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Ćirić's type theorems in abstract metric spaces

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Abstract

In this paper we extend fixed point theorems of Ćirić ([Lj. Ćirić, On a family of contractive maps and fixed points, Publ. Inst. Math., 17(31), (1974), 45-51; Some Recent Results in Metrical Fixed Point Theory, Beograd 2003.]) from the metric space to cone metric spaces. We do not impose the normality property on the cone, but suppose only that the cone P in the real ordered Banach space E has a nonempty interior. Thus our results generalize and extend fixed point theorems of contractive mappings in several aspect (see: Remark 3.3 and Corollaries 3.4-3.8). Three examples are given to illustrate the usability of our results.

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1 Introduction

Ordered normed spaces, cones and Topical functions have applications in applied mathematics, for instance, Newton approximation method [13]-[16], [26], [27]-[29] and optimization theory [7] and [20]. In these cases an order is introduced by using vector space cones [3], [19]. Baluev [4], [15], Huang and Zhang [8], used this approach and they have replaced the real numbers by ordering Banach space and define $P-$metric and $P-$normed space. The authors there described convergence in these spaces and introduced completeness. Then they proved some fixed point theorems of contractive mappings on $P-$metric spaces. Recently, in [1], [8]- [10], [21]-[23], [25] and [27] some common fixed point theorems were proved for maps on $P-$ metric spaces. Also, in these papers, the authors usually use the normality property of cones in their results. In this paper we do not impose the normality condition for the cones.

Consistent with [7] (see also [8], [12]-[16], [18], [26]-[29]) the following definitions and results will be needed in the sequel.

2 Preliminary Notes

Let E be a real Banach space. A subset P of E is called a cone whenever the following conditions hold: (**a**) P is closed, nonempty and $P \neq \{\theta\}$;(**b**) $a, b \in \mathbb{R}, a, b \geq 0$, and $x, y \in P$ imply $ax + by \in P$; (**c**) $P \cap (-P) = \{\theta\}$.

Given a cone $P \subset E$, we define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. We shall write $x \prec y$ to indicate that $x \preceq y$ but $x \neq y$, while $x \ll y$ will stand for $y - x \in intP$ (interior of P).

There exist two kinds of cones (see [7]): normal with normal constant $k \geq 1$ and nonnormal cones.

Let E be a real Banach space, $P \subset E$ a cone and $\preceq$ the partial ordering defined by P. Then P is called normal if

$$\inf \{\|x+y\| : x, y \in P \text{ and } \|x\| = \|y\| = 1\} > 0, \tag{1}$$

or equivalently, there is a number $k > 0$ such that for all $x, y \in P$,

$$\theta \preceq x \preceq y \text{ imply } \|x\| \leq k \|y\|, \tag{2}$$

or equivalently, if $(\forall n)$ $x_n \preceq y_n \preceq z_n$ and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = x \text{ imply } \lim_{n\to\infty} y_n = x. \tag{3}$$

The least positive number satisfying (2) the normal constant of P. It is clear that $k \geq 1$. From details see [7].

Example 2.1. [26] Let $E = C^1_{\mathbb{R}}[0,1]$ with $\|x\| = \|x\|_\infty + \|x'\|_\infty$ on $P = \{x \in E : x(t) \geq 0\}$. This cone is nonnormal. Consider, for example, $x_n(t) = \frac{t^n}{n}$ and $y_n(t) = \frac{1}{n}$. Then $0 \preceq x_n \preceq y_n$, and $\lim_{n\to\infty} y_n = \theta$, but

$$\|x_n\| = \max_{t\in[0,1]} \left|\frac{t^n}{n}\right| + \max_{t\in[0,1]} \left|t^{n-1}\right| = \frac{1}{n} + 1 > 1;$$

hence x_n does not converge to zero. It follows by (3) that P is a nonnormal cone.

