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Wada's Representations of the Pure Braid Group of High Degree

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Abstract

We consider the representation obtained by composing the embedding map of the pure braid group $P_n \to P_{n+k}$ and Wada's representation of degree $n+k$ to get a linear representation

$$P_n \to GL_{n+k}(C[t_1^{\pm 1}, \ldots, t_{n+k}^{\pm 1}]),$$

whose composition factors are to be determined. A similar work was done in a previous work in the case of the Gassner representation of P_n.

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1 Introduction

The braid group on n strands, denoted by B_n, is defined as an abstract group with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations $\sigma_i\sigma_j = \sigma_j\sigma_i$, $|i-j| \geq 2$, $1 \leq i,\ j \leq n-1$; $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for $1 \leq i \leq n-2$.

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The pure braid group, P_n, is a normal subgroup of the braid group, B_n, on n strings. It has many linear representations. One of them is Wada's representation, which is an embedding $P_n \to Aut(F_n)$, the automorphism group of a free group on n generators.

In [1], Abdulrahim has constructed an embedding of the pure braid group $P_n \to P_{n+k}$ and composed it with the Gassner representation of P_{n+k} to get a linear representation $P_n \to GL_{n+k}(C[t_1^{\pm 1}, ..., t_n^{\pm 1}, ..., t_{n+k}^{\pm 1}])$, where the composition factors were completely determined. In our work, we consider Wada's representation instead of the Gassner representation, where σ_i, takes $x_i \to x_i x_{i+1}^{-1} x_i$, $x_{i+1} \to x_i$; and fixes all other free generators. Our main theorem is similar to that obtained in [1], where the composition factors of the representation obtained by composing the embedding map and Wada's representation are to be determined. However, for the sake of our work, the embedding map $P_n \to P_{n+k}$ has to be defined in a different way, where a generator of P_n is mapped to another generator of P_{n+k} , rather than to a product of generators of P_{n+k} as in [1].

2 Preliminaries

Definition 2.1. *Let F_n be a free group of rank n, with free basis $x_1, \ldots, x_n$. We define for $j = 1, 2, ..., n$ the free derivatives on the group $\mathbb{Z}F_n$ by*

(i) $\frac{\partial x_i}{\partial xj} = \delta_{i,j}$,

(ii) $\frac{\partial x_i^{-1}}{\partial xj} = -\delta_{i,j} x_i^{-1}$,

(iii) $\frac{\partial}{\partial x_j}(uv) = \frac{\partial u}{\partial xj}\epsilon(v) + u\ \frac{\partial v}{\partial xj} \quad u, v \in \mathbb{Z}F.$

Here $\delta_{i,j}$ is the Kronecker symbol. For simplicity, we denote $\frac{\partial}{\partial x_j}$ by d_j.

Definition 2.2. *The pure braid group, P_n, is defined as the kernel of the homomorphism $B_n \to S_n$, defined by $\sigma_i \to (i, i+1), 1 \le i \le n-1$. It has the following generators:*

$$A_{i,,j} = \sigma_{j-1}^{-1}\sigma_{j-2}^{-1} \ldots \sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1} \ldots \sigma_{j-2}\sigma_{j-1},\ 1 \le i < j \le n$$

Definition 2.3. *Wada's representation is defined as the representation of the automorphism corresponding to the braid generator* σ_i*, takes* $x_i \rightarrow x_i x_{i+1}^{-1} x_i$, $x_{i+1} \rightarrow x_i$ *and fixes all other free generators.*

It is easy to see that the inverse σ_i^{-1}, takes $x_i \rightarrow x_{i+1}$, $x_{i+1} \rightarrow x_{i+1}^{-1} x_i x_{i+1}^{-1}$; and fixes all other generators. For more details, see [3].

3 Main Results

We determine the action of the automorphisms $A_{i,j}$ on the generators of the free group F_n. We then define an embedding of the pure braid group $P_n \rightarrow P_{n+k}$ and compose the embedding map and Wada's representation of the pure braid group P_{n+k}.

