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# A Note on Variational Principle of Subsets for Nonautonomous Dynamical Systems

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## Abstract

In this paper, we introduce measure-theoretic for Borel probability measures to characterize upper and lower Katok measure-theoretic entropies in discrete type and the measure-theoretic entropy for arbitrary Borel probability measure in nonautonomous case. Then we establish new variational principles for Bowen topological entropy for nonautonomous dynamical systems.

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# 1. Introduction

As an important invariant of topological conjugacy, the notion of topological entropy was introduced by Adler, Konheim and McAndrew [1] in 1965 [3]. Topological entropy is a key tool to measure the complexity of a classical dynamical system, i.e. the exponential growth rate of the number of distinguishable orbits of the iterates of an endomorphism of a compact metric space. In 1973, Bowen [2] introduced the topological entropy $h_{top}^{B}(T,Z)$ for any set $Z$ in a topological dynamical system $X$, in a way resembling Hausdorff dimension, where X is a compact metric space and $T:X\to X$ is a continuous self-map. Bowen topological entropy plays a key role in topological dynamics and dimension theory [2]. In 2012, Feng and Huang [6] showed that there is certain variational relation between Bowen topological entropy and measure-theoretic entropy for arbitrary non-invariant compact set of a topological dynamical system $(X,T)$. Following the idea of Brin and Katok [8], they defined the measure-theoretic entropy for Borel probability measure on X for their results.

In contrast with the autonomous discrete, in contrast with the autonomous discrete case [12], the properties of the entropies for the nonautonomous dynamical systems have not been fully investigated. In order to have a good understanding of the topological entropy of a skew product of dynamical systems (as we know that the calculation of its topological entropy can be transformed into that of its fibers), Kolyada and Snoha [4] proposed the concept of topological entropy in 1996 for a nonautonomous dynamical system determined by a sequence of maps.

A nonautonomous discrete dynamical system (in short: NADDS) is a natural generalization of classical dynamical systems, its dynamics are determined by a sequence of continuous self-maps $f_n:X\to X$ where $n\in\mathbf{N}$, defined on a compact metric space $X$.

By a nonautonomous dynamical system (NADDS for short) we understand a pair $\left(X,\left\{f_n\right\}_{n=1}^{\infty}\right)$, where $X$ is a compact metric space endowed with a metric $d$ and $\left\{f_n\right\}_{n=1}^{\infty}$, is a sequence of continuous maps from $X$ to $X$. In 2013, Kawan [10] generalized the classical notion of measure-theoretical entropy established by Kolomogorov and Sinai to NADSs, and proved that the measure-theoretical entropy can be estimated from above by its topological entropy. Following the idea of Brin and Katok [8] and Zhou [7] introduced the measure-theoretical entropy in nonautonomous case and established a variational principle for the first time. More results related to entropy for NADSs were developed in [11]. In this paper, We introduce ideas of Wang [9] to nonautonomous systems to establish new variational principles for Bowen topological entropy for nonautonomous dynamical systems.

Give a NADDS $\left(X,\left\{f_n\right\}_{n=1}^{\infty}\right)$. For each $n\in\mathbf{N}^{+}$, the Bowen metric $d_n$ on $X$ is defined by $d_n\left(x,y\right)=\max_{0\le i\le n-1}d\left(f_1^i\left(x\right),f_1^i\left(y\right)\right)$. For every $\varepsilon>0$, we denote by $B_n\left(x,\varepsilon\right)$ the open ball of radius $\varepsilon$ in the metric $d_n$ around $x$, i.e., $B_n\left(x,\varepsilon\right)=\left\{y\in X:d_n\left(x,y\right)<\varepsilon\right\}$.

We also consider a nonautonomous dynamical system (for short NADS) $\left(X,\phi\right)$ where $\left(X,d\right)$ is a compact metric space and $\phi:\left[0,\infty\right)\times X\to X$ is a continuous map with $\phi\left(0,x\right)=x$ for $x\in X$. We want to know whether there is certain variational relation of entropy for nonautonomous dynamical systems. For our study, we need to define the measure-theoretic entropy for arbitrary Borel probability measure in nonautonomous case.

Given a NADS $\left(X,\phi\right)$. For any $t\in\left[0,\infty\right)$, the $t$ Bowen metric $d_t^{\phi}$ on $X$ is defined by

$$d_t^{\phi}\left(x,y\right)=\max_{0\le i\le n-1}d\left(\phi\left(s,x\right),\phi\left(s,y\right)\right)$$

For every $\varepsilon>0$, we denote by $B_t^{\phi}\left(x,\varepsilon\right)$ the open ball of radius $\varepsilon$ in the metric $d_t^{\phi}$ around $x$, i.e.,

$$B_t^{\phi}\left(x,\varepsilon\right)=\left\{y\in X:d_t^{\phi}\left(x,y\right)<\varepsilon\right\}.$$

Write $\phi^i\left(x\right):=\phi\left(i,x\right)$ for $i=1,2,\cdots$ and $x\in X$.

In this case, we take $f_n\left(x\right)=\phi^n\left(x\right)$, then $\left\{\phi^n\right\}_{n=1}^{\infty}$ is a NADDS.

Let $M\left(X\right)$ denote the set of all Borel probability measures on $X$, $Z\subset X$ and $\mu\in M\left(X\right),\left(X,\left\{f_n\right\}_{n=1}^{\infty}\right)$ is a NADDS.