Definition 2.2. [8, 27] *Let X be a nonempty set. Suppose that the mapping $\rho : X \times X \to E$ satisfies:*

(d1) *$\theta \preceq \rho(x,y)$ for all $x, y \in X$ and $\rho(x,y) = \theta$ if and only if $x = y$;*

(d2) *$\rho(x,y) = \rho(y,x)$ for all $x, y \in X$;*

(d3) *$\rho(x,y) \preceq \rho(x,z) + \rho(z,y)$ for all $x, y, z \in X$.*

Then ρ is called a cone metric [8] or $P-$metric [27] on X and (X, ρ) is called a cone metric [8] or $P-$ metric space [27]. A concept of a $P-$ metric space is more general than that of a metric space, because each metric space is a $P-$metric space where $E = \mathbb{R}$ and $P = [0, +\infty)$.

Examples 2.3. $\mathbf{1}^0$ Let $E = \mathbb{R}^2, P = \{(x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\}, X = \mathbb{R}$ and $\rho : X \times X \to E$ defined by $\rho(x,y) = (|x-y|, \alpha|x-y|)$, where $\alpha \geq 0$ is a constant. Then (X,ρ) is a cone metric space [8] with normal cone P where $k = 1$.

$\mathbf{2}^0$ For other examples of a cone metric spaces, i.e., $P-$metric spaces reader can see [27], pp. 853 and 854.

Definition 2.4. *[8], [27]) Let (X,ρ) be a $P-$metric space. We say that $\{x_n\}$ is:*

(i) *a Cauchy sequence if for every c in E with $\theta \ll c$, there is an N such that for all $n, m > N$, $\rho(x_n, x_m) \ll c$;*

(ii) *a convergent sequence if for every c in E with $\theta \ll c$, there is an N such that for all $n > N$, $\rho(x_n, x) \ll c$ for some fixed x in X.*

(iii) *A $P-$metric space X is said to be complete if every Cauchy sequence in X is convergent in X.*

(iv) *Let $f : X \to X$ and $x_0 \in X$. Function f is a continuous at x_0 if for any sequence $x_n \to x_0$ we have $f(x_n) \to f(x_0)$, or equivalently, $\rho(x_n, x_0) \ll c$ implies that $\rho(fx_n, fx_0) \ll c$.*

The following remarks will be useful in the sequel.

Remark 2.5. **(1)** If $u \preceq v$ and $v \ll w$, then $u \ll w$.
(**2**) If $\theta \preceq u \ll c$ for each $c \in intP$ then $u = \theta$.

Remark 2.6. If $c \in intP$ and $a_n \to \theta$, then there exists n_0 such that for all $n > n_0$ we have $a_n \ll c$. From example 1.1. it follows that, in general, the converse is not true. Indeed, $x_n \nrightarrow \theta$ but $x_n \ll c$ for large n.
From this it follows that the sequence $\{x_n\}$ converges to $x \in X$ if $\rho(x_n, x) \to \theta$ as $n \to \infty$ and $\{x_n\}$ is a Cauchy if $\rho(x_n, x_m) \to \theta$ as $n, m \to \infty$. In the situation without the normality property we have only half of lemmas 1 and 4 from [8]. Also, the fact that $\rho(x_n, y_n) \to \rho(x, y)$ if $x_n \to x$ and $y_n \to y$ is not applicable.

Remark 2.7. Let $x \in X, \{x_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ two sequences in X and E, respectively, $\theta \ll c$ and $\theta \preceq \rho(x_n, x) \preceq b_n$ for all $n \geq 1$. If $b_n \to \theta$, then there exists a natural number N such that $\rho(x_n, x) \ll c$ for all $n \geq N$.

Proof. Follows from Remarks 2.6 and 2.5(1). □

Remark 2.8. If E is a real Banach space with a cone P and if $a \preceq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

In the sequel we assume that E is a real Banach space and P a cone in E with $intP \neq \emptyset$. The last assumption is necessary in order to obtain reasonable results connected with convergence and continuity. In particular, with this assumption the limit of a sequence is uniquely determined. The partial ordering induced by the cone P will be denoted by $\preceq$.

