Theoretical Mathematics & Applications, vol.2, no.2, 2012, 63-68
ISSN: 1792-9687 (print), 1792-9709 (online)
International Scientific Press, 2012

# Decomposition of Complete 3-uniform Hypergraphs $K_n^3$ into Cycles for $n = 7, 10$

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**Abstract**

A $k$-uniform hypergraph $H$ is a pair $(V, \varepsilon)$, where $V$ is a set $V = \{v_1, v_2, \ldots, v_n\}$ of $n$ vertices and $\varepsilon$ is a family of $k$-subset of $V$ called hyperedges. A cycle of length $l$ of $H$ is a sequence $(v_1, e_1, \ldots, v_l, e_l, v_1)$, where $v_1, v_2, \ldots, v_l$ are distinct vertices, and $e_1, e_2, \ldots, e_l$ are $k$-edges of $H$ and $v_i, v_{i+1} \in e_i, 1 \leq i \leq l$, where addition on the subscripts is modulo $n$, $e_i \neq e_j$ for $i \neq j$. We consider the problem of constructing such decompositions for complete uniform hypergraphs. In this paper we apply design theory to give the decomposition of complete 3-uniform hypergraph $K_n^3$ into cycles for $n = 7, 10$.

**Mathematics Subject Classification:** 05C65
**Keywords:** Hypergraph, $t$-design, Cycle, Hamilton Cycle

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Article Info: *Received* : March 9, 2012. *Revised* : April 8, 2012
*Published online* : June 30, 2012

# 1 Introduction

A decomposition of a graph $G = (V, E)$ is a partition of the edge-set $E$; a Hamiltonian decomposition of $G$ is a decomposition into Hamiltonian cycles. The problem of constructing Hamiltonian decompositions is a long-standing and well-studied one in graph theory; in particular, for the complete graph $K_n$, it was solved in the 1890s by Walecki [1]. Walecki showed that $K_n$ has a Hamiltonian decomposition if and only if $n$ is odd, while if $n$ is even $K_n$ has a decomposition into Hamiltonian cycles and a perfect matching. As with many problems in graph theory, it seems natural to attempt a generalisation to hypergraphs. Indeed, the notion of Hamiltonicity was first generalised to uniform hypergraphs by Berge in his 1970 book [2]. His definition of a Hamiltonian cycle in a hypergraph $H = (V, E)$ is a sequence $(v_0, e_1, v_1, e_2, \cdots, v_{n-1}, e_n, v_0)$, where $\{v_0, \cdots, v_n\} = V$, and $e_1, \cdots, e_n$ are distinct elements of $E$, such that the hyperedge $e_i$ contains both $v_{i+1}$ and $v_i (modulo\, n)$. The study of decompositions of complete 3-uniform hypergraphs into cycles of this type was begun by Bermond et al in the 1970s [3] and was completed by Verrall in 1994 [4]. A $k$-uniform hypergraph $H$ is a pair $(V, \varepsilon)$, where $V = \{v_1 v_2, \ldots, v_n\}$ is a set of $n$ vertices and $\varepsilon$ is a family of $k$-subset of $V$ called hyperedges. If $\varepsilon$ consists of all $k$-subsets of $V$, then $H$ is a complete $k$-uniform hypergraph on $n$ vertices and is denoted by $K_n^k$. At the same time we may refer a vertex $v_i \in V$ to $v_{i+n}$. A cycle of length $l$ of $H$ is a sequence of the form

$$(v_1, e_1, v_2, e_2, \ldots, v_l, e_l, v_1),$$

where $v_1, v_2, \ldots, v_l$ are distinct vertices, and $e_1, e_2, \ldots, e_l$ are $k$-edges of $H$, satisfying

(i) $v_i,\ v_{i+1} \in e_i$, $1 \leq i \leq l$, where addition on the subscripts is modulo $n$, and

(ii) $e_i \neq e_j$ for $i \neq j$. This cycle is known as a Berge cycle, having been introduced by Berge in [1]. A cycle of length $l$ decomposition of $H$ is a partition of the hyperedges of $H$ into cycles of length $l$.

The set of cycles of length $l$ of complete 3-uniform hypergraph $K_n^3$, say $C_1$, …, $C_m$, is called cycles of length $l$ decomposition if $\bigcup_{i=1}^{m} \varepsilon(C_i) = \varepsilon(K_n^3)$ and $\varepsilon(C_i) \cap \varepsilon(C_j) = \emptyset$ for $i \neq j$. In this paper, we apply design theory to give decomposition of complete 3-uniform hypergraph.