(1) A set $E\subset Z$ is said to be an $\left(n,\varepsilon,Z\right)$-separated set if $x,y\in E$ with $x\neq y$ implies $d_n^{\phi}\left(x,y\right)>\varepsilon$. Let $r_n\left(\varepsilon,Z\right)$ denote the maximum cardinality of $\left(n,\varepsilon,Z\right)$-separated set.

(2) A set $F\subset Z$ is said to be an $\left(n,\varepsilon,Z\right)$-spanning set if, for any $x\in X$, there exists $y\in F$ with $d_n^{\phi}\left(x,y\right)\le\varepsilon$. Let $s_n\left(\varepsilon,Z\right)$ denote the minimum cardinality of $\left(n,\varepsilon,Z\right)$-spanning sets.

(3) A set $F\subset Z$ is said to be a $\left(\mu,n,\varepsilon,\delta\right)$-spanning set if the union $\bigcup_{x\in F}B_n\left(x,\varepsilon\right)$ has $\mu$-measure more than or equal to $1-\delta$. Let $r_n\left(\mu,\varepsilon,\delta\right)$ denote the minimum cardinality of $\left(\mu,n,\varepsilon,\delta\right)$-spanning sets.

(4) We introduce a useful set: $X_{\mu,\delta}=\left\{Z\subset X:\mu(Z)\geq 1-\delta\right\}$.
Then it is clear that

$$r_n(\mu,\varepsilon,\delta)=\inf_{Z\in X_{\mu,\delta}} r_n(\varepsilon,Z)$$

An open cover of $X$ is a family of open subsets of $X$, whose union is $X$. For two covers $\mathcal{U}$ and $\mathcal{V}$ we say that $\mathcal{U}$ is a refinement of $\mathcal{V}$ if for each $U\in\mathcal{U}$ there is $V\in\mathcal{V}$ with $U\in V$. For $n\in\mathbf{N}$ and open covers $\mathcal{U}_1$/, $\mathcal{U}_2$ $,\cdots,$ $\mathcal{U}_n$/of $X$ we denote

$$\bigvee_{i=1}^{n}\mathcal{U}_i=\left\{A_1\bigcap A_2\bigcap\cdots\bigcap A_n : A_1\in\mathcal{U}_1, A_2\in\mathcal{U}_2,\cdots,A_n\in\mathcal{U}_n\right\}$$

Note that $\bigvee_{i=1}^{n}\mathcal{U}_i$ is also an open cover of $X$. We denote by $\mathcal{N}(\mathcal{U})$ the minimal cardinality of all subcovers chosen from $U$.
Set

$$f_i^0=id_X,\ f_i^n=f_{i+(n-1)}\circ f_{i+(n-2)}\circ\cdots\circ f_{i+1}\circ f_i,\ f_i^{-n}=\left(f_i^n\right)^{-1}$$

For all $i,n\in\mathbf{N}$, where $id_X$ is the identity map on $X$.
Let

$$h_{top}\left(\left\{f_n\right\}_{n=1}^{\infty},\mathcal{U}\right)=\limsup_{n\to\infty}\frac{\log\mathcal{N}\left(\bigvee_{i=0}^{n-1}f_1^{-i}\mathcal{U}_i\right)}{n}.$$

The topological entropy is defined by

$$h_{top}\left(X,\left\{f_n\right\}_{n=1}^{\infty}\right)=\left\{h_{top}\left(\left\{f_n\right\}_{n=1}^{\infty},\mathcal{U}\right):\mathcal{U}\text{ is an open cover of }X\right\}.$$

It was proved in \cite{AKM} that for every NADS, we have

$$h_{top}\left(X,\left\{f_n\right\}_{n=1}^{\infty}\right)=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{\log s_n(\varepsilon,X)}{n}=\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{\log r_n(\varepsilon,X)}{n}.$$

Following the idea of Katok \cite{AKM},we give the following.
Let $\mu\in M(X)$. The NADDS Katok measure-theoretical lower and upper entropies of $\mu$ are defined respectively by

$$\underline{h}_{\mu}^{K}\left(\left\{f_n\right\}_{n=1}^{\infty}\right)=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\liminf_{n\to\infty}\frac{1}{n}\log r_n(\mu,\varepsilon,\delta)$$

$$\overline{h}_{\mu}^{K}\left(\left\{f_n\right\}_{n=1}^{\infty}\right)=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\log r_n(\mu,\varepsilon,\delta)$$

In this paper, we introduce many quantities for Borel probability measure $\mu \in M(X)$, respectively denoted by $e_\mu\left(\{f_n\}_{n=1}^{\infty}\right)$, $\underline{e}_\mu\left(\{f_n\}_{n=1}^{\infty}\right)$, $\overline{e}_\mu\left(\{f_n\}_{n=1}^{\infty}\right)$, $e_\mu^*\left(\{f_n\}_{n=1}^{\infty}\right)$, and so on.

According to the relations of the several types of NADS topological entropies, it is natural to consider relationship of some new quantities and Katok measure-theoretical lower and upper entropies. Therefore, we have the first main result.

## 2. Main Results

**Theorem 2.1** Let $\left(X,\{f_n\}_{n=1}^{\infty}\right)$ *be a NADDS,* $\mu \in M(X)$.

Then following statements hold.