3 Main Results

In this section we shall prove one fixed point theorem of contractive mappings for $P-$metric space. We generalize results of [5] and [6], by omitting the assumption of normality in the results. We use only the definition of convergence in terms of the relation "$\ll$". The only assumption is that the interior of the cone P is nonempty-so we use neither continuity of the vector metric ρ, nor Sandwich Theorem. We begin with the following.

Let S be a nonempty set and let $\{f_a\}_{a\in A}$ be a family of self-mappings on S and A an indexing set. A point $u \in S$ is called a common fixed point for a family $\{f_a\}_{a\in A}$ if and only if $f_a u = u$ for each f_a.

Theorem 3.1. *Let (X, ρ) be a complete $P-$metric space and $\{f_a\}_{a\in A}$ a family of self-mappings of X. If there exists $b \in A$ such that for each $a \in A$ there exists $\lambda = \lambda(a) \in [0, 1)$ such that for each $x, y \in X$ there is $u \in M(f_a, f_b; X)$ for which*

$$\rho(f_a x, f_b y) \preceq \lambda \cdot u(x, y) \tag{4}$$

where

$$\begin{aligned} u &\in M(f_a, f_b; X) \\ &= \Big\{\rho(x, y), \rho(x, f_a x), \rho(y, f_b y), \\ &\qquad \frac{\rho(x, f_a x) + \rho(y, f_b y)}{2}, \frac{\rho(x, f_b y) + \rho(y, f_a x)}{2}\Big\}, \end{aligned}$$

then all f_a have a unique common fixed point, which is the unique fixed point of each $f_a, a \in A$.

Remark 3.2. In [5], instead of "$\in$" the author used "$\preceq$", because in metric spaces, the set $M(f_a, f_b, X)$ has minimum and maximum. Since in $P-$metric spaces the set $M(f_a, f_b, X)$ need not even have the supremum in ordered Banach space then, we use "$\in$". It is clear that "$\in$" can be used in metric spaces, while "$\preceq$" can not be used in general in $P-$metric spaces.

Remark 3.3. Also, it is worth mentioning that Theorem 2.1 and Theorem 2.2 from [2], that is., contractive condition (4) and the corresponding condition (2.4) from [2] are not comparable, in general. Indeed, this is obvious if $S \neq I_X$ or $T \neq I_X$ (an identity mapping on X). In the case that $S = T = I_X$ then we obtain that (2.4) from [2] implies (4).

Proof of the Theorem 3.1 Let $a \in A$ and $x \in X$ be arbitrary. Consider a sequence, defined inductively by

$$x_0 = x, x_{2n+1} = f_a x_{2n},\ x_{2n+2} = f_b x_{2n+1},\ n \geq 0.$$

We first show that

$$\rho(x_n, x_{n+1}) \preceq \lambda \rho(x_{n-1}, x_n) \text{ for } n = 1, 2, 3.... \tag{5}$$

From (4) we get

$$\rho(x_{2n+1}, x_{2n+2}) = \rho(f_a x_{2n}, f_b x_{2n+1}) \preceq \lambda \cdot u_1,$$

where

$$u_1 \in \begin{Bmatrix} \rho(x_{2n}, x_{2n+1}), \rho(x_{2n+1}, x_{2n+2}), \\ \frac{\rho(x_{2n}, x_{2n+1}) + \rho(x_{2n+1}, x_{2n+2})}{2}, \frac{\rho(x_{2n}, x_{2n+2})}{2} \end{Bmatrix} \tag{6}$$

and

$$\rho(x_{2n}, x_{2n+1}) = \rho(f_a x_{2n}, f_b x_{2n-1}) \preceq \lambda \cdot u_2,$$

where

$$u_2 \in \begin{Bmatrix} \rho(x_{2n-1}, x_{2n}), \rho(x_{2n}, x_{2n+1}), \\ \frac{\rho(x_{2n-1}, x_{2n}) + \rho(x_{2n}, x_{2n+1})}{2}, \frac{\rho(x_{2n-1}, x_{2n+1})}{2} \end{Bmatrix} \tag{7}$$