3.1 Action of the automorphisms on the free group

Lemma 3.1. *As automorphisms of the free group* F_n*, the generators,* $A_{i,j}$*, act on the free group* F_n *as follows:*

(i) $A_{i,j}(x_i) = x_i x_j^{-1} x_i x_j^{-1} x_i$

(ii) $A_{i,j}(x_j) = x_i x_j^{-1} x_i$

(iii) $A_{i,j}(x_r) = x_r$ *if* $1 \leq r < i$ *or* $j < r \leq n$

(iv) $A_{i,j}(x_r) = (x_i x_j^{-1} x_i x_j) x_r^{-1} (x_j x_i x_j^{-1} x_i)$ *if* $i < r < j$

Proof. We have that

$$A_{i,,j} = \sigma_{j-1}^{-1} \sigma_{j-2}^{-1} ... \sigma_{i+1}^{-1} \sigma_i^2 \sigma_{i+1} ... \sigma_{j-2} \sigma_{j-1}, \ 1 \leq i < j \leq n.$$

We need to consider $A_{i,j}$ as left automorphisms acting on the generators of F_n from the left.

To prove (i):

$\sigma_{j-1}^{-1} \sigma_{j-2}^{-1} ... \sigma_{i+1}^{-1} \sigma_i^2 \sigma_{i+1} \ldots \sigma_{j-2} \sigma_{j-1}(x_i)$

$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}...\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\ldots\sigma_{j-2}(x_i)$$
$$\vdots$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}\sigma_i(\sigma_i(x_i))$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}\sigma_i(x_ix_{i+1}^{-1}x_i)$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}(x_ix_{i+1}^{-1}x_ix_i^{-1}x_ix_{i+1}^{-1}x_i)$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}(x_ix_{i+1}^{-1}x_ix_{i+1}^{-1}x_i)$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+2}^{-1}(x_ix_{i+2}^{-1}x_ix_{i+2}^{-1}x_i)$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+3}^{-1}(x_ix_{i+3}^{-1}x_ix_{i+3}^{-1}x_i)$$
$$\vdots$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}(x_ix_{j-2}^{-1}x_ix_{j-2}^{-1}x_i)$$
$$= \sigma_{j-1}^{-1}(x_ix_{j-1}^{-1}x_ix_{j-1}^{-1}x_i)$$
$$= x_ix_j^{-1}x_ix_j^{-1}x_i$$

To prove (ii):

$$\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\ldots\sigma_{j-2}\sigma_{j-1}(x_j)$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\ldots\sigma_{j-2}(x_{j-1})$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}.\ldots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\ldots\sigma_{j-3}(x_{j-2})$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\ldots\sigma_{j-4}(x_{j-3})$$
$$\vdots$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}(x_{i+2})$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}\sigma_i^2(x_{i+1})$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+1}^{-1}\sigma_i(x_i)$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}.\ldots\sigma_{i+1}^{-1}(x_ix_{i+1}^{-1}x_i)$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+2}^{-1}(x_ix_{i+2}^{-1}x_i)$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+3}^{-1}(x_ix_{i+3}^{-1}x_i)$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\ldots\sigma_{i+4}^{-1}(x_ix_{i+4}^{-1}x_i)$$
$$\vdots$$
$$= \sigma_{j-1}^{-1}\sigma_{j-2}^{-1}(x_ix_{j-2}^{-1}x_i)$$
$$= \sigma_{j-1}^{-1}(x_ix_{j-1}^{-1}x_i)$$
$$= x_ix_j^{-1}x_i$$

To prove (iii):

Since $r > j$, that is, the smallest possible value of r is $j+1$, then $A_{i,j}(x_r) = x_r$ for $j < r \leq n$.

Now, since $r < i$, that is, the greatest possible value of r is $i-1$, then $A_{i,j}(x_r) = x_r$ for $1 \leq r < i$.

To prove (iv):

Since $i < r < j$, the largest index of x, namely r, is $j-1$ and the smallest index of x, namely r, is $i-1$. Then