(1) *For any* $Z \subseteq X$, $h_{top}^B\left(\{f_n\}_{n=1}^{\infty}, Z\right) \le h_{top}^P\left(\{f_n\}_{n=1}^{\infty}, Z\right)$.

(2) $\overline{h}_\mu^K\left(\{f_n\}_{n=1}^{\infty}\right) = \overline{e}_\mu\left(\{f_n\}_{n=1}^{\infty}\right)$.

(3) $\underline{h}_\mu^K\left(\{f_n\}_{n=1}^{\infty}\right) = \underline{e}_\mu\left(\{f_n\}_{n=1}^{\infty}\right)$.

(4) $e_\mu\left(\{f_n\}_{n=1}^{\infty}\right) \le \underline{e}_\mu\left(\{f_n\}_{n=1}^{\infty}\right) \le \overline{e}_\mu\left(\{f_n\}_{n=1}^{\infty}\right)$.

(5) $e_\mu\left(\{f_n\}_{n=1}^{\infty}\right) \le e_\mu^*\left(\{f_n\}_{n=1}^{\infty}\right) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf_{Z\in X_{\mu,\delta}} h_{top}^P\left(\{f_n\}_{n=1}^{\infty}, Z, \varepsilon\right)$.

where the definitions of these notions will be given in Section 3.

**Theorem 2.2** Let $\left(X,\{f_n\}_{n=1}^{\infty}\right)$ be a *NADDS. If* $K \subset X$ is a non-empty and compact, then

$$h_{top}^B\left(\{f_n\}_{n=1}^{\infty}, K\right) = \sup\left\{e_\mu\left(\{f_n\}_{n=1}^{\infty}\right) : \mu \in M(X), \mu(K) = 1\right\}.$$

**Theorem 2.3** Let $(X,\phi)$ *be a NADS,* $\mu \in M(X)$. Then following statements hold.

(1) For any $Z \subseteq X$, $h_{top}^B(\phi, Z) \le h_{top}^P(\phi, Z)$.

(2) $e_\mu(\phi) \le \underline{e}_\mu(\phi) \le \overline{e}_\mu(\phi)$.

(3) $e_\mu(\phi) \le e_\mu^*(\phi) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf_{Z\in X_{\mu,\delta}} h_{top}^P(\phi, Z, \varepsilon)$.

**Theorem 2.4** Let $(X,\phi)$ *be a NADS. If* $K\subset X$ is non-empty and compact, then

$$h_{top}^{B}(\phi,K)=\sup\{e_{\mu}(\phi):\mu\in M(X),\mu(K)=1\}.$$

# 3. Preliminary Notes

## 3.1 NADDS

In this subsection, let $\left(X,\{f_n\}_{n=1}^{\infty}\right)$ be a NADDS, next we introduced NADDS's entropies. Following, we give some definitions of several NADDS topological entropies of subsets.

**Definition 3.1** *Let* $Z\subset X$, $s\geq 0$, $N\in\mathbf{N}$ and $\varepsilon>0$, define

$$M_{N,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},Z\right)=\inf\sum_{i}\exp(-sn_i),$$

where the infimum is taken over all finite or countable families $\{B_{n_i}(x_i,\varepsilon)\}$ such that $x_i\in X$, $n_i\geq N$ *and* $\bigcup_i B_{n_i}(x_i,\varepsilon)\supseteq Z$. The quantity $M_{N,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},Z\right)$ does not decrease as $N$ increase and $\varepsilon$ decreases, hence the following limits exist:

$$M_{\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},Z\right)=\lim_{N\to\infty}M_{N,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},Z\right),$$

$$M^{s}\left(\{f_n\}_{n=1}^{\infty},Z\right)=\lim_{\varepsilon\to 0}M_{\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},Z\right)).$$

Bowen's topological entropy $h_{top}^{B}\left(\{f_n\}_{n=1}^{\infty},Z\right)$ is defined as a critical value of the parameters $s$, where $M^{s}\left(\{f_n\}_{n=1}^{\infty},Z\right)$ jumps from $\infty$ *to* $0$, *i.e.*

$$M^{s}\left(\{f_n\}_{n=1}^{\infty},Z\right)=\begin{cases}0 & ,s>h_{top}^{B}\left(\{f_n\}_{n=1}^{\infty},Z\right)\\ \infty & ,s<h_{top}^{B}\left(\{f_n\}_{n=1}^{\infty},Z\right)\end{cases}$$

**Definition 3.2** Let $Z\subset X$. *For* $s\geq 0$, $N\in\mathbf{N}$ *and* $\varepsilon>0$, define

$$P_{N,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},Z\right)=\sup\sum_{i}\exp(-sn_i),$$

where the supremum is taken over all finite or countable pairwise disjoint families $\left\{\overline{B}_{n_i}(x_i,\varepsilon)\right\}$ such that $x_i\in Z$, $n_i\geq N$ *for all* $i$,

where

$\overline{B}_{n_i}(x_i,\varepsilon):=\{y\in X:d_n(x,y)\leq\varepsilon\}$.

The quantity $P^s_{N,\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)$ does not decrease as $N$, $\varepsilon$ decrease.

