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Approximation of Common Fixed Points of a Finite Family of Asymptotically $\phi-$Demicontractive Maps Using a Composite Implicit Iteration Process

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Abstract

We prove that the modified form of the composite implicit iteration process introduced by Su and Li [1] can be used to approximate the common fixed points of a finite family of asymptotically $\phi-$demicontractive maps in real Hilbert spaces. Our results compliment the results of Su and Li [1], Osilike and Isiogugu [2], Igbokwe and Udofia [3], Igbokwe and Udo-Utun [4, 5], Igbokwe and Ini [6] and extend several others from asymptotically demicontractive maps to the more general class of asymptotically $\phi-$demicontractive maps (see for example [7-10]).

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1 Introduction

Let K be a nonempty subset of a real Hilbert space H. A mapping $T: K \to K$ is called asymptotically $\phi-$demicontractive with sequence $\{k_n\} \subseteq [1, \infty)$, $\lim_{n\to\infty} k_n = 1$ (see for example [3]), if $F(T) = \{x \in K : Tx = x\} \neq \phi$ and there exists an increasing continuous function $\phi: [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\left\|T^n x - p\right\|^2 = k_n \left\|x - p\right\|^2 + \left\|x - T^n x\right\|^2 - \phi\left(\left\|x - T^n x\right\|\right), \tag{1}$$

$\forall x \in K$, $p \in F(T)$ and $n \geq 1$.

A mapping $T: K \to K$ is called asymptotically demicontractive with sequence $\{a_n\} \subseteq [1, \infty)$, $\lim_{n\to\infty} a_n = 1$ (see for example [10]), if $F(T) \neq \phi$ and $\forall x \in K$, $p \in F(T)$, $\exists$ a $k \in [0, 1) \ni$

$$\left\|T^n x - p\right\|^2 \leq a_n^2 \left\|x - p\right\|^2 + k\left\|\left(I - T^n\right)x\right\|^2, \tag{2}$$

T is $k-$strictly asymptotically pseudocontractive with a sequence $\{a_n\} \subseteq [1, \infty)$, $\lim_{n\to\infty} a_n = 1$ if $\forall x \in K$, $n \in N$, $\exists$ a $k \in [0, 1) \ni$

$$\left\|T^n x - T^n y\right\|^2 \leq a_n^2 \left\|x - p\right\|^2 + k\left\|\left(I - T^n\right)x - \left(I - T^n\right)y\right\|^2, \tag{3}$$

where I is the identity operator. The class of $k-$strictly asymptotically pseudocontractive and asymptotically demicontractive maps were first introduced in Hilbert spaces by Qihou [10]. Observe that a $k-$strictly asymptotically pseudocontractive map with a nonempty fixed point set $F(T)$ is asymptotically demicontractive. An example of a $k-$strictly asymptotically pseudocontractive map is given in [11].

Furthermore, T is uniformly $L-$Lipschitzian if there exists a constant

$$L > 0 \ni \|T^n x - T^n y\| \leq L\|x - y\|.$$

Let K be a subset of an arbitrary real Banach space E, a mapping $T : D(T) \to E$ is said to be asymptotically $\phi-$demicontractive with sequence $\{a_n\} \subseteq [1, \infty)$, $\lim_{n\to\infty} a_n = 1$ (See for example [2]), if $F(T) \neq \phi$ and there exists a strictly increasing continuous function $\phi : [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle x - T^n x, j(x-p)\rangle \geq \phi(\|x - T^n x\|) - \tfrac{1}{2}(a_n^2 - 1)\|x - p\|^2 \tag{4}$$

$\forall x \in K$, $p \in F(T)$ and $n \in N$. j is the single – valued duality mapping from E to E^* given by

$$j(x) = \{ f \in E^* : \langle x, f\rangle = \|x\|^2 \ ; \ \|x\|^2 = \|f\|^2 \},$$

which holds in strictly convex dual spaces. E^* denotes the dual space of E and $\langle \, , \rangle$ denotes the generalized duality paring. The class of asymptotically $\phi-$demicontractive maps was first introduced in arbitrary real Banach spaces by Osilike and Isiogugu [2]. It is shown in [2] that the class of asymptotically demicontractive map is a proper subclass of the class of asymptotically $\phi-$demicontractive map while in [3], it is shown that every asymptotically demicontractive map is asymptotically $\phi-$demicontractive with $\phi : [0, \infty) \to [0, \infty)$ given by

$$\phi(t) = (1-k)t^2 - \tfrac{1}{2}(a_n^2 - 1)\|x - p\|^2.$$

These classes of operators have been studied by several authors (See for example [2,3, 7-10, 12]). In [2] Osilike and Isiogugu proved the convergence of the modified averaging iteration process of Mann [13] to the fixed points of asymptotically $\phi -$demicontractive maps. Specifically they proved the following:

Theorem 1.1 ([2], p.65) *Let E be a real Banach space and K a nonempty closed convex subset of E. Let $T : K \to K$ be a completely continuous uniformly $L -$Lipschitzian asymptotically $\phi -$demicontractive mapping with a sequence $\{a_n\} \subset [1, \infty) \ni \sum \left(a_n^2 - 1\right) < \infty$. Let $\{a_n\}$ be a real sequence satisfying*

(i) $0 < \alpha_n < 1$ (ii) $\sum \alpha_n = \infty$ (iii) $\sum \alpha_n^2 < \infty$.

Then the sequence $\{x_n\}_{n=1}^{\infty}$ generated from arbitrary $x_1 \in K$ by the modified averaging Mann iteration process

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n, \quad n \geq 1 \tag{5}$$

converges strongly to a common fixed point of T.

Similarly, in [3], using the modified averaging implicit iteration scheme $\{x_n\}$ of Sun [14], generated from an $x_1 \in K$, by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_i^k x_n, \; n \geq 1$$

where $n = (k-1)N + i, \, i \in I = \{1, 2, 3 \ldots, N\}$, Igbokwe and Udofia [3] proved that under certain conditions on the iteration sequence $\{\alpha_n\}$, the above iteration process $\{x_n\}$ converges strongly to the common fixed point of the family $\{T_i\}_{i=1}^{N}$ of N uniformly $L_i -$Lipschitzian asymptotically $\phi -$demicontractive self maps of nonempty closed convex subset of a Hilbert space H.

Recently, Su and Li [1] introduced the following iteration scheme and called it Composite Implicit Iteration Process. From $x_1 \in K$, the sequence $\{x_n\}_{n=1}^{\infty}$ is generated by

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n)T_n y_n$$
$$y_n = \beta_n x_{n-1} + (1 - \beta_n)T_n x_n, \quad \text{n} \geq 1 \tag{6}$$

where $\{\alpha_n\}, \{\beta_n\} \subseteq [0,1]$, $T_n = T_{n \bmod N}$.

Motivated by the results of Su and Li [1], very recently, Igbokwe and Ini [6] modified the iteration process (6) and applied the modified iteration process to approximate the common fixed points of a finite family of k-strictly asymptotically pseudocontractive maps. In compact form, the modified composite implicit iteration process of Igbokwe and Ini is expressed as follows:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T_i^k y_n$$

$$y_n = \beta_n x_{n-1} + (1-\beta_n)T_i^k x_n, \quad n \geq 1 \tag{7}$$

where $n = (k-1)N + i,\ i \in I = \{1, 2, 3 \ldots, N\}$. $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$.

Observe that, if $T : K \to K$ is uniformly $L-$Lipschitzian asymptotically $\phi-$Demicontractive map with sequence $\{a_n\} \subseteq [1, \infty)$ such that $\lim_{n\to\infty} a_n = 1$ then for every fixed $u \in K$ and $t \in \left(\frac{L}{1+L}, 1\right)$, the operator $S_{t,s,n} : K \to K$ defined for all $x \in K$ by

$$S_{t,s,n}x = tu + (1-t)T^n[su + (1-s)T^n x]$$

satisfies

$$\|S_{t,s,n}x - S_{t,s,n}y\| \leq (1-t)(1-s)L^2\|x-y\|, \quad \forall x, y \in K.$$

Thus, the composite implicit iteration process (7) is defined in K for the family $\{T_i\}_{i=1}^N$ of N uniformly $L-$Lipschitzian asymptotically $\phi-$Demicontractive mappings of nonempty closed convex subset K of a Hilbert space provided that $\{\alpha_n\}, \{\beta_n\} \subseteq (\eta, 1)$ for all $n \geq 1$, where $\eta = \dfrac{L}{1+L}$ and $L = \max_{1\leq i\leq N}\{L_i\}$.

