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Inequalities Involving Companion Matrix

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Abstract

We give several inequalities involving the Frobenius companion matrix of a polynomial P, and solve any equation in involving c, c^2, and c^3.

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1. Introduction

Let

$$P(z) = z^n + a_n z^{n-1} + \cdots + a_2 z + a_1$$

be a monic polynomial of degree $n \geq 2$ with complex coefficients $a_1, a_2, \ldots, a_n$, where $a_1 \neq 0$. Then the Frobenius companion matrix of p is given by:

$$C(P) = \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \ldots & 0 & 0 \\ 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix}$$

It is Well- Known that the zeros of p are exactly the eigenvalues of *C(P)*.

Now,

$$\lambda^n + a_n \lambda^{n-1} + \cdots + a_2 \lambda + a_1 = 0$$

So, by Cayley Hamilton we obtain.

$$c^n + a_n c^{n-1} + \cdots + a_2 c + a_1 \mathrm{I} = 0$$

Where *C* is the companion matrix of *P(z)*.

By similar way, we can write replace λ^2 instead of λ *for*

$$C^2 = \begin{bmatrix} b_n & b_{n-1} & \ldots & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 \end{bmatrix},$$

and we can write replace λ^3 instead of λ for

$$C^3 = \begin{bmatrix} c_n & c_{n-1} & \cdots & c_4 & c_3 & c_2 & c_1 \\ b_n & b_{n-1} & \cdots & b_4 & b_3 & b_2 & b_1 \\ -a_n & -a_{n-1} & \cdots & -a_4 & -a_3 & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & 0 \end{bmatrix},$$

where $b_j = a_n a_j - a_{j-1}$ and $c_j = -a_n b_j + a_{n-1} a_j - a_{j-2}$

for $j = 1,2, \ldots n$, with $a_0 = a_{-1} = 0$.

We can solve any equation in solving *C* such as the following examples.

Example: 1

Consider the following Clayey Hamilton polynomial

$$C^n + a_n C^{n-1} + \ldots. + a_2 C + a_1 = 0$$

the companion polynomial is

$$P(z) = z^n + a_n z^{n-1} + \ldots. + a_2 z + a_1$$

So, the Frobenius companion matrix of P is given by

$$C = \begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

By multiply the original equation by C^{1-n}, we have

$$C + a_n + a_{n-1}C^{-1} .... + a_2C^{2-n} + a_1C^{1-n} = 0$$

By substitution the values for n ,we can find the inverse powers for C.

For Example *if n=2*,we get

$$C + a_2 + a_1C^{-1} = 0$$

And hence, $$C^{-1} = \frac{-C - a_2I}{a_1}$$

Now, we have the following two cases for the companion polynomial.

Case One: Quadratic formula

Any Equation

$$C^2 + a_2C + a_1I = 0 .$$

has a solution

$$C = \begin{bmatrix} -a_2 & -a_1 \\ 1 & 0 \end{bmatrix}$$

With eigenvalues

$$\lambda = \frac{-a_2 \pm \sqrt{a_2^2 - 4a_1}}{2}$$

Case Two: Cube formula

Any Equation

$$C^3 + a_3C^2 + a_2C + a_1I = 0$$

has a solution

$$C = \begin{bmatrix} -a_3 & -a_2 & -a_1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

With eigenvalues λ satisfying the equation

$$\lambda^3 + a_3\lambda^2 + a_2\lambda + a_1 = 0$$

Example: 2

Consider the following Cayley Hamilton polynomial

$$C^3 - 5C^2 + 3C + I = 0$$

Then

$$C = \begin{bmatrix} 5 & -3 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

Now, we want to solve

$$\lambda^3 - 5\lambda^2 + 3\lambda + 1 = 0$$

$$\lambda=1\ ,\quad \lambda = 2+\sqrt{5}\ ,\quad \lambda= 2-\sqrt{5}$$

Multiply the equation by C^{-1},

$$= 0\ C^2 - 5C + 3I + C^{-1}$$

$$C^{-1} = -C^2 + 5C - 3I$$

$$C^{-1} = \begin{bmatrix} -22 & 16 & 5 \\ -5 & 3 & 1 \\ -1 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 25 & -15 & -5 \\ 5 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 5 & -3 \end{bmatrix}$$

With eigen values *1*,$\frac{1}{2+\sqrt{5}}$,$\frac{1}{2-\sqrt{5}}$

Conversely, if we have equation of C^{-1}, we can find the eigen values of *C*.

$$C^{-1} = C^2 + 5 - 3I$$

By multiplying both sides of the equation by *C*, we have

$$I = C^3 + 5C^2 - 3C$$

$$C^3 - 5C^2 + 3C + I = 0$$

So,

$$C = \begin{bmatrix} 5 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

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