Hence the following limit exists:

$$P^s_{\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)=\lim_{N\to\infty}P^s_{N,\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right).$$

Define

$$\mathrm{P}^s_{\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)=\inf\left\{\sum_{i=1}^{\infty}P^s_{\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z_i\right):\bigcup_{i=1}^{\infty}Z_i\supseteq Z\right\}.$$

There exists a critical value of the parameters $s$, which we will denote by $h^P_{top}\left(\left\{f_n\right\}_{n=1}^{\infty},Z,\varepsilon\right)$, *where* $\mathrm{P}^s_{\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)$ jumps from $\infty$ to $0$, *i.e.*

$$\mathrm{P}^s\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)=\begin{cases}0 & , s>h^P_{top}\left(\left\{f_n\right\}_{n=1}^{\infty},Z,\varepsilon\right)\\ \infty & , s<h^P_{top}\left(\left\{f_n\right\}_{n=1}^{\infty},Z,\varepsilon\right)\end{cases}$$

Note that $h^P_{top}\left(\left\{f_n\right\}_{n=1}^{\infty},Z,\varepsilon\right)$ increases when $\varepsilon$ decreases.

We call

$$h^P_{top}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right):=\lim_{\varepsilon\to0}h^P_{top}\left(\left\{f_n\right\}_{n=1}^{\infty},Z,\varepsilon\right)$$

the topological packing entropy of $Z$.

**Definition 3.3** Let $Z\subseteq X$. For $s\geq0$, $N\in\mathbf{N}$ *and* $\varepsilon>0$, define

$$R^s_{N,\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)=\inf\sum_i\exp\left(-sN\right)$$

where the infimum is taken over all finite or countable families $\left\{B_N\left(x_i,\varepsilon\right)\right\}$ such that $x_i\in X$, and $\bigcup_i B_N\left(x_i,\varepsilon\right)\supseteq Z$.

Let

$$\underline{R}^s_{\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)=\liminf_{N\to\infty}R^s_{N,\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right),$$

$$\overline{R}^s_{\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)=\limsup_{N\to\infty}R^s_{N,\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)$$

and

$$\underline{Ch}_Z\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon\right)=\inf\left\{s:\underline{R}^s_{\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)=0\right\}=\sup\left\{s:\underline{R}^s_{\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)=+\infty\right\},$$

$$\overline{Ch}_Z\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon\right)=\inf\left\{s:\overline{R}^s_{\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)=0\right\}=\sup\left\{s:\overline{R}^s_{\varepsilon}\left(\left\{f_n\right\}_{n=1}^{\infty},Z\right)=+\infty\right\}.$$

The lower and upper capacity topological entropies of $\{f_n\}_{n=1}^{\infty}$ restricted to $Z$ are defined respectively by

$$\underline{Ch}_Z\left(\{f_n\}_{n=1}^{\infty}\right)=\lim_{\varepsilon\to 0}\underline{Ch}_Z\left(\{f_n\}_{n=1}^{\infty},\varepsilon\right),$$

$$\overline{Ch}_Z\left(\{f_n\}_{n=1}^{\infty}\right)=\lim_{\varepsilon\to 0}\overline{Ch}_Z\left(\{f_n\}_{n=1}^{\infty},\varepsilon\right).$$

**Definition 3.4** Let $\mu\in M(X)$, $s\geq 0$, $N\in\mathbf{N}$, $\varepsilon>0$ and $0<\delta<1$, define

$$M_{N,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right)=\inf\sum_i\exp(-sn_i),$$

where the infimum is taken over all finite or countable families $\{B_{n_i}(x_i,\varepsilon)\}$ such that $x_i\in X$ , $n_i\geq N$ *and* $\mu\left(\bigcup_i B_{n_i}(x_i,\varepsilon)\right)\geq 1-\delta$ .

The quantity $M_{N,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right)$ does not decrease as $N$ increase, hence the following limit exist:

$$M_{\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right)=\lim_{N\to\infty}M_{N,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right)$$

Using standard method, we have following is well- defined:

$$e_{\mu}\left(\{f_n\}_{n=1}^{\infty},\varepsilon,\delta\right)=\inf\left\{s:M_{\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right)=0\right\}=\sup\left\{s:M_{\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right)=+\infty\right\}$$

Defined

$$e_{\mu}\left(\{f_n\}_{n=1}^{\infty}\right)=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}e_{\mu}\left(\{f_n\}_{n=1}^{\infty},\varepsilon,\delta\right).$$

**Definition 3.5** Let $\mu\in M(X)$, $s\geq 0$ , $N\in\mathbf{N}$, $\varepsilon>0$ and $0<\delta<1$, *put*

$$R_{N,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right)=\inf\sum_i\exp(-sN),$$

where the infimum is taken over all finite or countable families $\{B_N(x_i,\varepsilon)\}$ such that $x_i\in X$ , and $\mu\left(\bigcup_i B_N(x_i,\varepsilon)\right)\geq 1-\delta$ .

