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A Class of Generalized Cayley Digraphs Induced by Quasigroups

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Abstract

We generalize the results in [13] to produce new classes of generalized cayley graphs induced by quasigroups.

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1 Introduction

A *binary relation* on a set V is a subset E of $V \times V$. A *digraph* is a pair (V, E) where V is a non-empty set (called vertex set) and E is a binary relation on V. The elements of E are the *edges* of the digraph. A digraph (V, E) is called *vertex-transitive* if, given any two vertices a and b of V, there is some graph automorphism $f : V \longrightarrow V$ such that $f(a) = b$ [4]. In other words, a graph is *vertex-transitive* if its automorphism group acts transitively upon its

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vertices [4]. Whenever the word graph is used in this paper it will be referring to a digraph unless otherwise stated.

A non empty set G, together with a mapping $* : G \times G \longrightarrow G$ is called a *groupoid*. The mapping $*$ is called a *binary operation* on the set G. If $a, b \in G$, we use the symbol ab to denote $*(a, b)$. A groupoid $(G, *)$ is called a *quasigroup*, if for every $a, b \in G$, the equations, $ax = b$ and $ya = b$ are uniquely solvable in G [10]. This implies both left and right cancelation laws. Observe that a quasigroup is a weaker algebraic structure than a group.

Let G be a group and S be a subset of G. The cayley digraph of G with respect to S is defined as the digraph $X = (G, E)$, where E is a binary relation on G, such that

$$(x, y) \in E \quad \text{if and only if there is some } s \in S, \text{ such that } \; y = xs \;\; [8].$$

Informally, the vertices of the cayley digraphs are group elements, and two vertices are connected with an edge if and only if the second vertex is the product of an element from S and the first vertex. The Cayley digraph of G with respect to S is denoted by $\mathrm{Cay}(G, S)$. The set S is called the connection set of $\mathrm{Cay}(G, S)$.

In [13], *K. V. Anil* extended the definition of cayley graph and introduced a class of generalized cayley graphs induced by groups and obtained interesting relationship between properties of graphs and those of groups. In this paper, we introduce a class of digraphs induced by quasigroups. These digraphs can be considered as a generalization of those obtained in [13]. Moreover, we study various graph properties in terms of algebraic properties. Here, we need the following:

Definition 1.1. *Let G be a quasigroup, and let A be a subset of G. Then A said to be a $\mathcal{R}$ associative subset of G, if for every $x, y \in G$, $(xy)A = x(yA)$. This means, if $x, y \in G$ and $a \in A$, then $(xy)a = x(ya')$ for some $a' \in A$ [12]. Similarly, we can define $\mathcal{L}$ associative subset of G.*

Lemma 1.2. *Let A and B be $\mathcal{R}$ associative subsets of a quasigroup G. Then AB is also $\mathcal{R}$ associative [12].*

Lemma 1.3. *Let A and B be $\mathcal{L}$ associative subsets of a quasigroup G. Then AB is also $\mathcal{L}$ associative.*

2 Generalized cayley digraphs

In this section we generalize the results in [13] and introduce a bigger class of generalized cayley digraphs induced by quasigroups. These graphs can be considered as generalization of cayley digraphs induced by groups. Let $a_1, a_2, \ldots, a_n \in G$, then we may define the product $a_1 a_2 \ldots a_n$ as follows:

$$a_1 a_2 a_3 \ldots a_{n-1} a_n = (\ldots((a_1 a_2) a_3) \ldots a_{n-1}) a_n$$

We begin with the following definition:

Definition 2.1. *Let G be a quasigroup and let A and B be subquasigroups of G such that A is $\mathcal{L}$ associative and B is $\mathcal{R}$ associative. Let D and D^* be subsets of G such that D is $\mathcal{L}$ associative and D^* is $\mathcal{R}$ associative. Let a and b be fixed elements in A and B respectively. Let*

$$R_{D,D^*} = \{(x, y) : (ay)b = (z_1 x) z_2 \text{ for some } z_1 \in ADA, z_2 \in BD^*B\}.$$

Then the digraph (G, R_{D,D^*}) is called the generalized Cayley graph induced by the quasigroup G. The sets ADA and BD^*B are called the connection sets for (G, R_{D,D^*}).

In case G is a group, $A = B = D = \{1\}$ and $a = b = 1$, the generalized cayley graph (G, R_{D,D^*}) is the cayley graph $Cay(G, D^*)$.