It is our purpose in this paper to prove that the iteration process (7) converges to common fixed points of finite family of N uniformly $L-$Lipschitzian asymptotically $\phi-$Demicontractive mappings in Hilbert space. We show that the recent results of Osililke [2], Osililke, Aniagbosor and Akucku [9] concerning the iterative approximation of fixed points of asymptotically $\phi-$demicontractive and asymptotically demicontractive maps which are themselves generalizations of a theorem of Qihou [10], a result of Osililke [7] and a result of Osililke and

Aniagbosor [8] will be special cases of our results. Moreover, in Hilbert spaces, our present results extend the recent results of Igbokwe and Ini [6] from k-strictly asymptotically pseudocontraction to the much more general asymptotically $\phi-$demicontractive maps.

2 Preliminary Notes

In the sequel, we need the following:

Lemma 2.1 ([8], p.80) *Let* $\{a_n\}_{n=1}^{\infty}, \{b_n\}_{n=1}^{\infty}$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, n \ge 1.$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ *and* $\sum_{n=1}^{\infty} b_n < \infty$, *Then* $\lim_{n\to\infty} a_n$ *exists. If in addition* $\{a_n\}_{n=1}^{\infty}$ *has a* subsequence which converges strongly to zero, then $\lim_{n\to\infty} a_n = 0$.

Definition 2.1 *Let* K *be a closed subset of a real Banach space* E *and* $T: K \to K$ *be a mapping. T is said to be semicompact* (*see for example* [1]) *if for any bounded sequence* $\{x_n\}$ *in* K *such that* $\|x_n - T_n x_n\| \to 0$ *as* $n \to \infty$, *there exists a subsequence* $\{x_{n_k}\} \subset \{x_n\}$ *such that* $x_{n_k} \to x^* \in K$.

3 Main Results

Let K be a subset of a real Hilbert space H. We call the mapping $T: K \to K$ asymptotically $\phi-$demicontractive with sequence $\{a_{in}\} \subset [0,\infty)$, $\lim_{n\to\infty} a_n = 1$ if $F(T) \ne \phi$ and there exists a strictly increasing continuous function $\phi: [0, \infty) \to [0, \infty)$ with $\phi(0) = 0$ such that

$$\left\|T^n x - T^n p\right\|^2 \leq \left[1 + \tfrac{1}{2}\left(a_n^2 - 1\right)\right]\left\|x - p\right\|^2 + \left\|x - T^n x\right\|^2 - \phi\left(\left\|x - T^n x\right\|\right) \qquad (8)$$

for all $x \in K$, $p \in F(T)$ and $n \geq 1$.

Since in a Hilbert space, j is the identity map, for all $x \in K$, $p \in F(T)$ and $n \geq 1$. Using (4), we have:

$$\begin{aligned}
\left\|T^n x - T^n p\right\|^2 &= \left\|x - p - [\left(I - T^n\right)x - \left(I - T^n\right)p]\right\|^2 \\
&= \left\|x - p\right\|^2 - 2\left\langle\left(I - T^n\right)x - \left(I - T^n\right)p, (x - p)\right\rangle + \left\|\left(I - T^n\right)x - \left(I - T^n\right)p\right\|^2 \\
&= \left\|x - p\right\|^2 - 2\left\langle x - T^n x, (x - p)\right\rangle + \left\|x - T^n x\right\|^2 \\
&= \left\|x - p\right\|^2 - \phi\left(\left\|x - T^n x\right\|\right) + \tfrac{1}{2}\left(a_n^2 - 1\right)\left\|x - p\right\|^2 + \left\|x - T^n x\right\|^2 \\
&\leq \left[1 + \tfrac{1}{2}\left(a_n^2 - 1\right)\right]\left\|x - p\right\|^2 + \left\|x - T^n x\right\|^2 - \phi\left(\left\|x - T^n x\right\|\right), \text{ proving (8).}
\end{aligned}$$

Observe that if we set $1 + \tfrac{1}{2}\left(a_n^2 - 1\right) = \overline{k_n}$ in (8), then $\overline{k_n} \subseteq [1, \infty)$ and $\lim_{n\to\infty} \overline{k_n} = 1$.

