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A Brand new Approach to Collatz Conjecture

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Abstract

The aim of this paper is to propose a brand new approach on Collatz conjecture as well as a proof of it. The method is based on the fundamental theorem of arithmetics and on a definition of trajectories that implies a contradiction to the latter theorem when a divergence or looping of the algorithm is assumed. This contradiction proves Collatz conjecture.

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1. Introduction

Collatz conjecture (1937) [O'Connor and Robertson, 2009] states that for any natural integer *N* Collatz algorithm ends up at 1. This algorithm is as follows.

1- choose $N>0$; set current $n=N$;
2- if *n* is odd replace *n* by $3n+1$;
3- if not replace n by $n/2$;
4- repeat from step 2 until $n=1$.

Even if the algorithm can then loop with 1, 4, 2, 1, … after reaching 1 for the first time the algorithm is said to be done at the first occurrence of 1.
As of 2021 this conjecture was still an open problem.

2. Preliminary Notes

2.1 Fundamentals on Integers

Theorem 1 *Natural integers are infinitely many.*

Note. For the infinite set ℕ of natural integers the symbol of infinity (∞) symbolically represents an *unreachable and therefore an undefined integer*. Moreover, should infinity exist as an integer of ℕ, let's say

$$\infty = N_0 \tag{1}$$

one would immediately fall into the impossibility

$$N_0 + 1 > \infty \tag{2}$$

2.2 Fundamental Theorem of Arithmetics

Initiated with lemmas circa 300 BCE by Euclid in his Elements [Heath, 1956] and proved by Gauss [Gauss, 1801] with Proposition 16 in book 1 (of 13) entitled *Disquisitiones Arithmeticae,* this theorem follows.

Theorem 2 *Every composite integer (greater than one) can be expressed uniquely (up to the order) as a product of powers of primes.*

$$\text{With } n>1 \text{ in } N,\ n = \Pi_{j=1,Jn}\, p_j^{ej} \tag{3}$$

where j, Jn, ej are integers, p_j are prime numbers and Jn the number of prime numbers necessary to factorize *n.*

2.3 Extension of Domain

Theorem 1 is easily extended to set ℤ of signed integers but theorem 2 can only be extended to ℤ** (ℤ without 0 and 1) so that only ℕ** (ℕ without 0 and 1) will be further used here to avoid dealing with signs, zero and prime considerations on number 1.

2.4 Factorization is a Discrete Function

Theorem 2 means that factorization sets up a one-to-one correspondence between any number in ℕ** and a unique product of prime numbers. Factorization can then be written as a discrete function valid only for natural integers

$$F(n) = n = \Pi_{j=1,Jn}\, p_j^{ej} \tag{4}$$

where J_n is not infinite. As this function is multiplicative one has

$$F(a\times b) = F(a) \times F(b) \tag{5}$$

2.5 Factorization is also an Algorithm

Factorization is obtained by an algorithm consisting in doing successive divisions by prime numbers *p*, these divisions being based on the fact that any integer *n* in ℕ** can be written.

$$n = pq + r \tag{6}$$

where the prime *p* is used as a test divisor, *q* is the integer quotient of *n* divided by *p* and *r* is the remainder of the division of *n* by *p*. A prime number *p* is validated as a factor of *n only when r is null*. The algorithm terminates when q=1 and r=0. Example of the algorithm for *n*=312:

312 = 2×156+0	→	2 is a factor of 312	→	312 = 2×156
156 = 2×78+0	→	2 is a factor of 156	→	312 = 2^2×78
78 = 2×39+0	→	2 is a factor of 78	→	312 = 2^3×39
39 = 2×19+1 thus		2 is no more a factor of 312		
39 = 3×13+0	→	3 is a factor of 39	→	312 = 2^3×3×13
13 = 3×4+1 thus		3 is no more a factor of 39	→	
13 = 5×2+3 thus		5 is not a factor of 13		
13 = 7×1+6 thus		7 is not a factor of 13	→	
13 = 11×1+2	→	11 is not a factor of 13	→	
13 = 13×1+0	→	13 is a factor of 13	→	312 = 2^3×3×13

And as last *q*=1 and last *r*=0 (which together are the stop alert) the algorithm ends up and gives the factorization

$$F(312) = 2^3\times 3\times 13 \tag{7}$$

as well as the *trajectory* of the prime factorization algorithm

312, 156, 78, 39, 3, 13, 1

which always ends up at 1 as, according to the fundamental theorem of arithmetics, a unique factorization always exists for any natural integer *N* in ℕ**.
Remark. The trajectory of the prime factorization has only two phases: the first is a sequence obtained by divisions by two of even numbers and the second is an ending-by-1 sequence obtained by factoring the first encountered odd number into powers of increasing primes.