$$\begin{aligned}
&\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\dots.\sigma_{j-2}\sigma_{j-1}(x_r)\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\dots\sigma_{r-1}\sigma_r(x_r)\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\dots\sigma_{r-1}(x_r x_{r+1}^{-1}x_r)\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}\dots\sigma_{r-2}(x_{r-1}x_{r+1}^{-1}x_{r-1})\\
&\vdots\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+1}^{-1}\sigma_i^2\sigma_{i+1}(x_{i+2}x_{r+1}^{-1}x_{i+2})\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+1}^{-1}\sigma_i^2(x_{i+1}x_{r+1}^{-1}x_{i+1})\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+1}^{-1}\sigma_i(x_i x_{r+1}^{-1}x_i)\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+1}^{-1}(x_i x_{i+1}^{-1}x_i x_{r+1}^{-1}x_i x_{i+1}^{-1}x_i)\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+2}^{-1}(x_i x_{i+2}^{-1}x_i x_{r+1}^{-1}x_i x_{i+2}^{-1}x_i)\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{i+3}^{-1}(x_i x_{i+3}^{-1}x_i x_{r+1}^{-1}x_i x_{i+3}^{-1}x_i)\\
&\vdots\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{r}^{-1}(x_i x_{r}^{-1}x_i x_{r+1}^{-1}x_i x_{r}^{-1}x_i)\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{r+1}^{-1}(x_i x_{r+1}^{-1}x_i x_{r+1}x_r^{-1}x_{r+1}x_i x_{r+1}^{-1}x_i)\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}\dots\sigma_{r+2}^{-1}(x_i x_{r+2}^{-1}x_i x_{r+2}x_r^{-1}x_{r+2}x_i x_{r+2}^{-1}x_i)\\
&\vdots\\
&=\sigma_{j-1}^{-1}\sigma_{j-2}^{-1}(x_i x_{j-2}^{-1}x_i x_{j-2}x_r^{-1}x_{j-2}x_i x_{j-2}^{-1}x_i)\\
&=\sigma_{j-1}^{-1}(x_i x_{j-1}^{-1}x_i x_{j-1}x_r^{-1}x_{j-1}x_i x_{j-1}^{-1}x_i)\\
&=x_i x_j^{-1}x_i x_j x_r^{-1}x_j x_i x_j^{-1}x_i\\
&=(x_i x_j^{-1}x_i x_j)x_r^{-1}(x_j x_i x_j^{-1}x_i) \qquad \square
\end{aligned}$$

3.2 The embedding $P_n \to P_{n+k}$

In [1], the embedding of the pure braid group was defined in a way that the generators $A_{1,j}$ were mapped to a product of generators of P_{n+k} and other generators $A_{i,j}$ to $A_{i+k,j+k}$. Whereas in our work, we require that the generator of P_n is to be mapped to another generator of P_{n+k}. More precisely, we define the following map:

$$\Psi : P_n \to P_{n+k}$$

as

$$\Psi(A_{i,j}) = \begin{cases} A_{1,j+k} \ , & i=1 \ \ and \ \ 2 \le j \le n \\ A_{i+k,j+k} \ , & 2 \le i < j \le n, \end{cases}$$

where $\Psi(A_{i,j})$ is a generator of P_{n+k}, that is an automorphism of F_{n+k} whose action on the free generator $x_1, x_2, \ldots, x_n, \ldots, x_{n+k}$ is defined in Lemma 3.1. We compose the map above with the embedding $P_{n+k} \to Aut(F_{n+k})$. The image of the generators under this embedding is treated as left automorphisms of the free froup F_{n+k}. As a basis for the free group F_{n+k}, we take the following generators:

$$y_1 = x_1, \ \ y_2 = x_{k+2}, \ \ y_3 = x_{k+3}, \ldots, y_n = x_{k+n},$$

$$y_{n+1} = x_2, \ \ y_{n+2} = x_3, \ldots, y_{n+k} = x_{k+1}$$

Using the action of the automorphism, σ_i, on the basis of F_{n+k}, we have the following lemmas about the images of the generators of P_n, namely $\Psi(A_{i,j})$ for $1 \le i < j \le n$.

Lemma 3.2. *For* $1 < j \le n$, *the action of the images of the generators,* $A_{1,j}$ *on the basis of* F_{n+k} *is given by*

(i) $\Psi(A_{1,j})(y_1) = y_1(y_j^{-1}y_1y_j^{-1}y_1)$,
(ii) $\Psi(A_{1,j})(y_j) = y_1y_j^{-1}y_1$,
(iii) $\Psi(A_{1,j})(y_r) = y_r$ *for* $j < r \le n$,
(iv) $\Psi(A_{1,j})(y_r) = (y_1y_j^{-1}y_1y_j)y_r^{-1}(y_jy_1y_j^{-1}y_1)$ *for* $1 < r < j$,
(v) $\Psi(A_{1,j})(y_{n+r}) = (y_1y_j^{-1}y_1y_j)y_{n+r}^{-1}(y_jy_1y_j^{-1}y_1)$ *for* $1 \le r \le k$.