So that equivalently, we have,

$$\left\|T^n x - T^n p\right\|^2 \leq \overline{k_n}\left\|x - p\right\|^2 + \left\|x - T^n x\right\|^2 - \phi\left(\left\|x - T^n x\right\|\right) \qquad (9)$$

Lemma 3.1 *Let E be a normed space and K a nonempty convex subset of E. Let $\{T_i\}_{i=1}^N$ be N uniformly $L_i -$Lipschitzian self mappings of K such that $L = \max\{L_i\}$, L_i the Lipschitzian constant of T_i, $i = 1, 2, \ldots, N$. Let $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in such that*

(i) $\sum_{n=1}^{\infty}\left(1 - \alpha_n\right) = \infty$ (ii) $\sum_{n=1}^{\infty}\left(1 - \alpha_n\right)^2 < \infty$ (iii) $\sum_{n=1}^{\infty}\left(1 - \beta_n\right) < \infty$.

For arbitrary $x_1 \in K$, generate the $\{x_n\}$ by

$$x_n = \alpha_n x_{n-1} + \left(1 - \alpha_n\right)T_i^k y_n$$

$$y_n = \beta_n x_{n-1} + \left(1 - \beta_n\right)T_i^k x_n, \quad n \geq 1$$

Then,

$$\begin{aligned}\|T_i x_n - x_n\| \quad &\leq \quad 2\left[2L^3(2+L)+\left(1+L^2\right)\right]\left\|T_i^k x_n - x_n\right\| \\ &+2L\left\|T_i^{k-1}x_{n-1}-x_{n-1}\right\| + 4\left(1-\alpha_n\right)L^2\left[1+2L(1+L)\right]\|x_n - x_{n-1}\|\end{aligned}$$

Proof. Let $\lambda_{in} = \left\|x_n - T_i^k x_n\right\|$. Then, applying (7) and the fact that $L = \max(L_i)$ we obtain :

$$\begin{aligned}\|x_n - T_i x_n\| \quad &\leq \quad \left\|x_n - T_i^k x_n\right\| + L\left\|T_i^{k-1}x_{n-1} - x_n\right\| \\ &\leq \quad \lambda_{in} + L^2\|x_n - x_{n-1}\| + L\left\|T_i^{k-1}x_n - x_n\right\| \\ &\leq \quad \left[1+\left(1-\alpha_n\right)L^2\right]\lambda_{in} + L\lambda_{in-1} + 2\left(1-\alpha_n\right)L^2\|x_n - x_{n-1}\| \\ &+\left(1-\alpha_n\right)L^2\|T_i x_n - x_n\| + 2\left(1-\alpha_n\right)L^3\|y_n - x_n\|.\end{aligned}$$

That is

$$\begin{aligned}\left[1-\left(1-\alpha_n\right)L^2\right]\|x_n - T_i x_n\| \quad &\leq \quad \left[1+L^2\right]\lambda_{in} + L\lambda_{in-1} + 2\left(1-\alpha_n\right)L^2\|x_n - x_{n-1}\| \\ &+2\left(1-\alpha_n\right)L^3\|y_n - x_n\|. \qquad (10)\end{aligned}$$

Observe that

$$\|y_n - x_n\| \quad \leq \quad (2+L)\lambda_{in} + 2(1+L)\|x_n - x_{n-1}\|. \qquad (11)$$

Substituting (11) into (10), we obtain

$$\begin{aligned}\|T_i x_n - x_n\| \quad &\leq \quad \frac{1}{\left[1-\left(1-\alpha_n\right)L^2\right]}\left\{\left[2L^3(2+L)+\left(1+L^2\right)\right]\left\|T_i^k x_n - x_n\right\|\right. \\ &\left.+L\left\|T_i^{k-1}x_{n-1}-x_{n-1}\right\| + 2\left(1-\alpha_n\right)L^2\left[1+2L(1+L)\right]\|x_n - x_{n-1}\|\right\}.\end{aligned}$$

Since from condition (ii) $\lim_{n\to\infty}\left(1-\alpha_n\right) = 0$, then there exists an $N_1 > 0$ such that $\forall n \geq N_1$,

$$1-\left(1-\alpha_n\right)L^2 \geq \frac{1}{2}.$$

Therefore,

$$\begin{aligned}\|T_i x_n - x_n\| \quad &\leq \quad 2\left[2L^3(2+L)+\left(1+L^2\right)\right]\left\|T_i^k x_n - x_n\right\| \\ &+2L\left\|T_i^{k-1}x_{n-1}-x_{n-1}\right\| + 4\left(1-\alpha_n\right)L^2\left[1+2L(1+L)\right]\|x_n - x_{n-1}\|,\end{aligned}$$