2.1 Examples of generalized cayley graphs

In this section we give some examples of generalized cayley graphs. We prove that the complete bipartite graph $K_{6,6}$, the disjoint union of two copies of $\bar{K_6}$(complete graph of order 6), disjoint union of two copies of $K_{3,3}$ and disjoint union of 4 copies of $\bar{K_3}$ are generalized cayley graphs. In general, we prove that the graphs $K_{n,n}$ and $K_{n,n,\cdots,n}$ are generalized cayley graphs.

Example 1. Let $G = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. Define a binary operation in G as follows:

*	0	1	2	3	4	5	6	7	8	9	10	11
0	1	2	0	4	5	3	7	6	8	10	11	9
1	2	0	1	5	3	4	6	8	7	11	9	10
2	0	1	2	3	4	5	8	7	6	9	10	11
3	5	4	3	0	1	2	9	10	11	7	8	6
4	3	5	4	1	2	0	10	11	9	8	6	7
5	4	3	5	2	0	1	11	9	10	6	7	8
6	6	7	8	9	10	11	1	2	0	4	3	5
7	7	8	6	10	11	9	0	1	2	5	4	3
8	8	6	7	11	9	10	2	0	1	3	5	4
9	10	11	9	7	6	8	3	4	5	2	1	0
10	9	10	11	6	8	7	4	5	3	1	0	2
11	11	9	10	8	7	6	5	3	4	0	2	1

Under this operation G is a quasigroup.

(i) Let $A = \{0,1,2\}$, $B = \{0,1,2,3,4,5\}$, $D = \{6,7,8,9,10,11\}$, and $D^* = \{3,4,5\}$. We find that A and B are respectively $\mathcal{L}$ associative and $\mathcal{R}$ associative subquasigroups of G. Furthermore, D and D^* are respectively $\mathcal{L}$ associative and $\mathcal{R}$ associative subsets of G. Take $a = 0$ in A and $b = 3$ in B. On examination we find that

$$
\begin{aligned}
&ADA = \{6,7,8,9,10,11\}, BD^*B = \{0,1,2,3,4,5\}, \text{ and} \\
&R_{D,D^*} = \{(0,6),(6,0),(0,7),(7,0),(0,8),(8,0),(0,9),(9,0),(0,10),(10,0), \\
&\quad (0,11),(11,0),(1,6),(6,1),(1,7),(7,1),(1,8),(8,1),(1,9),(9,1), \\
&\quad (1,10),(10,1),(1,11),(11,1),(2,6),(6,2),(2,7),(7,2),(2,8),(8,2), \\
&\quad (2,9),(9,2),(2,10),(10,2),(2,11),(11,2),(3,6),(6,3),(3,7),(7,3), \\
&\quad (3,8),(8,3),(3,9),(9,3),(3,10),(10,3),(3,11),(11,3),(4,6),(6,4), \\
&\quad (4,7),(7,4),(4,8),(8,4),(4,9),(9,4),(4,10),(10,4),(4,11),(11,4), \\
&\quad (5,6),(6,5),(5,7),(7,5),(5,8),(8,5),(5,9),(9,5),(5,10),(10,5), \\
&\quad (5,11),(11,5)\}.
\end{aligned}
$$

Observe that (G, R_{D,D^*}) is an undirected bipartite graph. A graphical representation of (G, R_{D,D^*}) is shown in Figure 1.

(ii) Let $A = \{0,1,2\}$, $B = \{0,1,2,3,4,5\}$, $D = \{6,7,8\}$ and $D^* = \{9,10,11\}$. Then A is a $\mathcal{L}$ associative subquasigroup of G, B is a $\mathcal{R}$ associative subquasigroup of G, D is a $\mathcal{L}$ associative subset of G and D^* is a $\mathcal{R}$

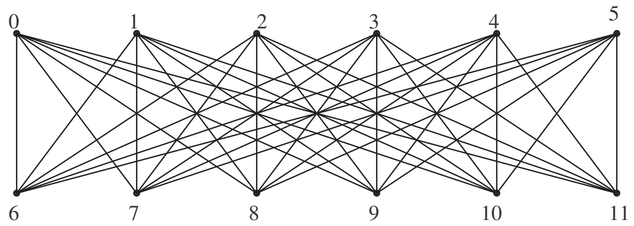

Figure 1: The graph representing (G, R_{D,D^*}) with connection sets $ADA = \{6, 7, 8, 9, 10, 11\}$, $BD^*B = \{0, 1, 2, 3, 4, 5\}$.