Proof. Since the image of $A_{1,j}$ under Ψ is a generator of P_{n+k} namely $A_{1,j+k}$, it suffices only to prove (v). We have that

$$\Psi(A_{1,j}) = A_{1,j+k} = \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3\ldots\sigma_k\sigma_{k+1}\sigma_{k+2}\ldots\sigma_{j+k-1}.$$

To prove (v): Let $y_{n+r} = x_{r+1}$, for $1 \le r \le k$. Then $1 \le r < j+k-1$ and we have that

$$\sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3\ldots\sigma_k\sigma_{k+1}\sigma_{k+2}\ldots\sigma_{j+k-1}(x_{r+1})$$
$$= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3\ldots\sigma_k\sigma_{k+1}\sigma_{k+2}\ldots\sigma_r\sigma_{r+1}(x_{r+1})$$

$$
\begin{aligned}
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3\ldots\sigma_k\sigma_{k+1}\sigma_{k+2}\ldots\sigma_r(x_{r+1}x_{r+2}^{-1}x_{r+1})\\
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3\ldots\sigma_k\sigma_{k+1}\sigma_{k+2}\ldots\sigma_{r-1}(x_r x_{r+2}^{-1}x_r)\\
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2\sigma_2\sigma_3\ldots\sigma_k\sigma_{k+1}\sigma_{k+2}\ldots\sigma_{r-2}(x_{r-1}x_{r+2}^{-1}x_{r-1})\\
&\vdots\\
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_3^{-1}\sigma_2^{-1}\sigma_1^2(x_2x_{r+2}^{-1}x_2)\\
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_3^{-1}\sigma_2^{-1}\sigma_1(x_1x_{r+2}^{-1}x_1)\\
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_3^{-1}\sigma_2^{-1}(x_1x_2^{-1}x_1x_{r+2}^{-1}x_1x_2^{-1}x_1)\\
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_3^{-1}(x_1x_3^{-1}x_1x_{r+2}^{-1}x_1x_3^{-1}x_1)\\
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_4^{-1}(x_1x_4^{-1}x_1x_{r+2}^{-1}x_1x_4^{-1}x_1)\\
&\vdots\\
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_{r+2}^{-1}\sigma_{r+1}^{-1}(x_1x_{r+1}^{-1}x_1x_{r+2}^{-1}x_1x_{r+1}^{-1}x_1)\\
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_{r+2}^{-1}(x_1x_{r+2}^{-1}x_1x_{r+2}x_{r+1}^{-1}x_{r+2}x_1x_{r+2}^{-1}x_1)\\
&= \sigma_{j+k-1}^{-1}\sigma_{j+k-2}^{-1}\ldots\sigma_{r+3}^{-1}(x_1x_{r+3}^{-1}x_1x_{r+3}x_{r+1}^{-1}x_{r+3}x_1x_{r+3}^{-1}x_1)\\
&\vdots\\
&= \sigma_{j+k-1}^{-1}(x_1x_{j+k-1}^{-1}x_1x_{j+k-1}x_{r+1}^{-1}x_{j+k-1}x_1x_{j+k-1}^{-1}x_1)\\
&= x_1x_{j+k}^{-1}x_1x_{j+k}x_{r+1}^{-1}x_{j+k}x_1x_{j+k}^{-1}x_1\\
&= (y_1y_j^{-1}y_1y_j)y_{n+r}^{-1}(y_jy_1y_j^{-1}y_1) \qquad \square
\end{aligned}
$$

For $1 < i < j \leq n$, we have that $\Psi(A_{i,j}) = A_{i+k,j+k}$. Acting on the generators of F_{n+k} , namely, $x_1, \ldots, x_{n+k}$ subject to the rules stated in Lemma 3.1, we easily verify the following lemma.