If $u_1 = \rho(x_{2n}, x_{2n+1})$ then clearly (5) holds. If $u_1 = \rho(x_{2n+1}, x_{2n+2})$ then according to Remark 2.8 $u_1 = 0$ and (5) is immediate. For

$$u_1 = \frac{1}{2}(\rho(x_{2n}, x_{2n+1}) + \rho(x_{2n+1}, x_{2n+2})),$$

we have

$$\begin{aligned} \rho(x_{2n+1}, x_{2n+2}) &\preceq \frac{\lambda}{2}\rho(x_{2n}, x_{2n+1}) + \frac{1}{2}\rho(x_{2n+1}, x_{2n+2})\text{; i.e.,} \\ \rho(x_{2n+1}, x_{2n+2}) &\preceq \lambda\rho(x_{2n}, x_{2n+1})\text{; that is (5) holds.} \end{aligned}$$

In the case that $u_1 = \frac{1}{2}\rho(x_{2n}, x_{2n+2})$, then from (4) with $x = x_{2n}$ and $y = x_{2n+2}$, as $\lambda < 1$, we have

$$\begin{aligned} \rho(x_{2n+1}, x_{2n+2}) &\preceq \lambda\frac{\rho(x_{2n}, x_{2n+1}) + \rho(x_{2n+1}, x_{2n+2})}{2} \\ &\preceq \lambda\frac{\rho(x_{2n}, x_{2n+1})}{2} + \frac{\rho(x_{2n+1}, x_{2n+2})}{2}. \end{aligned}$$

Hence, $\rho(x_{2n+1}, x_{2n+2}) \preceq \lambda\rho(x_{2n}, x_{2n+1})$, i.e., (5) holds.

In the second case, if $u_2 = \rho(x_{2n-1}, x_{2n})$ then (5) holds. If we assume that $u_2 = \rho(x_{2n}, x_{2n+1})$ then according to Remark 2.8 $u_2 = 0$ and (5) is immediate. For

$$u_2 = \frac{1}{2}(\rho(x_{2n-1}, x_{2n}) + \rho(x_{2n}, x_{2n+1})),$$

it follows

$$\begin{aligned} \rho(x_{2n}, x_{2n+1}) &\preceq \frac{\lambda}{2}\rho(x_{2n-1}, x_{2n}) + \frac{1}{2}\rho(x_{2n}, x_{2n+1}) \text{ i.e.,} \\ \rho(x_{2n}, x_{2n+1}) &\preceq \lambda\rho(x_{2n-1}, x_{2n})\text{; that is (5) holds.} \end{aligned}$$

In the case $u_2 = \frac{1}{2}\rho(x_{2n-1}, x_{2n+1})$, then from (4) with $x = x_{2n-1}$ and $y = x_{2n+1}$, as $\lambda < 1$, we obtain

$$\begin{aligned} \rho(x_{2n}, x_{2n+1}) &\preceq \lambda\frac{\rho(x_{2n}, x_{2n-1}) + \rho(x_{2n-1}, x_{2n+1})}{2} \\ &\preceq \lambda\frac{\rho(x_{2n-1}, x_{2n})}{2} + \frac{\rho(x_{2n}, x_{2n+1})}{2}. \end{aligned}$$

Hence, $\rho(x_{2n}, x_{2n+1}) \preceq \lambda\rho(x_{2n-1}, x_{2n})$; i.e., (5) holds. The proof that (5) holds is completed.