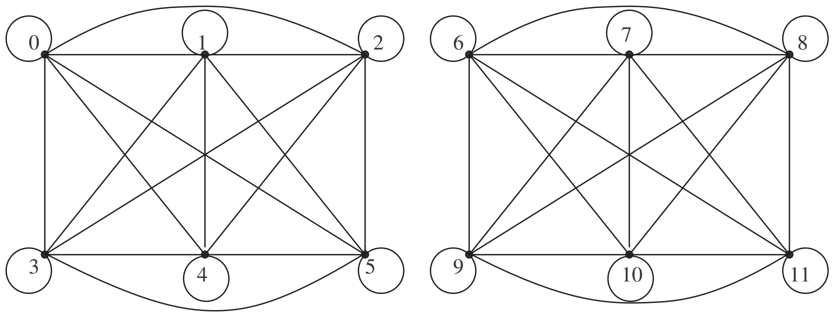

Figure 2: The graph representing (G, R_{D,D^*}) with connection sets $ADA = \{6, 7, 8\}$ and $BD^*B = \{6, 7, 8, 9, 10, 11\}$

associative subset of G. Let $a = 0$ and $b = 3$. One can easily verify that

$$ADA = \{6, 7, 8\}, BD^*B = \{6, 7, 8, 9, 10, 11\}, \text{ and}$$

$$R_{D,D^*} = \{0, 1, 2, 3, 4, 5\} \times \{0, 1, 2, 3, 4, 5\} \cup \{6, 7, 8, 9, 10, 11\} \times \{6, 7, 8, 9, 10, 11\}$$

Observe that (G, R_{D,D^*}) is the disjoint union of two complete graphs. A graphical representation of (G, R_{D,D^*}) is shown in Figure 2.

(iii) Take $A = B = \{0, 1, 2\}$, $D = \{3, 4, 5\}$, $D^* = \{6, 7, 8\}$. Then the subquasigroup A is both $\mathcal{L}$ and $\mathcal{R}$ associative. Furthermore, D and D^* are respectively, $\mathcal{L}$ associative and $\mathcal{R}$ associative subsets of G. If we take $a = b = 0$,

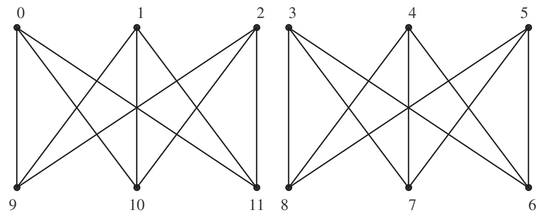

Figure 3: The graph representing (G, R_{D,D^*}) with connection sets $ADA = \{3,4,5\}$ and $BD^*B = \{6,7,8\}$

it is not difficult to verify that

$$\begin{aligned} ADA &= \{3,4,5\}, BD^*B = \{6,7,8\} \text{ and} \\ R_{D,D^*} &= \{(0,9),(9,0),(0,10),(10,0),(0,11),(11,0),(1,9),(9,1),(1,10),(10,1), \\ &\quad (1,11),(11,1),(2,9),(9,2),(2,10),(10,2),(2,11),(11,2),(3,6),(6,3), \\ &\quad (3,7),(7,3),(3,8),(8,3),(4,6),(6,4),(4,7),(7,4),(4,8),(8,4),(5,6), \\ &\quad (6,5),(5,7),(7,5)\} \end{aligned}$$

Observe that (G, R_{D,D^*}) is the disjoint union of two complete bipartite graphs. The graphical representation this graph is shown in Figure 3.

(iv) Let $A = B = D = D^* = \{0,1,2\}$, $a = b = 0$. Then A, B, D and D^* are $\mathcal{L}$ as well as $\mathcal{R}$ subquasigroups of G. It can be easily verify that

$$\begin{aligned} ADA &= BD^*B = \{0,1,2\} \\ &= \{(0,0),(0,1),(1,0),(1,2),(2,1),(2,0),(0,2),(1,1),(2,2),(3,3),(3,4), \\ &\quad (4,3),(4,5),(5,4),(3,5),(5,3),(4,4),(5,5),(6,6),(6,7),(7,6),(7,8), \\ &\quad (8,7),(8,6),(6,8),(7,7),(8,8),(9,9),(9,10),(10,9),(9,11),(11,9), \\ &\quad (10,11),(11,10),(10,10),(11,11)\} \end{aligned}$$

Observe that (G, R_{D,D^*}) is the disjoint union of 4 complete graphs. A graphical representation of (G, R_{D,D^*}) is shown in Figure 4.