completing the proof. □

Theorem 3.1 *Let* H *be a real Hilbert space and Let* K *be a nonempty closed convex subset of* H *. Let* $\{T_i\}_{i=1}^{N}$ *be* N *uniformly* $L-$*Lipschitzian asymptotically* $\phi-$*demicontractive self maps of* K *with sequence* $\{a_{in}\}\subseteq[1,\infty)$ *such that*

$$\sum_{n=1}^{\infty}(a_{in}-1)<\infty \text{ for all } i\in I\text{ , and } F=\bigcap_{i=1}^{N}F(T_i)\neq\phi$$

where $F(T_i)=\{x\in K: T_i x = x\}$. *Let one member of the family* $\{T_i\}_{i=1}^{N}$ *be semicompact. Let* $\{\alpha_n\}_{n=1}^{\infty}, \{\beta_n\}_{n=1}^{\infty}\subset[\eta,1]$ *be two real sequences satisfying the conditions;*

(i) $\sum_{n=1}^{\infty}(1-\alpha_n)=\infty$

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2<\infty$

(iii) $\sum_{n=1}^{\infty}(1-\beta_n)<\infty,\ (1-\alpha_n)(1-\beta_n)L^2<1,\ 0<\beta<\beta_n\leq\Omega\leq1,$

where $\eta=\frac{L}{1+L}$ *and* $L=\max\{L_i\}$, L_i *uniform Lipschitz constants of* T_i, $i=1,2,\ldots,N$. *Let* $x_1\in K$ *be arbitrary, the modified implicit iteration sequence* $\{x_n\}_{n=1}^{\infty}$ *generated by*

$$x_n=\alpha_n x_{n-1}+(1-\alpha_n)T_i^k y_n$$
$$y_n=\beta_n x_{n-1}+(1-\beta_n)T_i^k x_n,\quad n\geq1 \tag{12}$$

exists in K *and converges strongly to a common fixed point* p *of the family* $\{T_i\}_{i=1}^{N}$.

Proof. We use the well known result of Reinermann [15] (See also Osilike and Igbokwe [16]).

$$\|tx+(1-t)y\|^2 = t\|x\|^2+(1-t)\|y\|^2-t(1-t)\|x-y\|^2 \tag{13}$$

which holds for all $x,\ y \in H,\ t \in [0,1]$.

Using (12) and (13), we obtain

$$
\begin{aligned}
\|x_n - p\|^2 &= \|\alpha_n(x_{n-1} - p) + (1-\alpha_n)(T_i^k y_n - p)\|^2 \\
&= \alpha_n\|x_{n-1} - p\|^2 + (1-\alpha_n)\|T_i^k y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1} - T_i^k y_n\|^2 \\
&\leq \alpha_n\|x_{n-1} - p\|^2 + L^2(1-\alpha_n)\|y_n - p\|^2 - \alpha_n(1-\alpha_n)\|x_{n-1} - T_i^k y_n\|^2 \\
&\leq \alpha_n\|x_{n-1} - p\|^2 + L^2(1-\alpha_n)\|y_n - p\|^2 + \alpha_n(1-\alpha_n)\|x_{n-1} - T_i^k y_n\|^2 \\
&= \alpha_n\|x_{n-1} - p\|^2 + L^2(1-\alpha_n)\|\beta_n(x_{n-1} - p) + (1-\beta_n)(T_i^k x_n - p)\| \\
&\quad + \alpha_n(1-\alpha_n)\|x_{n-1} - T_i^k y_n\|^2 \\
&= \alpha_n\|x_{n-1} - p\|^2 + L^2(1-\alpha_n)\{\ \beta_n\|x_{n-1} - p\|^2 \\
&\quad + (1-\beta_n)\|T_i^k x_n - p\|^2 - \beta_n(1-\beta_n)\|x_{n-1} - T_i^k x_n\|^2\ \} \\
&\quad + \alpha_n(1-\alpha_n)\|x_{n-1} - T_i^k y_n\|^2 \\
&\leq \alpha_n\|x_{n-1} - p\|^2 + L^2\beta_n(1-\alpha_n)\|x_{n-1} - p\|^2 + L^2(1-\alpha_n)(1-\beta_n)\|T_i^k x_n - p\|^2 \\
&\quad - L^2\beta_n(1-\alpha_n)_n(1-\beta_n)\|x_{n-1} - T_i^k x_n\|^2 + (1-\alpha_n)\|x_{n-1} - T_i^k y_n\|^2. \qquad (14)
\end{aligned}
$$

$$
\begin{aligned}
\|x_{n-1} - T_i^k y_n\| &\leq \left[(L\beta_n + 1)^2 + L^2(1-\beta_n)(L\beta_n + 1)\right]\|x_{n-1} - p\|^2 \\
&\quad + \left[L^2(1-\beta_n)(L\beta_n + 1) + L^4(1-\beta_n)^2\right]\|x_n - p\|^2. \qquad (15)
\end{aligned}
$$