Let

$$\underline{R}_{\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right)=\liminf_{N\to\infty}R_{N,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right),$$

$$\overline{R}_{\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right)=\limsup_{N\to\infty}R_{N,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right).$$

Using standard method, we have following is well- defined:

$$\underline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)=\inf\left\{s:\underline{R}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)=0\right\}=\sup\left\{s:\underline{R}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)=+\infty\right\}$$

$$\overline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)=\inf\left\{s:\overline{R}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)=0\right\}=\sup\left\{s:\overline{R}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)=+\infty\right\}$$

Define

$$\underline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty}\right)=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\underline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right),$$

$$\overline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty}\right)=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\overline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right).$$

**Definition 3.6** Let $\mu\in M\left(X\right)$, $s\geq 0$, $N\in\mathbf{N}$, $\varepsilon>0$ and $0<\delta<1$, put

$$\mathrm{P}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)=\inf\left\{\sum_{i=1}^{\infty}P_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},Z_i\right):\mu\left(\bigcup_{i=1}^{\infty}Z_i\right)\geq 1-\delta\right\},$$

where $P_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},Z_i\right)$ is defined in Definition 2.2. There exists a critical value of $s$ such that

$$e_{\mu}^{*}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)=\left\{s:\mathrm{P}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)=0\right\}=\sup\left\{s:\mathrm{P}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)=+\infty\right\}.$$

Define

$$e_{\mu}^{*}\left(\left\{f_n\right\}_{n=1}^{\infty}\right)=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}e_{\mu}^{*}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right).$$

### 3.2 NADS

In this subsection, let $\left(X,\phi\right)$ be a NADS, next we introduced NADS's entropies.

**Definition 3.7** *Let* $Z\subset X$, $s\geq 0$, $N\in\mathbf{N}$ *and* $\varepsilon>0$,
define

$$M_{N,\varepsilon}^{s}\left(\phi,Z\right)=\inf\sum_{i}\exp\left(-st_i\right),$$

where the infimum is taken over all finite or countable families $\left\{B_{t_i}^{\phi}\left(x_i,\varepsilon\right)\right\}$ such that $x_i\in X$, $t_i\geq N$ and $\bigcup_{i}B_{t_i}^{\phi}\left(x_i,\varepsilon\right)\supseteq Z$. The quantity $M_{N,\varepsilon}^{s}\left(\phi,Z\right)$ does not decrease as $N$ increase and $\varepsilon$ decreases.

Hence the following limits exist:

$$M_{\varepsilon}^{s}\left(\phi,Z\right)=\lim_{N\to\infty}M_{N,\varepsilon}^{s}\left(\phi,Z\right),$$

$$M^{s}\left(\phi,Z\right)=\lim_{\varepsilon\to 0}M_{\varepsilon}^{s}\left(\phi,Z\right).$$

Bowen's topological entropy $h_{top}^{B}(\phi,Z)$ is defined as a critical value of the parameters $s$, where $M^{s}(\phi,Z)$ jumps from $\infty$ to $0$, i.e.

$$M^{s}(\phi,Z)=\begin{cases}0 & ,s>h_{top}^{B}(\phi,Z)\\ \infty & ,s<h_{top}^{B}(\phi,Z)\end{cases}$$

Other topological entropy definitions are similar to the discrete case definition.

**Definition 3.8** Let $\mu\in M(X)$, $s\geq 0$, $N\in\mathbf{N}$, $\varepsilon>0$ and $0<\delta<1$, define

$$M_{N,\varepsilon}^{s}(\phi,\mu,\delta)=\inf\sum_{i}\exp(-st_i),$$

where the infimum is taken over all finite or countable families $\left\{B_{t_i}^{\phi}(x_i,\varepsilon)\right\}$ such that $x_i\in X$, $t_i\geq N$ and $\mu\left(\bigcup_i B_{t_i}^{\phi}(x_i,\varepsilon)\right)\geq 1-\delta$. The quantity $M_{N,\varepsilon}^{s}(\phi,\mu,\delta)$ does not decrease as $N$ increase, hence the following limit exist:

$$M_{\varepsilon}^{s}(\phi,\mu,\delta)=\lim_{N\to\infty}M_{N,\varepsilon}^{s}(\phi,\mu,\delta).$$

Using standard method, we have following is well- defined:

$$e_{\mu}(\phi,\varepsilon,\delta)=\inf\left\{s:M_{\varepsilon}^{s}(\phi,\mu,\delta)=0\right\}=\sup\left\{s:M_{\varepsilon}^{s}(\phi,\mu,\delta)=+\infty\right\},$$

defined

$$e_{\mu}(\phi)=\lim_{\delta\to 0}\lim_{\varepsilon\to 0}e_{\mu}(\phi,\varepsilon,\delta).$$

# 4. Proof of Theorem

## 4.1 Proof of Theorem 2.1

**Proposition 4.1** *Let* $0<\delta<1$, $\mu\in M(X)$, $\{Z_i\}_{i=1}^{\infty}$ be a family of Borel subsets of $X$ with $\mu\left(\bigcup_i Z_i\right)\geq 1-\delta$. For any $\varepsilon>0$,

$$M_{\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},\mu,\delta\right)\leq\sum_{i=1}^{\infty}M_{\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},Z_i\right).$$

**Poof** For any $\varepsilon>0$, $N$, $i\in\mathbb{N}$, there exists $N_i>N$ such that

$$M_{N_i,\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},Z_i\right)<M_{\varepsilon}^{s}\left(\{f_n\}_{n=1}^{\infty},Z_i\right)+\frac{\varepsilon}{2^i}.\$\$$$

Hence, there exists a countable family $\left\{B_{n_j^i}\left(x_j^i,\varepsilon\right)\right\}_{j=1}^{\infty}$ such that $n_j^i \geq N_i$, $x_j^i \in X$, $\left\{B_{n_j^i}\left(x_j^i,\varepsilon\right)\right\}_{j=1}^{\infty} \supseteq Z_i$,

$$\sum_{j=1}^{\infty}\exp\left(-sn_j^i\right) < M_\varepsilon^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z_i\right) + \frac{\varepsilon}{2^i}.$$