Example 2. Let $G = \{1,2,3,\ldots,2n\}$. Let $N_1 = \{1,2,3,\ldots,n\}$ and $N_2 = \{n+1, n+2, \ldots, 2n\}$ be a partition of G. Define the product in G as follows with the condition that the equations $ax = b$ and $ya = b$ have unique solutions in G:

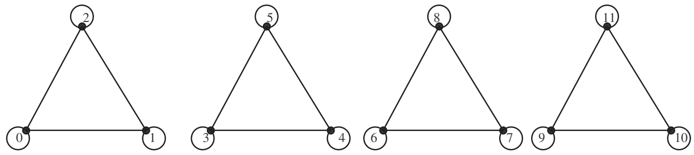

Figure 4: The graph representing (G, R_{D,D^*}) with connection sets $ADA = BD^*B = \{0, 1, 2\}$

$*$	N_1	N_2
N_1	N_1	N_2
N_2	N_2	N_1

Take $A = B = N_1$, $D^* = N_2$, $a = b = 1$. Then we find that $ADA = N_1$ and $BD^*B = N_2$ and

$$R_{D,D^*} = (N_1 \times N_2) \cup (N_2 \times N_1).$$

Hence (G, R_{D,D^*}) is the complete bipartite graph $K_{n,n}$. Thus every complete bipartite graph is a generalized cayley graph.

Example 3. Let $G = \{0, 1, 2, \ldots, mn - 1\}$. For $i = 0, 1, 2, \ldots, n - 1$, define $N_i = \{im, im + 1, \ldots, im + m - 1\}$. Observe that $\{N_0, N_1, \ldots, N_{n-1}\}$ is a partition of G. Define a multiplication in G as follows with the condition that the equations $ax = b$ and $ya = b$ have unique solutions in G.

$$N_iN_j = N_{(i+j)\text{mod}(n)} \text{ for all } i = j = 0, 1, 2, \ldots, n - 1.$$

Take $A = B = D = N_0$, $D^* = G \setminus N_0$ and $a = b = 0$. Then one can easily verify that A, B, D and N_0 are $\mathcal{L}$ as well as $\mathcal{R}$ associative subquasigroups. Moreover, $G \setminus N_0$ is a $\mathcal{R}$ associative subset of G. We find that

$$R_{D,D^*} = \bigcup_{i \neq j} (N_i \times N_j)$$

Observe that (G, R_{D,D^*}) is a complete n partite graph $K_{m,m,\ldots,m}$. As a consequence $K_{m,m,\ldots,m}$ is a generalized cayley graph.

2.2 Basic Results

Next, we prove some interesting relationship between the properties of quasigroups and those of (G, R_{D,D^*}). In this sequel, need the following lemma:

Lemma 2.2. *If $z \in ADA, t \in BD^*B$, and $x \in G$ then we have*

(i) $z = z_1 a$ *for some* $z_1 \in ADA$, *and* $za \in ADA$, $az \in ADA$ *for all* $a \in A$.

(ii) $t = bz_2$ *for some* $z_2 \in BD^*B$, *and* $tb \in BD^*B$, $bt \in BD^*B$ *for all* $b \in B$

(iii) $(zx)t = (z^*((ax)b))t^*$ *for some* $z^* \in ADA$ *and* $t^* \in BD^*B$.

(iv) $(z(ab))t = z^*t^*$ *for some* $z^* \in ADA$, $t^* \in BD^*B$ *and* $zt = (u(ab))v$ *for some* $u \in ADA$, $v \in BD^*B$.

Proof. Proof is trivial. □

Let M and N be subsets of a quasigroup G and let a and b be fixed elements in G. We will use the following notations:

(1) $[M|N]_a^b = \{x \in G : (ax)b = (z_1x)z_2$ for some $z_1 \in M, z_2 \in N\}$.

(2) $M_aN_b = \{x \in G : (a(ab))b = (z_1x)z_2$ for some $z_1 \in M, z_2 \in N\}$.

(3) $[M_a|N_b] = \{x \in G : (ax)b = z_1z_2$ for some $z_1 \in M, z_2 \in N\}$.