From (5) we get

$$\rho(x_n, x_{n+1}) \preceq \lambda\rho(x_{n-1}, x_n) \preceq \ldots \preceq \lambda^n\rho(x_0, x_1). \tag{8}$$

We show now that $\{x_n\}$ is a Cauchy sequence. For this, by the triangle inequality and from (8), for $n > m$ we have:

$$\begin{aligned}
\rho(x_n, x_m) &\preceq \rho(x_n, x_{n-1}) + \rho(x_{n-1}, x_{n-2}) + \cdots + \rho(x_{m+1}, x_m) \\
&\preceq \left(\lambda^{n-1} + \lambda^{n-2} + \cdots + \lambda^m\right) \rho(x_0, x_1) \\
&= \lambda^m \left(1 + \lambda + \lambda^2 + \cdots + \lambda^{n-m-1}\right) \rho(x_0, x_1) \\
&\preceq \frac{\lambda^m}{1-\lambda} \rho(x_0, x_1) \to \theta, \text{ as } m \to \infty.
\end{aligned}$$

Remark 2.6 implies that for $\theta \ll c$ and large $m : \lambda^m (1-\lambda)^{-1} \rho(x_0, x_1) \ll c$; thus, according to Remark 2.5(1) $\rho(x_n, x_m) \ll c$. Hence, by Definition 2.4 (i) $\{x_n\}$ is a Cauchy sequence. Since (X, ρ) is complete $P-$metric space there is an v in X such that $x_n \to v$. Now, we shall show that $f_b v = v$. By the triangle inequality and (4) we have

$$\begin{aligned}
\rho(f_b v, v) &\preceq \rho(f_b v, x_{2n+1}) + \rho(x_{2n+1}, v) \\
&= \rho(f_a x_{2n}, f_b v) + \rho(x_{2n+1}, v) \preceq \lambda \cdot u_n + \rho(x_{2n+1}, v),
\end{aligned}$$

where

$$u_n \in \begin{array}{l} \{\rho(x_{2n}, v), \rho(v, f_b v), \rho(x_{2n}, x_{2n+1}), \\ \frac{\rho(x_{2n}, f_a x_{2n}) + \rho(v, f_b v)}{2}, \frac{\rho(v, x_{2n+1}) + \rho(x_{2n}, f_b v)}{2}\} \end{array}$$

Let $\theta \ll c$. It is clear that at least one of the following five cases holds for infinitely many n :

1) If $u_n = \rho(x_{2n}, v)$ then

$$\rho(f_b v, v) \preceq \lambda \cdot \rho(x_{2n}, v) + \rho(x_{2n+1}, v) \ll \lambda \frac{c}{2\lambda} + \frac{c}{2} = c.$$

2) For $u_n = \rho(v, f_b v)$ it follows

$$\begin{aligned}
\rho(f_b v, v) &\preceq \lambda \cdot \rho(v, f_b v) + \rho(x_{2n+1}, v); \\
\text{that is } \rho(f_b v, v) &\ll \frac{1}{1-\lambda} \cdot c(1-\lambda) = c.
\end{aligned}$$

3) If $u_n = \rho(x_{2n}, x_{2n+1})$ then

$$\begin{aligned}
\rho(f_b v, v) &\preceq \lambda\rho(x_{2n}, x_{2n+1}) + \rho(x_{2n+1}, v) \\
&\preceq \lambda\rho(x_{2n}, v) + \lambda\rho(v, x_{2n+1}) + \rho(x_{2n+1}, v) \\
&\ll \lambda \frac{c}{3\lambda} + \lambda \frac{c}{3\lambda} + \frac{c}{3} = c.
\end{aligned}$$

4) If $u_n = \frac{1}{2}(\rho(x_{2n}, f_a x_{2n}) + \rho(v, f_b v))$ we obtain

$$\begin{aligned}
\rho(f_b v, v) &\preceq \lambda \frac{1}{2}(\rho(x_{2n}, f_a x_{2n}) + \rho(v, f_b v)) + \rho(x_{2n+1}, v) \\
&= \lambda \frac{1}{2}(\rho(x_{2n}, x_{2n+1}) + \rho(v, f_b v)) + \rho(x_{2n+1}, v) \\
&\preceq \frac{\lambda}{2}\rho(x_{2n}, v) + \frac{\lambda}{2}\rho(v, x_{2n+1}) + \frac{1}{2}\rho(v, f_b v) + \rho(x_{2n+1}, v); \text{ i.e.,} \\
\rho(f_b v, v) &\ll \lambda \frac{c}{2\lambda} + (\lambda + 2)\frac{c}{2(\lambda + 2)} = c.
\end{aligned}$$