Since $\mu\left(\bigcup_i Z_i\right) \geq 1-\delta$, we have $\mu\left(\bigcup_{i\geq 1}\bigcup_{j\geq 1} B_{n_j^i}\left(x_j^i,\varepsilon\right)\right) \geq 1-\delta$. Hence

$$M_\varepsilon^s\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right) \leq \sum_{i\geq 1}\sum_{j\geq 1}\exp\left(-sn_j^i\right) < \sum_{i=1}^{\infty} M_\varepsilon^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z_i\right).$$

**Proof** (1) Let $Z \subseteq X$ and assume be $0 < s < h_{top}^B\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right)$. For any $n \in \mathbf{N}$ and $\varepsilon > 0$, let $R = R_n\left(\left\{f_n\right\}_{n=1}^{\infty}, Z, \varepsilon\right)$ be the largest number so that there is a disjoint family $\left\{\overline{B}_n\left(x_i,\varepsilon\right)\right\}_{i=1}^{R}$ with $x_i \in Z$. Then it is easy to see that for any $\delta > 0$,

$$\bigcup_{i=1}^{R}\overline{B}_n\left(x_i, 2\varepsilon+\delta\right) \supseteq Z,$$

which implies that

$$M_{n,2\varepsilon+\delta}^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right) \leq R\cdot\exp\left(-ns\right) \leq P_{n,\varepsilon}^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right)$$

for any $s \geq 0$, and hence $M_{2\varepsilon+\delta}^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right) \leq P_\varepsilon^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right)$, we have $M_{2\varepsilon+\delta}^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right) \leq \mathrm{P}_\varepsilon^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right)$. Since $0 < s < h_{top}^B\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right)$, we have $M^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right) = \infty$ and thus $M_{2\varepsilon+\delta}^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right) \geq 1$ when $\varepsilon$ and $\delta$ are small enough. Hence $\mathrm{P}_\varepsilon^s\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right) \geq 1$ and $h_{top}^P\left(\left\{f_n\right\}_{n=1}^{\infty}, Z, \varepsilon\right) \geq s$ when $\varepsilon$ is small. Therefore $h_{top}^P\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right) = \lim_{\varepsilon\to 0} h_{top}^P\left(\left\{f_n\right\}_{n=1}^{\infty}, Z, \varepsilon\right) \geq s$.

This implies that $h_{top}^B\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right) \leq h_{top}^P\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right)$.

(2) Denote

$$\overline{h}_\mu^K\left(\left\{f_n\right\}_{n=1}^{\infty}, \varepsilon, \delta\right) = \limsup_{n\to\infty}\frac{1}{n}\log r_n\left(\mu,\varepsilon,\delta\right)$$

then $\overline{h}_\mu^K\left(\left\{f_n\right\}_{n=1}^{\infty}\right) = \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\overline{h}_\mu^K\left(\left\{f_n\right\}_{n=1}^{\infty}, \varepsilon, \delta\right)$.

We first prove that

$$\overline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)\le \overline{h}_{\mu}^{K}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)$$

for any $0<\delta<1$ and $\varepsilon>0$, using like-Huasdorff dimension method. For any $s>\overline{h}_{\mu}^{K}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)$ and $Z\in X_{\mu,\delta}$, let $F$ is a $(n,\varepsilon,Z)$-spanning set, then

$$R_{n,\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)\le\sum_{x\in F}\exp(-sn)=\#F\cdot\exp(-sn)\}$$

which follows that

$$R_{n,\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)\le\exp(-sn)\cdot\inf_{Z\in X_{\mu,\delta}} r_n(\varepsilon,Z).$$

Hence

$$R_{n,\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)\le\exp(-sn)\cdot r_n(\mu,\varepsilon,\delta)=e^{-n\left(s-\frac{1}{n}\log r_n(\mu,\varepsilon,\delta)\right)}.$$

Since $\overline{h}_{\mu}^{K}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)=\limsup\limits_{n\to\infty}\frac{1}{n}\log r_n(\mu,\varepsilon,\delta)<s$, we have

$$\limsup_{n\to\infty}R_{n,\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)=0.$$

For $s>\overline{h}_{\mu}^{K}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)$ we get $\overline{R}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)=0$ and $\overline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)\le s$.

Hence $\overline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)\le\overline{h}_{\mu}^{K}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)$.

Next we prove $\overline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)\ge\overline{h}_{\mu}^{K}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)$ for any $0<\delta<1$ and $\varepsilon>0$ by showing $\overline{h}_{\mu}^{K}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)\le s$ whenever $s>\overline{e}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)$. For such a $s$, we have $\overline{R}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)=0$. Then there exists $N\in\mathbf{N}$ such that $R_{n,\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty},\mu,\delta\right)<1$ for any $n\ge N$. Fix $n\ge N$, we can find a finite family $\left\{B_n(x_i,\varepsilon)\right\}_{i\in I}$ such that $x_i\in X$,

$$\mu\left(\bigcup_{i\in I}B_n(x_i,\varepsilon)\right)\ge 1-\delta \quad\text{and}\quad \#I\cdot e^{-sn}<1$$

So $r_n(\mu,\varepsilon,\delta)\le e^{sn}$ for any $n\ge N$. Hence $\overline{h}_{\mu}^{K}\left(\left\{f_n\right\}_{n=1}^{\infty},\varepsilon,\delta\right)\le s$.