(4) $[[M|N][_a^b= \{x \in G : (ax)b = (z_1(z_2 \ldots (z_{n-1}(z_nt_n)t_{n-1})t_{n-1} \ldots)t_1,$ for some $z_i \in M, t_i \in N\}$.

(5) $[[M_aN_b]] = \{x \in G : (a(ab))b = (z_1(z_2 \ldots (z_{n-1}(z_nxt_n))t_{n-1} \ldots)t_1,$ for some $z_i \in M, t_i \in N\}$.

(6) $[M|N] = \{z_1(z_2 \cdots ((z_{n-1}(z_nt_n))t_{n-1}) \cdots t_2)t_1 : z_i \in M, t_i \in N, n = 1, 2, 3, \ldots\}$.

Proposition 2.3. *The graph (G, R_{D,D^*}) is an empty(i. e., $R_{D,D^*} = \emptyset$) if and only if $D = \emptyset$ or $D^* = \emptyset$.*

Proof. By definition, (G, R_{D,D^*}) is trivial if and only if $R_{D,D^*} = \emptyset$. Since A and B are nonempty subquasigroups of G, $D = \emptyset$ or $D^* = \emptyset$. □

Proposition 2.4. *The graph* $(G, , R_{D,D^*})$ *is a reflexive if and only if* $G = [ADA|BD^*B]_a^b$.

Proof. First assume that (G, R_{D,D^*}) is reflexive and let $x \in G$. Then, $(x, x) \in R_{D,D^*}$. Hence by definition,

$$(ax)b = (z_1x)z_2 \quad \text{for some } z_1 \in ADA \text{ and } z_2 \in BD^*B.$$

This implies that $x \in [ADA|BD^*B]_a^b$. Since x is an arbitrary element in G, we have $G = [ADA|BD^*B]_a^b$. The proof of the converse is trivial. □

Proposition 2.5. *If* (G, R_{D,D^*}) *is a symmetric graph (i.e.,* $R_{D,D^*} = R^{-1}_{D,D^*}$*), then* $[ADA_a|BD^*B_b] = (ADA)_a(BD^*B)_b$.

Proof. Suppose (G, R_{D,D^*}) is symmetric. Observe that

$$\begin{aligned} x \in [ADA_a|BD^*B_b] &\Leftrightarrow (ax)b = z_1z_2, \text{ for some } z_1 \in ADA, z_2 \in BD^*B \\ &\Leftrightarrow (ax)b = (z_1^*(ab))z_2^* \text{ for some } z_1^* \in ADA, z_2^* \in BD^*B \\ &\Leftrightarrow (ab, x) \in R_{D,D^*} (\text{by the definition of } R_{D,D^*}) \\ &\Leftrightarrow (x, ab) \in R_{D,D^*} (\text{ since } R_{D,D^*} \text{ is symmetric}) \\ &\Leftrightarrow (a(ab))b = (t_1x)t_2 \text{ for some } t_1 \in ADA, t_2 \in BD^*B \\ &\Leftrightarrow x \in (ADA)_a(BD^*B)_b. \end{aligned}$$

This implies that $[(ADA)_a(BD^*B)_b] = (ADA)_a(BD^*B)_b.$ □

Proposition 2.6. (G, R_{D,D^*}) *is a transitive graph (i.e.,* $R_{D,D^*} \circ R_{D,D^*} \subseteq R_{D,D^*}$*), then* $(ADA)^2(BD^*B)^2 \subseteq (ADA)(BD^*B)$.

Proof. Assume (G, R_{D,D^*}) is transitive graph. Let $z_1, z_2 \in ADA, z_3$ and $z_4 \in BD^*B$. Note that

$$\begin{aligned} (a(z_1z_3))b &= ((a_1z_1))z_3)b \text{ for some } a_1 \in A \ (\because A \text{ is } \mathcal{L} \text{ associative}) \\ &= (z_5z_3)b \text{ for some } z_5 \in B \text{ (is by Lemma 2.2)} \\ &= z_5(z_3b_1) \text{ for some } b_1 \in B \ (\because B \text{ is } \mathcal{R} \text{ associative}) \\ &= z_5z_6 \text{ for some } z_6 \in B \text{ (by Lemma 2.2)} \\ &= (z_7(ab))z_8 \text{ for some } z_7 \in ADA, z_8 \in BD^*B \text{ (by Lemma 2.2).} \end{aligned}$$