5) Finally, if $u_n = \frac{1}{2}(\rho(v, x_{2n+1}) + \rho(x_{2n}, f_b v))$ we have

$$\begin{aligned}
\rho(f_b v, v) &\preceq \lambda \frac{1}{2}(\rho(v, x_{2n+1}) + \rho(x_{2n}, f_b v)) + \rho(x_{2n+1}, v) \\
&\preceq \lambda \frac{1}{2}\rho(v, x_{2n+1}) + \lambda \frac{1}{2}\rho(x_{2n}, v) + \lambda \frac{1}{2}\rho(v, f_b v) + \rho(x_{2n+1}, v) \\
&\preceq \frac{3}{2}\rho(x_{2n+1}, v) + \frac{1}{2}\rho(x_{2n}, v) + \frac{1}{2}\rho(v, f_b v);
\end{aligned}$$

that is.,

$$\rho(f_b v, v) \ll 3\frac{c}{6} + \frac{c}{2} = c.$$

In all cases, we obtain $\rho(f_b v, v) \ll c$ for each $c \in intP$. Using Remark 2.5(2) it follows that $\rho(f_b v, v) = \theta$, or $f_b v = v$, that is v is a fixed point of f_b.

We shall show that v is a fixed point of all $\{f_a\}_{\alpha \in J}$. Let $a \in A$ be arbitrary. Then from (4) with $x = y = v = f_b v$, we have

$$\rho(f_a v, v) = \rho(f_a v, f_b v) \preceq \lambda(a) \cdot u,$$

where $u \in \left\{\rho(f_a v, v), \frac{1}{2}\rho(f_a v, v)\right\}$. Hence, we have two cases:

$$\rho(f_a v, v) \preceq \lambda\rho(f_a v, v) \text{ and } \rho(f_a v, v) \preceq \frac{\lambda}{2}\rho(f_a v, v) \prec \lambda\rho(f_a v, v).$$

According to Remark 2.8. it follows that $f_a v = v$. Thus, all f_a have a common fixed point. Suppose that w is also a fixed point of f_b. Then it follows, as above, that w is a common fixed point of all $\{f_a\}_{a \in A}$. Thus, from (4) we get

$$\rho(w, v) = \rho(f_b w, f_a v) \preceq \lambda \cdot u$$

where

$$\begin{aligned} u &\in \begin{array}{c}\left\{\rho(w,v), \rho(v, f_a v), \rho(w, f_b w), \frac{1}{2}(\rho(v, f_a v) + \rho(w, f_b w)),\right. \\ \left.\frac{1}{2}(\rho(v, f_b w) + \rho(w, f_a v))\right\}\end{array} \\ &= \Big\{\rho(w,v), \rho(v,v), \rho(w,w), \frac{1}{2}(\rho(v,v) + \rho(w,w)), \\ &\qquad \frac{1}{2}(\rho(v,w) + \rho(w,v))\Big\} \\ &= \{0, \rho(v,w)\}. \end{aligned}$$

Hence, $\rho(w,v) \preceq \lambda \cdot \rho(w,v)$. By Remark 2.8. it follows that $\rho(w,v) = \theta$; that is $v = w$ is a unique common fixed point of all $\{f_a\}_{a\in A}$. The theorem is proved. □

We now list some corollaries of Theorem 3.1.