(3) The proof of (3) is similar to (2).
(4) The proof of (4) is a consequence of definition.
(5) We first show that $e_\mu\left(\{f_n\}_{n=1}^\infty\right) \le e_\mu^*\left(\{f_n\}_{n=1}^\infty\right)$. Let $s < e_\mu\left(\{f_n\}_{n=1}^\infty\right)$, $0<\delta<1$ and $\{Z_i\}_{i=1}^\infty$ be a family of Borel subsets of $X$ with $\mu\left(\bigcup_{i=1}^\infty Z_i\right) \ge 1-\delta$. For any $i,n \in \mathbf{N}$ and $\varepsilon>0$, let $R_n^i = R(Z_i,\varepsilon)$ be the largest number such that there is a disjoint family $\left\{\overline{B}_n\left(x_j^i,\varepsilon\right)\right\}_{j=1}^{R_n^i}$ with $x_j^i \in Z_i$. Then we can verify that for any $\theta>0$,

$$\left\{B_{n_j^i}\left(x_j^i, 2\varepsilon+\theta\right)\right\} \supseteq Z_i .$$

It following that $M_{n,2\varepsilon+\theta}^s\left(\{f_n\}_{n=1}^\infty, Z_i\right) \le R_n^i \cdot e^{-sn} \le P_{n,\varepsilon}^s\left(\{f_n\}_{n=1}^\infty, Z_i\right)$ and $M_{2\varepsilon+\theta}^s\left(\{f_n\}_{n=1}^\infty, Z_i\right) \le \mathrm{P}_\varepsilon^s\left(\{f_n\}_{n=1}^\infty, Z_i\right)$. Therefore, by the Proposition4.1, we have $M_{2\varepsilon+\theta}^s\left(\{f_n\}_{n=1}^\infty, \mu,\delta\right) \le \mathrm{P}_\varepsilon^s\left(\{f_n\}_{n=1}^\infty, \mu,\delta\right)$. As $s<e_\mu\left(\{f_n\}_{n=1}^\infty\right)$, we can get $s<e_\mu\left(\{f_n\}_{n=1}^\infty, 2\varepsilon+\theta,\delta\right)$ when $\varepsilon$ , $\theta$ , $\delta$ are small enough. This implies that $M_{2\varepsilon+\theta}^s\left(\{f_n\}_{n=1}^\infty, \mu,\delta\right)=\infty$ and thus $\mathrm{P}_\varepsilon^s\left(\{f_n\}_{n=1}^\infty, \mu,\delta\right)=\infty$. Therefore, it can be deduced that $e_\mu^*\left(\{f_n\}_{n=1}^\infty\right) \ge s$. So the desired inequality holds.

Now we proved that $e_\mu^*\left(\{f_n\}_{n=1}^\infty\right) = \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf_{Z\in X_{\mu,\delta}} h_{top}^P\left(\{f_n\}_{n=1}^\infty, Z,\varepsilon\right)$.

Let $e_\mu^*\left(\{f_n\}_{n=1}^\infty\right) > s$, then there exists $\varepsilon', \delta' > 0$ such that $e_\mu^*\left(\{f_n\}_{n=1}^\infty, \varepsilon,\delta\right) \ge s$ for any $\varepsilon \in (0,\varepsilon')$ and $\delta \in (0,\delta')$. Thus $\mathrm{P}_\varepsilon^s\left(\{f_n\}_{n=1}^\infty, \mu,\delta\right)=\infty$.

For any $Z \in X_{\mu,\delta}$ and any $\{Z_i\}_{i\ge 1}$ with $\bigcup_{i=1}^\infty Z_i \supseteq Z$, we have $\mu\left(\bigcup_{i=1}^\infty Z_i\right) \ge 1-\delta$.

It follows from $\mathrm{P}_\varepsilon^s\left(\{f_n\}_{n=1}^\infty, \mu,\delta\right)=\infty$ that $\sum_{i=1}^\infty P_\varepsilon^s\left(\{f_n\}_{n=1}^\infty, Z_i\right)=\infty$.

So $\mathrm{P}_\varepsilon^s\left(\{f_n\}_{n=1}^\infty, Z\right)=\infty$, which gives that $h_{top}^P\left(\{f_n\}_{n=1}^\infty, Z,\varepsilon\right) \ge s$.

On the other hand, let $s < \lim_{\varepsilon\to 0}\lim_{\delta\to 0}\inf_{Z\in X_{\mu,\delta}} h_{top}^P\left(\{f_n\}_{n=1}^\infty, Z,\varepsilon\right)$. Then there exist $\varepsilon'$, $\delta'>0$ such that $h_{top}^P\left(\{f_n\}_{n=1}^\infty, Z,\varepsilon\right)>s$ for any $\varepsilon\in(0,\varepsilon')$ , $\delta\in(0,\delta')$ and

$Z \in X_{\mu,\delta}$. Thus, we have $\mathrm{P}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty}, Z\right)=\infty$. Fix $\left\{Z_i\right\}_{i\geq 1}$ with $\mu\left(\bigcup_{i=1}^{\infty} Z_i\right) \geq 1-\delta$ and write $Z=\bigcup_{i=1}^{\infty} Z_i$, then $Z \in X_{\mu,\delta}$. So $\sum_{i=1}^{\infty} P_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty}, Z_i\right)=\infty$, which yields that $\mathrm{P}_{\varepsilon}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty}, \mu, \delta\right)=\infty$. Furthermore, we can get $e_{\mu}^{*}\left(\left\{f_n\right\}_{n=1}^{\infty}, \varepsilon, \delta\right) \geq s$ and $e_{\mu}^{*}\left(\left\{f_n\right\}_{n=1}^{\infty}\right) \geq s$.