This implies that $((ab), z_1z_3) \in R_{D,D^*}$. Let $t_1 = z_1z_3$. Then

$$\begin{aligned}(a((z_2t_1)z_4))b &= ((a_2(z_2t_1))z_4)b \text{ for some } a_2 \in A\ (\because A \text{ is } \mathcal{L} \text{ associative})\\ &= (((a_3z_2)t_1)z_4)b \text{ for some } a_3 \in A\ (\because A \text{ is } \mathcal{L} \text{ associative})\\ &= ((z_9t_1)z_4)b \text{ for some } z_9 \in B \text{ (by Lemma 2.2)}\\ &= (z_9t_1)(z_4b_2) \text{ for some } b_2 \in B\ (\because B \text{ is } \mathcal{R} \text{ associative})\\ &= (z_9t_1)z_{10} \text{ for some } z_{10} \in BD^*B \text{ (by Lemma 2.2)}.\end{aligned}$$

This implies that $(t_1, ((z_2t_1)z_4)) \in R_{D,D^*}$. Since (G, R_{D,D^*}) is transitive, we have $(ab, ((z_2t_1)z_4)) \in R_{D,D^*}$. This means that

$$(a(z_2t_1)z_4)b = (t_3(ab))t_4 \text{ for some } t_3 \in ADA, t_4 \in BD^*B.$$

That is,

$$(z_{11}t_1)z_{12} = (t_3(ab))t_4 \text{ for some } z_{11} \in ADA, z_{12} \in BD^*B.$$

Equivalently,

$$(ADA)^2(BD^*B)^2 \subseteq (ADA)(BD^*B).$$

□

Proposition 2.7. *Assume that* $(ADA)^2 \subseteq ADA$ *and* $(BD^*B)^2 \subseteq BD^*B$. *Then* (G, R_{D,D^*}) *is a transitive graph.*

Proof. Let x, y and $z \in G$ such that $(x, y) \in R_{D,D^*}$ and $(y, z) \in R_{D,D^*}$. Then by the definition of R_{D,D^*}, we have

$$(ay)b = (z_1x)z_2 \text{ for some } z_1 \in ADA, z_2 \in BD^*B, \tag{1}$$

$$(az)b = (z_3y)z_4 \text{ for some } z_3 \in ADA, z_4 \in BD^*B. \tag{2}$$

Using lemma 2.2, equation (2) can be written as:

$$\begin{aligned}(az)b &= ((z_5a)y)(bz_6) \text{ for some } z_5 \in ADA,\ z_6 \in BD^*B\\ &= ((z_6(ay))b)z_7 \text{ for some } z_7 \in BD^*B\ (\because BD^*B \text{ is } \mathcal{R} \text{ associative})\\ &= (z_8((ay)b))z_7 \text{ for some } z_8 \in ADA\ (\because ADA \text{ is } \mathcal{L} \text{ associative}).\end{aligned} \tag{3}$$

Inserting the value of $(ay)b$ in equation (3), we get

$$\begin{aligned}
(az)b &= (z_8((z_1x)z_2))z_7 \\
&= ((z_9(z_1x))z_2)z_7 \ \text{ for some } z_9 \in ADA \ (\because ADA \text{ is } \mathcal{L} \text{ associative}) \\
&= (((z_{10}z_1)x)z_2)z_7 \ \text{ for some } z_{10} \in ADA \ (\because ADA \text{ is } \mathcal{L} \text{ associative}) \\
&= ((t_1x)z_2)z_7 \text{ where } t_1 = z_{10}z_1 \in (ADA)(ADA) \\
&= (t_1x)(z_2z_{11}) \ \text{ for some } z_{12} \in BD^*B \ (\because BD^*B \text{ is } \mathcal{R} \text{ associative}) \\
&= (t_1x)t_2 \ \text{ where } \ t_2 = z_2z_{11} \in (BD^*B)(BD^*B). \qquad (4)
\end{aligned}$$

From the fact that $(ADA)(ADA) = ADAADA \subseteq ADA$ and $(BD^*B)(BD^*B) = BD^*BD^*B \subseteq BD^*B$, equation (8) implies that $(x, z) \in R_{D,D^*}$. Hence (G, R_{D,D^*}) is a transitive graph. □