Corollary 3.4. *In Theorem 3.1 by setting* $E = \mathbb{R}, P = [0, +\infty[, \|x\| = |x|, x \in E$, *we get Ćirić's result [5] for a family of contractive maps.*

Corollary 3.5. *Taking* $a = b, \lambda = \lambda(a) \in [0,1), f_a = f_b = f, u(x,y) = \rho(x,y)$ *we obtain the cone version of Banach contraction principle ([8], Th.1.).*

Corollary 3.6. *Taking* $a = b, \lambda = \lambda(a) \in [0,1), f_a = f_b = f,$

$$u(x,y) = \frac{1}{2}(\rho(x, fx) + \rho(y, fy)),$$

in the Theorem 3.1 we have the cone version of Kannan contraction ([24] (19) (ii), [8], Th. 3.).

Corollary 3.7. *Taking* $a = b, \lambda = \lambda(a) \in [0,1), f_a = f_b = f,$

$$u(x,y) = \frac{1}{2}(\rho(x, fy) + \rho(y, fx)),$$

in the Theorem 2.1. we have the cone version of Chatterjea contraction ([24] (19) (ii), [8], Th. 4.).

Corollary 3.8. *Taking* $M(f_a, f_b; X) = \{\rho(x,y), \rho(x, f_a x), \rho(y, f_b y)\}$ *in the Theorem 3.1 we obtain the cone version of Ćirić's result from ([6], Theorem 4.19.).*

Now, we add an example with Banach type contraction on a $P-$metric space with nonnormal cone (see Corollaries 3.4 and 3.8).

Example 3.9. Let $X = [0,1], E = C^1_{\mathbb{R}}[0,1], P = \{\varphi \in E : \varphi(t) \geq 0\}$. Define $\rho : X \times X \to E$ by $\rho(x,y)(t) := |x-y| \cdot \varphi(t)$ where $\varphi : [0,1] \to \mathbb{R}$ such that $\varphi(t) = 4^t$. It is easy to see that ρ is a $P-$metric on X. Consider the mappings $f, g : X \to X$ in the following manner:

$$fx = \begin{cases} ax + 1 - a, x \neq 0 \\ 0, x = 0 \end{cases} \quad \text{and } gx = x,$$

where $a \in (0,1)$. It is clear that

$$\rho(fx, fy) \preceq \lambda\rho(gx, gy) = \lambda\rho(x,y),$$

for all $x, y \in X$, where $\lambda \in [0,1)$. All the conditions of the Corollaries 3.4 and 3.8 hold, and f and g have a common fixed point.

This example verifies that Theorems 3.1 is a proper extension of the known results from [5] and [6]. Indeed, we know (see Example 1.1.) that the cone P is nonnormal. So, in this case [5, Theorem 4.5] and [6, Theorem 4.19] cannot be applied. This shows that Theorems 3.1 is more general, i.e., the main theorems from [5] and [6] can be obtained as its special cases taking $\|.\| = |.|, E = \mathbb{R}$ and $P = [0, +\infty)$.

Remark 3.10. We finish the present article with the following problem, to which we have no answer yet: Does Theorem 3.1 remain true if

$$M(f_a, f_b; X) = \left\{ d(x,y), \frac{1}{2}d(x, f_a x), \frac{1}{2}d(y, f_b y), d(x, f_b y), d(y, f_a x) \right\}?$$

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References

[1] M. Abbas and G. Jungck, Common fixed point results for noncommuting mappings without continuity in cone metric spaces, *J. Math. Anal. Appl.*, **341**, (2008), 416-420.

[2] M. Abbas, B.E.Rhoades and T. Nazir, Common fixed points for four maps in cone metric spaces, *Applied Mathematics and Computation*, **216**, (2010), 80-86.

[3] C.D. Aliprantis and R. Tourky, *Cones and Duality*, vol. **84** of Graduates Studies in Mathematics, American Mathematical Society, Providence, RI, USA, 2007.

[4] A.N. Baluev, Application of a semi-ordered norms in approximate solution of nonlinear equations, *Leningrad. Gos. Univ. Uchen. Zap. Ser. Mat. Nauk*, **33**, (1985), 18-27.

[5] Lj. B. Ćirić, On a family of contractive maps and fixed points, *Publ. Inst. Math.*, **17**(31), (1974), 45-51.

