = u = f_1(x,u,A)$$
$$A''(x) = u' = -b(x)A(x) + \frac{c_1^4}{A^3(x)} = f_2(x,u,A).$$

Since $A(x)$ and $b(x)$ are continuous functions, f_1 and f_2 are also continuous, and Lipschitz condition gives

$$\left| f_2(x,u_1,A_1) - f_2(x,u_2,A_2) \right| = \left| c_1^4 \frac{A_2^3 - A_1^3}{(A_1A_2)^3} + b(x)(A_2 - A_1) \right| \le$$
$$\le |A_1 - A_2| \left(c_1^4 \frac{\max(|A_1|,|A_2|)}{\min(|A_1||A_2|)^3} + |b(x)| \right) \le k|A_1 - A_2|.$$

Here with procedure described in [6,7,8], it is easy to find $A(x)$. □

Theorem 2.2. *Nonlinear equation* (1.12) *on condition that* $G'(x) \neq 0$ *and* $\phi(x)$ *continuous function, has solution which approximately can be found by Picard's successive approximations method.*

Proof. For given $a(x)$ and $b(x)$ let's solve first equation of the system (2.2) according to $A(x)$

$$A(x) = \sqrt{c_1^2}\,(G'(x))^{-1/2} \exp\left(-\frac{1}{2} \int a(x)\,dx \right). \tag{2.5}$$

When this value, along with $A'(x)$ and $A''(x)$ we replace in second equation of the system (2.2), after necessary calculation and cancellation, we obtain nonlinear differential equation of the third order for frequency function $G(x)$

$$b(x) = (G'(x))^2 + \frac{1}{2}\frac{G'''(x)}{G'(x)} - \frac{3}{4}\left(\frac{G''(x)}{G'(x)} \right)^2 + \frac{1}{4}a^2(x) + \frac{1}{2}a'(x). \tag{2.6}$$

From here, by solving we obtain not just frequency function $G(x) = G(a(x), b(x))$, but also periods of possible equally-amplitudinal solutions of general linear homogeneous differential equation (1.1).

As in formula (2.3) we already have constants c_1 and c_2, by solving (2.6) we would obtain two more new constants c_5 and c_6 by what we are finalizing this problem.

Let now $a(x)=0$, i.e. let condition (1.14) applies and its consequence is (1.15). Then (2.6) becomes nonlinear equation of the third order

$$b(x)=\left(G'(x)\right)^2+\frac{1}{2}\frac{G'''(x)}{G'(x)}-\frac{3}{4}\left(\frac{G''(x)}{G'(x)}\right)^2=\phi(x). \tag{2.7}$$

Since is difficult to solve this equation with quadratures, the only thing left is to estimate the existence of possible solutions. Substituting $G'(x)=v$ from (2.7) we obtain system of two differential equations of the first order

$$G''(x)=v'=w=f_1(x,v,w)$$

$$G'''(x)=v''=w'=2\phi(x)v+\frac{3}{2}\frac{w^2}{v}-2v^3=f_2(x,v,w).$$

Let's notice that second equation of the system is nonlinear, but is not unbounded due to condition $G'(x)=v\neq 0$. This system satisfies Lipschitz condition, which can be seen from the evaluation

$$\left|f_2(x,v_1,w_1)-f_2(x,v_2,w_2)\right|\leq\left|v_1-v_2\right|\left|2\phi(x)-2\left(v_2^2+v_1v_2+v_1^2\right)\right|+\frac{3}{2}\left|\frac{w_1^2}{v_1}-\frac{w_2^2}{v_2}\right|\leq$$

$$\leq\left|v_1-v_2\right|\left(2\left|\phi(x)\right|+2\left|v_2^2+v_1v_2+v_1^2\right|\right)+\frac{3}{2}\left|w_1^2\max\frac{1}{v_1}-w_2^2\min\frac{1}{v_2}\right|.$$

Let N is the least by module common multiple of $\left(\max\frac{1}{v_1},\min\frac{1}{v_2}\right)$. Since $G'(x)=v$, and $\phi(x)=b(x)$ is continuous and bounded function, so due to $v\neq 0$ function w is like that too. Then $2\left|\phi(x)\right|+2\left|v_2^2+v_1v_2+v_1^2\right|<k_1$, so afore mentioned estimation finally becomes

$$\left|f_2(x,v_1,w_1)-f_2(x,v_2,w_2)\right|\le k_1\left|v_1-v_2\right|+\frac{3}{2}N\left|w_1^2-w_2^2\right|<q\left(\left|v_1-v_2\right|+\left|w_1-w_2\right|\right).$$

Thus, equation (2.7) satisfies Lipschitz condition. Now we can apply a series iteration method and we can easely find solution. □

3 Classification of problems. Three types of problems

Solving of system of differential equations (2.2), for amplitude $A(x)$ or frequency function $G(x)$, and for given $a(x)$ and $b(x)$, has sense only for some subclasses of these equations. That means, that we can take additional conditions but in a way that equations (2.4) and (2.6) apply.