Proposition 2.8. *If (G, R_{D,D^*}) is a complete graph, then*

$$G = [(ADA)_a|(BD^*B)_b].$$

Proof. Suppose (G, R_{D,D^*}) is a complete graph and let $x \in G$. Then $(ab, x) \in R_{D,D^*}$. This implies that $(ax)b = (z_1(ab))z_2$, for some $z_1 \in ADA$ and $z_2 \in BD^*B$. That is, $(ax)b = z_1^*z_2^*$, for some $z_1^* \in ADA$ and $z_2^* \in BD^*B$. Equivalently, $x \in [(ADA)_a|(BD^*B)_b]$. Since x is an arbitrary element of G,

$$G = [(ADA)_a|(BD^*B)_b]$$

This completes the proof. □

Proposition 2.9. *If (G, R_{D,D^*}) is connected, then $G = [[ADA|BD^*B]]_a^b$.*

Proof. Suppose that (G, R_{D,D^*}) is connected and let $x \in G$. Then there is a path from ab to x, say:

$$(ab, x_1, x_2, \cdots, x_n, x)$$

Then we have the following:

$$\begin{aligned}
(ax_1)b &= (z_1(ab))t_1 \text{ for some } z_1 \in ADA \text{ and } t_1 \in BD^*B \\
&= z_1^*t_1^* \text{ for some } z_1^* \in ADA \text{ and } t_1^* \in BD^*B \quad \text{(by Lemma 2.2)}, \quad (5)\\
(ax_2)b &= (z_2x_1)t_2 \text{ for some } z_2 \in ADA \text{ and } t_2 \in BD^*B \\
&= (((z_3a)x_1)(bt_3)) \text{ for some } z_3 \in ADA \text{ and } t_3 \in BD^*B \\
&= (z_4((ax_1)b))t_4 \text{ for some } z_4 \in ADA \text{ and } t_4 \in BD^*B \\
&= (z_4(z_1^*t_1^*))t_4 \quad \text{(by equation (5))}, \quad (6)\\
&\vdots \\
(ax)b &= (z_{n+1}x_n)t_{n+1} \\
&= (z_{n+1}^*((ax_n)b))t_{n+1}^* \text{for some } z_{n+1}^* \in ADA, t_{n+1}^* \in BD^*B) \\
&= (z_{n+1}^*((z_nx_{n-1})t_n))t_{n+1}^* = \cdots = (z_{n+1}^*(\cdots(z_2^*(z_1^*t_1^*))t_2^*\cdots))t_{n+1}^*. \quad (7)
\end{aligned}$$

From equation (7), it follows that

$$G = [[ADA|BD^*B]]_a^b.$$

This completes the proof. □

Proposition 2.10. *If* (G, R_{D,D^*}) *is locally connected, then*

$$[AADA|BD^*B] = [[(ADA)_a(BD^*B)_b]]$$

.

Proof. Assume that (G, R_{D,D^*}) is locally connected. Let $x \in [ADA|BD^*B]$. Then

$$x = (z_1(z_2 \dots (z_{n-1}(z_nt_n))t_{n-1} \dots t_2)t_1$$

for some $z_i \in ADA$ and $t_i \in BD^*B$. Let

$$x_1 = z_nt_n, x_2 = (z_{n-1}x_1)t_{n-1}, \dots, x_n = (z_1x_{n-1})t_1$$

Using Lemma 2.2, the above equation can be re-written as:

$$(ax_1)b = (z_n^*(ab))t_n^*, (ax_2)b = (z_{n-1}^*(ab)x_1^*)t_{n-1}^*, \dots, (ax_n^*)b = (z_1^*(ab)x_{n-1}^*)t_1^*.$$

for some $z_i^* \in AADA$ and $t_i^* \in BD^*B$. Then $(ab, x_1, \ldots, x_n, x_n)$ is a path from ab to x. Since (G, R_{D,D^*}) is locally connected, there exits a path from x to ab, say:

$$(x, y_1, \ldots, y_m, ab)$$

This implies that $x \in [[(ADA)_a(D^*B)_b]]$. Hence

$$[ADA|D^*B] \subseteq [[(ADA)_a(BD^*B)_b]]$$

. Similarly, $[[(ADA)_a(D^*B)_b]] \subseteq [ADA|D^*B]$. □

Proposition 2.11. *If (G, R_{D,D^*}) is semi connected, then*

$$G = [[ADA|D^*B]]_a^b \cup [[(ADA)_a(BD^*B)_b]].$$

Proof. Assume that (G, R_{D,D^*}) is semi connected and let $x \in G$. Then there exits a path from ab to x, say

$$(ab, x_1, \ldots, x_n, x)$$

or a path from x to ab, say

$$(x, y_1, \ldots, y_m, ab)$$

This implies that

$$x \in [[ADA|D^*B][_a^b \cup [[(ADA)_a(BD^*B)_b]].$$

Since x is arbitrary, it follows that

$$G = [[ADA|D^*B][_a^b \cup [[(ADA)_a(BD^*B)_b]].$$

□

Proposition 2.12. *(G, R_{D,D^*}) is a hasse- diagram, if and only if*

$$(ADA)^n \cap (ADA) = \emptyset \quad or \quad (BD^*B)^n \cap (BD^*B) = \emptyset, \quad n \geq 2.$$