[6] Lj. B. Ćirić, *Some recent results in metrical fixed point theory*, Beograd, 2003.

[7] K. Deimling, *Nonlinear Functional Analysis*, Springer-Verlag, 1985.

[8] L.G. Huang and X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings, *J. Math. Anal. Appl.*, **332**(2), (2007), 1468-1476.

[9] D. Ilić and V. Rakočević, Common fixed points for maps on cone metric space, *J. Math. Anal. Appl.*, **341**, (2008), 876-882.

[10] D. Ilić and V. Rakočević, Quasi-contraction on a cone metric spaces, *Appl. Math. Lett.*, **22**, (2009), 728-731.

[11] S. Janković, Z. Kadelburg and S. Radenović, On cone metric spaces: A survey, *Nonlinear Anal.*, **74**, (2011), 2591-2601.

[12] L.V. Kantorovich, Lineare halbeordnete Raume, *Mat. Sb.*, **2**(44), (1937), 121-168.

[13] L.V. Kantorovich, The method of successive approximations for functional equations, *Acta Math.*, **71**, (1939), 63-77.

[14] L.V. Kantorovich, Functional Analysis and applied mathematics, *Uspehi Mat. nauk (N.S.)*, **3**(6), (1948), 89-185.

[15] L.V. Kantorovich, The majorant principle and Newton"s method, *Dokl. Akad. Nauk SSSR (N.S.)*, **76**, (1951), 17-20.

[16] L.V. Kantorovich, On some further applications of the Newton approximation method, *Vestnik Leningrad Univ. Ser. mat. Meh. Astr.*, **12**(7), (1957), 68-103.

[17] M.G. Krein and M.A. Rutman, Linear operators leaving invariant a cone in a Banach spaces, *Uspehi Math. Nauk (N.S)*, **3**(1), (1948), 3-95.

[18] M.A. Krasnoseljski and P.P. Zabreiko, *Geometrical Methods in Nonlinear Analysis*, Springer, 1984.

[19] M.A. Krasnoseljski, E.A. Lifshitz and A.V. Sobolev, *Positive linear systems*, Moscow, Fizmatigz, 1985.

[20] H. Mohebi, *Topical functions and their properties in a class of ordered Banach spaces*, Part II of the book "Continuous Optimization, Curent Trends and Modern Applications", 343-361, Springer, 2005.

[21] Mirjana Pavlović, S. Radenović and Slobodan Radojević, Abstract metric spaces and Sehgal-Guseman-type theorems, *Comput. Math. Appl.*, **60**, (2010), 865-872.

[22] Sh. Rezapour and R.H. Haghi, Some notes on the paper "Cone metric spaces and fixed point theorems of contractive mappings", *J. Math. Anal. Appl.*, **345**, (2008), 719-724.

[23] S. Radenović, Common fixed points under contractive conditions in cone metric spaces, *Comput. Math. Appl.*, **58**, (2009), 1273-1278.

[24] B.E. Rhoades, A comparison of Various Definitions of Contractive Mappings, *Trans. Amer. Math. Soc.*, **226**, (1997), 257-290.

[25] B. Samet, Common Fixed point under contractive condition of Ćirić's type in cone metric spaces, *Appl. Anal. Discrete Math.*, (to appear), DOI: 10.2298/AADM110206007S

[26] J. Vandergraft, Newton method for convex operators in partially ordered spaces, *SIAM J. Numer. Anal.*, **4**, (1967), 406-432.

[27] P.P. Zabreiko, K-metric and K-normed spaces: survey, *Collect. Math.*, **48**, (1997), 852-859.

[28] P.P. Zabreiko and T.A. Makarevich, Fixed point principle and L.V. Ovsjanikov"s theorem, *Vesnik Belorusek. Gos. Univ.* **3**, (1987), 53-55.

[29] P.P. Zabreiko and J.V. Radyno, Application of a fixed point theory to Cauchy problem for equations with deteriorating operators, *Differentsialnye Uravnenija*, **23**(2), (1987), 345-348.