Lemma 3.3. *For $1 < i < j \leq n$, the action of the images of the generators $A_{i,j}$ on the basis of F_{n+k} is as follows:*

(i) $\Psi(A_{i,j})(y_i) = y_i(y_j^{-1}y_iy_j^{-1}y_i)$
(ii) $\Psi(A_{i,j})(y_j) = y_iy_j^{-1}y_i$
(iii) $\Psi(A_{i,j})(y_r) = y_r$ *for* $1 \leq r < i$ *or* $j < r \leq n$
(iv)$\Psi(A_{i,j})(y_r) = (y_iy_j^{-1}y_iy_j)y_r^{-1}(y_jy_iy_j^{-1}y_i)$ *for* $i < r < j$
(v) $\Psi(A_{i,j})(y_{n+r}) = y_{n+r}$ *for* $1 \leq r \leq k$

Proof. As in Lemma 3.2, we only need to prove (v): Let $y_{n+r} = x_{r+1}$, for $1 \leq r \leq k$. Since $r \leq k$, that is, the greatest possible value of $r+1$ is $k+1$, it follows that $\Psi(A_{i,j})(y_{n+r}) = \Psi(A_{i,j})(x_{r+1}) = A_{i+k,j+k}(x_{r+1}) = x_{r+1} = y_{n+r}$. $\square$

Let ϕ be a homomorphism from F_{n+k} to $(\mathbb{C}^*)^{n+k}$ defined by $\phi(y_i) = t_i$, for $1 \leq i \leq n+k$. Let $D_i = \phi\frac{\partial}{\partial y_i}$. Our objective now is to determine the Jacobian matrix of the image of the generator $A_{i,j}$ under the map, namely the automorphism $\Psi(A_{i,j})$ on the free group F_{n+k} defined in Lemma 3.2 and Lemma 3.3, so that we can find the linear representation obtained by composing the map $P_n \to P_{n+k}$ with Wada's representation. By intuition, the order of the generators of F_{n+k} is:

$$y_1, \ldots, y_n, y_{n+1}, \ldots, y_{n+k}$$

Consider $\Psi(A_{i,j})$, the image of $A_{i,j}$ under the map $P_n \to P_{n+k}$, and call it $A_{i,j}$ for simplicity. Then we define the jacobian matrix as follows:

$$J(A_{i,j}) = \begin{bmatrix} D_1(A_{i,j}(y_1)) & \ldots & D_{n+k}(A_{i,j}(y_1)) \\ \vdots & & \vdots \\ D_1(A_{i,j}(y_{n+k})) & \ldots & D_{n+k}(A_{i,j}(y_{n+k})) \end{bmatrix}$$

The construction used here is the Magnus representation of P_{n+k} [2, p.115-119].

We now prove our main theorem.

Theorem 3.4. *By composing the embedding $P_n \to P_{n+k}$ with Wada's representation of P_{n+k}, we get a linear representation of degree $n+k$ whose one of the composition factors is isomorphic to Wada's representation of P_n and the other is a diagonal representation. The matrix that corresponds to the image of $A_{i,j}$ has the following form:*

$$\begin{bmatrix} \gamma(A_{i,j}) & 0 \\ * & M_k \end{bmatrix},$$

where $\gamma(A_{i,j})$ is the image of $A_{i,j}$ under Wada's representation of degree n and M_k is a diagonal representation whose diagonal entries are all ones in the case $1 < i \leq n$ and $-t_1^2 t_{n+r}^{-1}$ when $i = 1$ and $1 \leq r \leq k$.

Proof. From Lemma 3.2 and Lemma 3.3, we easily verify that statements (i), (ii), (iii) and (iv) coincide with the definition of the image of $A_{i,j}$ under the

Wada's representation of P_n specified by the basis $\{y_1, \ldots, y_n\}$ (See Lemma 3.1). Furthermore, statement (v) in Lemma 3.2 asserts that

$$\Psi(A_{1,j})(y_{n+r}) = (y_1 y_j^{-1} y_1 y_j) y_{n+r}^{-1} (y_j y_1 y_j^{-1} y_1),$$

for any $1 \leq r \leq k,$ which in turn implies that

$$D_{n+r}(\Psi(A_{1,j})(y_{n+r})) = -t_1^2 t_{n+r}^{-1}.$$

Statement (v) in Lemma 3.3 asserts that for $1 < i < j \leq n,$ we have that $\Psi(A_{i,j})(y_{n+r}) = y_{n+r}$ for $1 \leq r \leq k,$ which implies that

$$D_{n+r}(\Psi(A_{i,j})(y_{n+r})) = 1.$$

This completes the proof. □

4 Conclusion

In our work, we have completely determined the composition factors of the representation obtained by composing the embedding map $P_n \to P_{n+k}$ and Wada's representation of the pure braid group P_{n+k}.

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