Substitute (15) into (14) to obtain

$$
\begin{aligned}
&\|x_n - p\|^2 \leq \\
&[\alpha_n + L^2\beta_n(1-\alpha_n) + (1-\alpha_n)\left[(L\beta_n + 1)^2 + L^2(1-\beta_n)(L\beta_n + 1)\right]]\|x_{n-1} - p\|^2 \\
&\quad + L^2(1-\alpha_n)(1-\beta_n)\|T_i^k x_n - p\|^2 - L^2\beta_n(1-\alpha_n)(1-\beta_n)\|x_{n-1} - T_i^k x_n\|^2 \\
&\qquad + (1-\alpha_n)\left[L^2(1-\beta_n)(L\beta_n + 1) + L^4(1-\beta_n)^2\right]\|x_n - p\|^2. \qquad (16)
\end{aligned}
$$

From (8), we have

$$\left\|T_i^k x_n - p\right\|^2 \le \left[(L+1)^2 + 1 + \tfrac{1}{2}\left(a_{in}^2 - 1\right)\right]\|x_n - p\|^2 - \phi\left(\left\|x_n - T_i^k x_n\right\|\right). \quad (17)$$

Substitute (17) into (16) to obtain

$$\begin{aligned}\|x_n - p\|^2 \le\ & [\alpha_n + L^2\beta_n(1-\alpha_n) + (1-\alpha_n)[(L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)]]\|x_{n-1} - p\|^2 \\ & - L^2(1-\alpha_n)(1-\beta_n)\phi\left(\left\|x_n - T_i^k x_n\right\|\right) - L^2\beta_n(1-\alpha_n)(1-\beta_n)\left\|x_{n-1} - T_i^k x_n\right\|^2 \\ & + (1-\alpha_n)\{ L^2(1-\beta_n)(L\beta_n+1) + L^4(1-\beta_n)^2 \\ & + L^2(1-\beta_n)[(L+1)^2 + 1 + \tfrac{1}{2}(a_{in}^2 - 1)\ \}]\|x_n - p\|^2.\end{aligned}$$

That is

$$\begin{aligned}&\{\ 1-(1-\beta_n)\left[L^2(L\beta_n+1) + L^4(1-\beta_n) + L^2\left[(L+1)^2 + 1 + \tfrac{1}{2}\left(a_{in}^2 - 1\right)\right]\right] \\ &+ (1-\beta_n)\left[L^2(L\beta_n+1) + L^4(1-\beta_n) + L^2\left[(L+1)^2 + 1 + \tfrac{1}{2}\left(a_{in}^2 - 1\right)\right]\right]\beta_n - \beta_n\ \}\|x_n - p\|^2 \\ &\le (1-\beta_n)[\alpha_n + L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)]\|x_{n-1} - p\|^2 \\ &\quad - L^2(1-\alpha_n)(1-\beta_n)^2\phi\left(\left\|x_n - T_i^k x_n\right\|\right) - L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\left\|x_{n-1} - T_i^k x_n\right\|^2 \quad (18)\end{aligned}$$

Observed that $L^2(L\beta_n+1) + L^4(1-\beta_n) + L^2\left[(L+1)^2 + 1 + \tfrac{1}{2}\left(a_{in}^2 - 1\right)\right] > 1$. Using the fact that $(1-\beta_n) \le 1$, then,

$$\begin{aligned}&(1-\beta_n)\times\left\{(1-\beta_n)\left[L^2(L\beta_n+1) + L^4(1-\beta_n) + L^2\left[(L+1)^2 + 1 + \tfrac{1}{2}\left(a_{in}^2 - 1\right)\right]\right]\beta_n - \beta_n\right\} \\ &\le (1-\beta_n)\left[L^2(L\beta_n+1) + L^4(1-\beta_n) + L^2\left[(L+1)^2 + 1 + \tfrac{1}{2}\left(a_{in}^2 - 1\right)\right]\right]\beta_n - \beta_n\ .\end{aligned}$$