Example 3.1. The most important place in classification of problems have equally amplitudinal oscillations, i.e. harmonic oscillations.

Let the linear function $G(x)=\alpha x$ is given. From $G'(x)=\alpha$, $G''(x)=G'''(x)=0$ and from (2.6) we have $b(x)=\alpha^2+\frac{1}{4}a^2(x)+\frac{1}{2}a'(x)$. Especially, for $a(x)=0$, (see (2.2)), follows $b(x)=\alpha^2$. Now, canonic equation $y''+\alpha^2 y=0$ has solutions $y_1=\cos\alpha x, y_2=\sin\alpha x$ which are harmonic oscillations.

From $a(x)=0$, that is, from $A^2(x)G'(x)=c_1^2=const.$ we obtain one solution $A(x)=\frac{c_1}{\sqrt{\alpha}}=const.$ for equation of amplitude (2.1).

Example 3.2. Let linear function $G(x)=\alpha x$ is given, $a(x)=0$ and $b(x)=\phi(x)=\alpha^2=const.$. The equation of the second order (2.1), for amplitude, becomes $A''(x)+\alpha^2 A(x)=\dfrac{1}{A^3(x)}$, or equivalent to

$$A''(x)=\frac{1-\alpha^2 A^4(x)}{A^3(x)}. \tag{3.1}$$

This is differential equation which explicitly does not contains x. Substituting $A'(x)=p$ and $A''(x)=p\dfrac{dp}{dA(x)}$ equation (3.1) becomes equation with separate variables: $pdp=\left(\dfrac{1}{A^3(x)}-\alpha^2 A(x)\right)dA$. From here we obtain first integral $p^2=k_1-\dfrac{1}{A^2(x)}-\alpha^2 A^2(x)$, where k_1 is integrating constant. Further we have $p=\pm\sqrt{k_1-\dfrac{1}{A^2(x)}-\alpha^2 A^2(x)}=\dfrac{dA(x)}{dx}$. Second separation of variables, as second integral, gives $x\pm k_2=\int\dfrac{A(x)dA(x)}{\sqrt{k_1A^2(x)-1-\alpha^2A^4(x)}}$. Finally, by solving we obtain $A^2(x)=\dfrac{k_1}{2\alpha^2}+\sqrt{\dfrac{k_1^2-4\alpha^2}{4\alpha^2}}\left[\dfrac{1}{\alpha}\sin 2\alpha(x\pm k_2)\right]$, where k_1 and k_2 are integration constants. Equally amplitudinal solutions are $y_1=A_{1,2}(x)\cos\alpha x$ and $y_2=A_{1,2}(x)\sin\alpha x$. $A(x)$ can have zeros if $\sin 2\alpha(x\pm k_2)$ and $A(x)$ does not exist when this sinus is less then zero. This is not in opposition to continuality guarantied by Theorem 2.2 (for $A(x)\neq 0$). Namely, then we have opposite solutions y_1 and y_2, and zeros $A(x)$ mean vertical tangents in solutions. As a matter of fact, those are connections of two solutions y_1 and y_2. It means, when y_1 stops, further follows continuation with y_2 which is of opposite sign. Therefore, continuality continues and sustains, even when $A(x)\to 0$.

We, therefore, conclude that three types of problems are possible:

1° The simplest is to suppose that frequency $G(x)$ is known, because then $F(x)$ can be only found by derivatives $G'(x), G''(x), G'''(x)$. $A(x)$ can be easily found from (1.15), but solutions in more complex cases are also possible, as shown in Example 3.2. Knowing of form $A(x)$ can be assumed, but in a way that equation (2.1) for $A(x)$ is satisfied. From here $\phi(x)$ is determined, and from (2.7) and $G(x)$.

2° Knowing of $\phi(x)$ can also be assumed, but then $A(x)$ must be found from nonlinear equation (2.1), which is not trivial at all. It is more difficult to find $G(x)$ since for $G(x)$ we have nonlinear equation of the third order.

3° Without some, even minimal accuracy and specialization, or for $A(x)$ or for $G(x)$, this nonlinear problem is quadrature unsolvable in general case. For Physics, however, the most important is connection given by $a(x)=0$, so, if $A^2(x)G'(x)=c_1^2=const.$

ACKNOWLEDGEMENTS. The authors are thankful to the Ministry of Science and Environmental Protection of Serbia.

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