## 4.2 Proof of Theorem 2.2

**Proposition 4.2** For $\mu \in M(X)$, it holds that

$$\underline{h}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty}\right) \leq e_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty}\right) \leq \inf\left\{h_{top}^{B}\left(\left\{f_n\right\}_{n=1}^{\infty}, K\right): \mu(K)=1\right\}.$$

**Proof** The second inequality is a direct consequence of the definition and we only deduce the first one. For $s>0$ with $\underline{h}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty}\right)>s$. By a standard procedure, there exist $A \subset X$ with $\mu(A)>0$ and $N \in \mathbf{N}$ such that

$$\mu\left(B_n(x, \varepsilon)\right)<e^{-sn}, \forall x \in A, n \geq N$$

Pick $\delta \in (0, \mu(A))$. Let $\left\{B_{n_i}\left(x_i, \frac{\varepsilon}{2}\right)\right\}_{i\in I}$ be a countable family such that $n_i \geq N$, $x_i \in X$ and $\mu\left(\bigcup_{i\in I} B_{n_i}\left(x_i, \frac{\varepsilon}{2}\right)\right) \geq 1-\delta$ that intersects $A$, if taking $y_i \in B_{n_i}\left(x_i, \frac{\varepsilon}{2}\right) \cap A$, then one has $B_{n_i}\left(x_i, \frac{\varepsilon}{2}\right) \subseteq B_{n_i}\left(y_i, \varepsilon\right)$ and thus

$$\mu\left(B_{n_i}\left(x_i, \frac{\varepsilon}{2}\right)\right) \leq \mu\left(B_{n_i}\left(y_i, \varepsilon\right)\right) \leq e^{-sn_i}$$

Then we have

$$\sum_{i\in I} e^{-sn_i} \geq \sum_{i\in I} \mu\left(B_{n_i}\left(y_i, \varepsilon\right) \cap A\right) \geq \sum_{i\in I} \mu\left(B_{n_i}\left(x_i, \frac{\varepsilon}{2}\right) \cap A\right)$$
$$\geq \mu\left(\bigcup_{i\in I} B_{n_i}\left(x_i, \frac{\varepsilon}{2}\right) \cap A\right)=\mu(A)>0$$

Hence $M_{\frac{\varepsilon}{2}}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty}, \mu, \delta\right) \geq M_{N,\frac{\varepsilon}{2}}^{s}\left(\left\{f_n\right\}_{n=1}^{\infty}, \mu, \delta\right) \geq \mu(A)$. By Bowen's definition, we can derive that $e_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty}, \frac{\varepsilon}{2}, \delta\right) \geq s$ and moreover $\underline{h}_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty}\right) \leq e_{\mu}\left(\left\{f_n\right\}_{n=1}^{\infty}\right)$.

**Definition 4.3** Let $\mu \in M(X)$. The NADS $(X,\phi)$ measure-theoretical lower entropies of $\mu$ is defined by

$$\underline{h}_{\mu}(\phi) = \int \underline{h}_{\mu}(\phi, x)\, d\mu(x)$$

where

$$\underline{h}_{\mu}(\phi, x) = \lim_{\varepsilon \to 0} \liminf_{t \to \infty} -\frac{1}{t} \log \mu\left(B_t^{\phi}(x, \varepsilon)\right).$$

**Lemma 4.4** ([5,theorem1.4]) Let $(X, \{f_n\}_{n=1}^{\infty})$ be a NADDS. If $K \subseteq X$ is non-empty and compact, then

$$h_{top}^{B}\left(\{f_n\}_{n=1}^{\infty}, K\right) = \sup\left\{\underline{h}_{\mu}\left(\{f_n\}_{n=1}^{\infty}\right) : \mu \in M(X), \mu(K) = 1\right\}.$$

**Proof** By the Proposition, we have

$$\sup\left\{\underline{h}_{\mu}\left(\{f_n\}_{n=1}^{\infty}\right) : \mu \in M(X), \mu(K) = 1\right\} \le \sup\left\{e_{\mu}\left(\{f_n\}_{n=1}^{\infty}\right) : \mu \in M(X), \mu(K) = 1\right\}$$

$$\le \inf\left\{h_{top}^{B}\left(\{f_n\}_{n=1}^{\infty}, K\right) : \mu \in M(X), \mu(K) = 1\right\}$$

Combining with lemma,

$$h_{top}^{B}\left(\{f_n\}_{n=1}^{\infty}, K\right) = \sup\left\{\underline{h}_{\mu}\left(\{f_n\}_{n=1}^{\infty}\right) : \mu \in M(X), \mu(K) = 1\right\}$$

the conclusion can be proved.

Using the same proof method of Theorem 2.1 and, Theorem 2.2, we have result of Theorem 2.3 and Theorem 2.4.

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