So we obtain,

$$\begin{aligned}&\left[1-(1-\beta_n)\xi_{in} + (1-\beta_n)^2\beta_n\xi_{in} - (1-\beta_n)\beta_n\right]\|x_n - p\|^2 \le \\ &\quad (1-\beta_n)[\alpha_n + L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)]\|x_{n-1} - p\|^2 \\ &\quad - L^2(1-\alpha_n)(1-\beta_n)^2\phi\left(\left\|x_n - T_i^k x_n\right\|\right) - L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\left\|x_{n-1} - T_i^k x_n\right\|^2,\end{aligned}$$

where $\xi_{in} = L^2(L\beta_n + 1) + L^4(1-\beta_n) + L^2\left[(L+1)^2 + 1 + \frac{1}{2}(a_{in}^2 - 1)\right] > 0$.

So that

$$\|x_n - p\|^2 \le \left\{1 + \frac{(1-\beta_n)\{L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)\} + (1-\beta_n)\xi_{in} - (1-\beta_n)^2\beta_n\xi_{in}}{1-(1-\beta_n)[\xi_{in}(1-(1-\beta_n)\beta_n) + \beta_n]}\right\}\|x_{n-1} - p\|^2$$

$$- \frac{L^2(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)[\xi_{in}(1-(1-\beta_n)\beta_n)+\beta_n]}\phi\left(\|x_n - T_i^k x_n\|\right)$$

$$- \frac{L^2\beta_n(1-\alpha_n)(1-\beta_n)^2}{1-(1-\beta_n)[\xi_{in}(1-(1-\beta_n)\beta_n)+\beta_n]}\|x_{n-1} - T_i^k x_n\|^2. \quad (19)$$

Since $\lim_{n\to\infty}(1-\beta_n) = 0$, then there exists a natural number N_2, such that $\forall\ n > N_2$

$$1-(1-\beta_n)[\xi_{in}(1-(1-\beta_n)\beta_n)+\beta_n] \ge \frac{1}{2}.$$

Therefore, it follows from (19) that

$$\begin{aligned}\|x_n - p\|^2 \le\ & [1 + 2[(1-\beta_n)\{L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)\} \\ & + (1-\beta_n)\xi_{in} - (1-\beta_n)^2\beta_n\xi_{in}]\|x_{n-1} - p\|^2 \\ & - L^2(1-\alpha_n)(1-\beta_n)^2\phi\left(\|x_n - T_i^k x_n\|\right) \\ & - L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\|x_{n-1} - T_i^k x_n\|^2. \\ \le\ & [1+\delta_{in}]\|x_{n-1} - p\|^2 - L^2(1-\alpha_n)(1-\beta_n)^2\phi\left(\|x_n - T_i^k x_n\|\right) \\ & - L^2\beta_n(1-\alpha_n)(1-\beta_n)^2\|x_{n-1} - T_n x_n\|^2, \end{aligned} \quad (20)$$

where

$$\begin{aligned}\delta_{in} = 2[&(1-\beta_n)\left\{L^2\beta_n + (L\beta_n+1)^2 + L^2(1-\beta_n)(L\beta_n+1)\right\} \\ & + (1-\beta_n)\xi_{in} - (1-\beta_n)^2\beta_n\xi_{in}]. \end{aligned} \quad (21)$$

Since from condition (iii) $\sum_{n=1}^{\infty}\delta_{in} < \infty$, it follows from (21) and Lemma 2.1 that $\lim_{n\to\infty}\|x_{n-1} - p\|$ exists. Hence $\lim_{n\to\infty}\|x_n - p\|$ exists. So there exists $M > 0$ such that $\|x_n - p\| \leq M$, $\forall\ n \geq 1$, we obtain from (20) that

$$L^2(1-\alpha_n)(1-\beta_n)^2\phi(\|x_n - T_i^k x_n\|) \leq [1+\delta_{in}]\|x_{n-1} - p\|^2 - \|x_n - p\|^2$$

$$\leq \|x_{n-1} - p\|^2 - \|x_n - p\|^2 + M^2\delta_{in}$$

$$L^2(1-\Omega)^2\sum_{j=N+1}^{\infty}(1-\alpha_j)\phi(\|x_j - T_j^k x_j\|) \leq \|x_N - p\|^2 + M^2\sum_{j=N+1}^{\infty}\delta_{ij}$$

$$\sum_{n=1}^{\infty}(1-\alpha_n)\phi(\|x_n - T_i^k x_n\|) \leq \|x_N - p\|^2 + M^2\sum_{n=1}^{\infty}\delta_{in} < \infty .$$

Since $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$, it follows that $\lim_{n\to\infty}\inf\ \phi\ (\|x_n - T_i^k x_n\|) = 0$.