2.6 A First Look at Collatz Algorithm

Let's begin with an example for which we do not know if Collatz algorithm ends up at 1. For n=312 this algorithm begins with

312, 156, 78, 39; 118, 59; 178, 89; 268, 134, 67; 202, …

where commas indicate divisions by 2 and semi-colons *jumps a=3n+1* which separate and define series of numbers or *branches*. Let's notice that at the end of branch B, if J is the number of used jumps, one has

$$B=1+J \tag{8}$$

due to the fact that the first branch does not begins by a jump but directly by the chosen N.
This proves that Collatz algorithm begins like the factorization of a number N that includes a power of 2 or not.
The difference with the usual prime factorization begins after the first encountered odd number (here n=39).

2.7 Trajectories

A *normal* or *long* trajectory is obtained when the algorithm uses jumps defined as

$$a = 3n + 1 \tag{9}$$

Some of the numerous studies use *short* trajectories obtained by using jumps defined as

$$a' = (3n+1) / 2 \tag{10}$$

This shows that the choice of a trajectory is fundamental to solve the conjecture.

2.8 The Fundamental Questions

Here come the usual two cases and three questions covering all cases that can happen to Collatz algorithm:

1- will it always end up at 1?
2- or will it sometimes not end up at 1?
 - by diverging to infinity?
 - or by entering an endless loop excluding number 1 (the stop alert)?

Answers are given in the next sections.

2.9 Collatz Algorithm is a Special Factorization

Let's prove that Collatz algorithm is a special factorization by running it on another example ending up at 1.
Proof. For N=28 one gets the long trajectory

28, 14, 7; 22, 11; 34, 17; 52, 26, 13; 40, 20, 10, 5; 16, 8, 4, 2, 1

Or the short version of it

27, 14, 7; 11; 17; 26, 13; 20, 10, 5; 16, 8, 4, 2, 1

Here a *new type of trajectory* will be used, defined for $i \geq 2$ (the current branch index) by taking only the first number of each branch of the long trajectory so that

$$a_1 = N \tag{11}$$

$$a_i = 3n_{i-1} + 1 \tag{12}$$

This then gives the new trajectory

28; 22; 34; 52; 40; 16

Noticing that all of the a_i's can also be factorized as

$$a_i = 2^{d_i} n_i \tag{13}$$

where d_i are the number of divisions in each branch i and n_i their last odd number we now define the property $T(N)$ of this trajectory made of B branches as the product of these numbers

$$T(N) = \Pi_{i=1,B}(a_i) \tag{14}$$

For N=28 this gives

$$T(N) = 28\times22\times34\times52\times40\times16 = 697\,016\,320 \tag{15}$$

From (28) this trajectory can also be written as

$$T(N) = \Pi_{i=1,B}(2^{d_i} n_i) \tag{16}$$

so that

$$T(N) = 2^d K \tag{17}$$

with

$$d = \Sigma_{i=1,B}(d_i) \tag{18}$$

$$K = \Pi_{i=1,B}(n_i) \tag{19}$$

For N=28 this gives

$$T(28) = 2^{13}(7\times11\times17\times13\times5) \tag{20}$$

This proves that Collatz algorithm is a special factorization of property $T(N)$ that secondarily gives the exact number of divisions by 2 needed by the whole trajectory to end up at 1 (here d=13).

3. Main Results

Proof. We just have seen that according to the fundamental theorem of arithmetics the factorizations $F(N)$ and $F(T(N))$ always exist.

Hypothesis. Let's now suppose that Collatz algorithm diverges or loops (excluding number 1) for one particular number N_0 in ℕ**. We would then simultaneously have

- On one hand (the hypothesis being inactive) the infinitely many existing factorizations $F(N \neq N_0)$ of all $N>1$ except N_0 and particularly those of all their multiples including $F(T(N)=KN)$;
- And on the other hand (the hypothesis being active for a particular N_0) the existing factorization $F(N_0)$ and those of all its multiples *except* $F(K_0N_0)$. This is because the assumed hypothesis of a diverging or looping algorithm implies that the number B of branches becomes infinity and is therefore *undefined* according to section 2.1. This in turn implies that

$$K_0 = \Pi_{i=1,\infty}(n_i) \qquad (21)$$

is also *undefined* as well as the particular multiple K_0N_0 of N_0 so that the prime factorization

$$F(T(N_0)) = F(K_0N_0) \qquad (22)$$

is also *undefined*.

This is a contradiction to the fundamental theorem of arithmetics which states that there always exists a unique factorization for each integer number greater than 1 and particularly for N_0 and its multiple $T(N_0) = K_0N_0$.

This contradiction proves that Collatz algorithm always ends up at 1.

4. Conclusion

This paper proves Collatz conjecture by addressing the three fundamental questions about Collatz algorithm and by defining a new type of trajectory that avoids the need to find a formula (or an approximation) for the usual long or short trajectories that (still in 2021) seem to be unfitted for a solution.

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