Proof. First, assume that (G, R_{D,D^*}) is a hasse- diagram. Then for any vertices $x_0, x_1, \ldots, x_n \in G$ with $(x_i, x_{i+1}) \in R_{D,D^*}$ for all $i = 0, 1, 2, \ldots, n-1$

implies that $(x_0, x_n) \notin R_{D,D^*}$. Observe that $(x_i, x_{i+1}) \in R_{D,D^*}$ for all $i = 0, 1, 2, \dots, n-1$ implies that

$$(ax_{i+1})b = (z_i x_0)t_i \text{ for some } z_i \in ADA \text{ and } t_i \in BD^*B \tag{8}$$

for $i = 0, 1, 2, \dots, n-1$. Putting $n = 0, 1, 2, \dots (n-1)$ successively in equation (8), we get

$$\begin{aligned}(ax_1)b &= (z_1 x_0)t_1\\ (ax_2)b &= (z_2 x_1)t_2\\ (ax_3)b &= (z_3 x_2)t_3\\ &\vdots\\ (ax_n)b &= (z_n x_{n-1})t_n\end{aligned}$$

Using Lemma 4.1, above equations can be re-written as:

$$\begin{aligned}(ax_2)b &= (u_1((ax_1)b))v_1 \text{ for some } u_1 \in ADA \text{ and } v_1 \in BD^*B\\ &= (u_1((z_1x_0)t_1))v_1\\ &= ((u_2(z_1x_0))t_1)v_1 \text{ for some } u_2 \in ADA \text{ and } v_1 \in BD^*B\\ &= ((u_3z_1)x_0)t_1)v_1 \text{ for some } z_1 in ADA \text{ and } v_1 \in BD^*B\\ &= ((u_3z_1)x_0)(t_1v_2)\\ &= (r_1x_0)s_1 \text{ where } r_1 = u_3z_1 \in (AA)^2 \text{ and } s_1 = t_1v_1 \in (BD^*B)^2.\end{aligned}$$

Similarly,

$$(ax_3)b = (r_2x_0)s_2 \text{ where } r_2 \in (ADA)^3 \text{ and } s_2 \in (BD^*B)^3$$

Proceeding like this, we get

$$(ax_n)b = r_n x_0 s_n \text{ for some } r_n \in (ADA)^n \text{ and } s_n \in (BD^*B)^n$$

Since $(x_0, x_n) \notin R_{D,D^*}$, therefore

$$(ADA)^n \cap (ADA) = \emptyset \text{ or } (BD^*B)^n \cap (BD^*B) = \emptyset.$$

Conversely, assume that

$$(ADA)^n \cap (ADA) = \emptyset \text{ or } (BD^*B)^n \cap (BD^*B) = \emptyset, n \geq 2.$$

We will show that (G, R_{D,D^*}) is a hasse-diagram. Let

$$x_0, x_1, \ldots, x_n$$

be any $(n+1)$ elements of G with $n \geq 2$, and $(x_i, xi+1) \in R_{D,D^*}$ for all $i = 0, 1, 2, \ldots, n-1$. Then we have

$$(ax_n)b = (zx_0)t \text{ for some } z \in (ADA)^n \text{ and } t \in (D^*B)^n$$

Since $(ADA)^n \cap (ADA) = \emptyset$ or $(D^*B)^n \cap (D^*B) = \emptyset$, $(x_0, x_n) \notin R_{D,D^*}$. Hence (G, R_{D,D^*}) is a hasse-diagram. □

3 Open Problem

In this paper we have introduced a class of generalized cayley digraphs induced by quasigroups. It is well known that all cayley graphs induced by groups are vertex transitive graphs. One can naturally ask the question: are the generalized cayley di-graphs induced by quasigroups vertex transitive? So we conclude this section with the following problem:

Problem 3.1. *Prove or disprove that* (G, R_{D,D^*}) *is vertex transitive.*

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