Since ϕ is an increasing and continuous function, then $\lim_{n\to\infty}\inf\ \|x_n - T_i^k x_n\| = 0$.

Furthermore, since $\{x_n\}$ is bounded and $\lim_{n\to\infty}(1-\alpha_n) = 0$, it follows from

Lemma 2.1 that $\lim_{n\to\infty}\inf\ \|x_n - T_i^k x_n\| = 0$ for all $i \in I$.

Since one member of the family $\{T_i\}_{i=1}^{N}$ is semicompact, there exists a subsequence $\{x_{n_j}\}_{j=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ which converges strongly to u and furthermore, $\|u - T_i u\| = \underset{n\to\infty}{Lim}\ \|x_n - T_i x_n\| = 0$ for all $i \in I$. Thus $u \in F$. Since $\{x_{n_j}\}_{j=1}^{\infty}$ converges to u and $\underset{n\to\infty}{Lim}\ \|u - T_i u\|$ exists, It follows from Lemma 2.1, that $\{x_n\}_{n=1}^{\infty}$ converges strongly to u and hence the proof. □

Since every asymptotically demicontractive maps T is asymptotically $\phi-$demicontractive map (see for example [2]), we have the following:

Corollary 3.1 *Let K be a nonempty closed convex subset of a real Hilbert space H. Let $\{T_i\}_{i=1}^N$ be N uniformly $L-$Lipschitzian asymptotically demicontractive self maps of K with sequence $\{a_{in}\} \subseteq [1,\infty)$ such that $\sum_{n=1}^{\infty}(a_{in}-1) < \infty$ for all $i \in I$. Let $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$ where $F(T_i) = \{x \in K, T_i x = x\}$. Let one member of the family $\{T_i\}_{i=1}^N$ be semicompact. Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \subset [\eta, 1]$ be two real sequences satisfying the conditions:*

(i) $\sum_{n=1}^{\infty}(1-\alpha_n) = \infty$ (ii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty$ (iii) $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$.

(iv) $(1-\beta_n)(1-\alpha_n)L^2 < 1$, $0 < \beta \leq \alpha_n \leq \alpha < 1$,

where $\eta = \dfrac{L}{1+L}$ and $L = \max_{1 \leq i \leq N}\{L_i\}$, L_i the Lipschitzian constants of $\{T_i\}_{i=1}^N$. Let $\{x_n\}$ be the implicit iteration sequence generated by (12). Then $\{x_n\}_{n=1}^{\infty}$ exists in K and converges strongly to a common fixed point p of the family $\{T_i\}_{i=1}^N$.

Remark

If we set $\beta_n = 1$, the iteration scheme takes the non-implicit form:

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T^k x_{n-1}. \tag{22}$$

In the case of N=1, (22) becomes the modified Mann iteration process [13] given by

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)T^k x_{n-1}. \tag{23}$$

The conclusion of Theorem 3.1 and Corollary 3.1 are still valid. Hence, we state the following theorem without proof:

Theorem 3.2 *Let K be a nonempty closed convex subset of real Hilbert space H. Let T be an $L-$Lipschitzian asymptotically $\phi-$demicontractive self map of K*

with sequence $\{a_{in}\} \subseteq [1,\infty)$ *such that*

$$\sum_{n=1}^{\infty}(a_{in}-1)<\infty \quad \text{and} \quad F(T)=\{x \in K,\ Tx=x\} \neq \phi.$$

Let $\{\alpha_n\}_{n=1}^{\infty} \subseteq (0,1)$ *be a real sequence satisfying the conditions:*

(i) $\sum_{n=1}^{\infty}(1-\alpha_n)=\infty$

(ii) $\sum_{n=1}^{\infty}(1-\alpha_n)^2 < \infty, \quad 0<\alpha \le \alpha_n \le \beta <1.$

For arbitrary $x_1 \in K$*, let* $\{x_n\}$ *be the averaging Mann iteration process generated by (23). If* T *is semicompact, Then* $\{x_n\}_{n=1}^{\infty}$ *exists in* K *and converges strongly to a common fixed point* p *of the family* T *.*

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