## 2 Main results

We notice a Hamiltonian cycle in $K_n^k$ is an example of a 1-$(n, k, k)$ design; clearly, each vertex (i.e. point) lies in exactly $k$ edges. Therefore a Hamiltonian decomposition of $K_n^k$ is, in the language of design theory, a large set of 1-$(n, k, k)$ designs. So one may ask what known results in the design theory literature may be of use to us here.

**Definition 2.1.** *Let $v, k$ and $\lambda$ be integers such that $v \geq k \geq 2$ and $\lambda \geq 1$. Let $X$ be a finite set of elements, called points, and let $\mathcal{B}$ be a finite collection of subsets of $X$, called blocks. The pairs $(X, \mathcal{B})$ is called a $(v, k, \lambda)$ balanced incomplete block design or, simply, a $(v, k, \lambda)-BIBD$, if the following conditions hold:*

*(i) $|X| = v$.*

*(ii) $|B| = k$ for all $B \in \mathcal{B}$.*

*(iii) Every pairs of distinct points is contained in exactly $\lambda$ blocks.*

*The set $\{v, k, \lambda\}$ is called the set of parameters of the $BIBD$ $(X, \mathcal{B})$. We also use notation $\mathcal{D} = (X, \mathcal{B})$.*

**Definition 2.2.** *Let $v, k$ and $\lambda$ be integers such that $v \geq k \geq 2$ and $\lambda \geq 1$. Let $X$ be a finite set of elements, called points, and let $\mathcal{B}$ be a finite collection of subsets of $X$, called blocks. The pairs $(X, \mathcal{B})$ is called a $t-(v, k, \lambda)$ design or, simply, a $t-$design, if the following conditions hold:*

*(i) $|X| = v$.*

*(ii) $|B| = k$ for all $B \in \mathcal{B}$.*

*(iii) Every subset of $t$ distinct points is contained in exactly $\lambda$ blocks. The set $\{t; v, k, \lambda\}$ is called the set of parameters of the $t-design$ $(X, \mathcal{B})$.*

We have a 3-(7,5,1) design as followed:

{1,2,3,5,6}
{2,3,4,6,7}
{1,3,4,5,7}
{1,3,5,6,7}
{2,3,4,6,7}

$$\{1,2,4,5,6\}$$
$$\{1,2,4,5,7\}$$

The sets above are happen to be a decomposition of complete 3-uniform hypergraph $K_7^3$, which has seven cycles of length 5. We can easily see from the above Definition 2.2 when $k = \lambda, t = 1$, a $1-(v,k,k)$ design happen to be a Hamiltonian cycle of a k-uniform hypergraph.

For the simpleness, we omit the set sign $\{\cdots\}$.

Complete 3-uniform hypergraph $K_7^3$ can decompose into 5 Hamiltonin cycles of length 7 or 7 cycles of length 5.

Let vertex set be $\{1,2,3,4,5,6,7\}$, we have five $1-(7,3,3)$ designs as followed

013,135,356,456,246,024,012
015,125,235,236,346,046,014
023,234,345,145,156,016,026
034,134,146,145,126,025,035
045,245,124,123,136,036,056

All the 35 3-subsets divided into 5 lines, every 3-subset of any line is a hyperedge of hypergraph, every line happen to be a Hamiltonian cycle, 5 lines are the Hamiltonian decomposition of complete 3-uniform hypergraph $K_7^3$. We also can arrange 35 3-subsets as followed:

123,235,356,156,126
234,346,046,026,023
345,045,015,013,134
135,035,056,016,136
034,036,236,246,024
456,146,124,125,256
145,245,025,012,014

Every line happen to be a cycle of length 5, so we have a decomposition of complete 3-uniform hypergraph $K_7^3$ into cycles length 5.

**Definition 2.3.** *t-wise $\Gamma$ balance design, is a pair $(X, \mathcal{B})$, $\Gamma$ is a set consisting of t-uniform hypergraphs, $\Omega$ is set consisting of complete t-uniform hypergraph, $X$ is a finite set with $v$ vertices, $\mathcal{B}$ is a hypergraph on the subsets of $X$,* *such that for any block $B \in \mathcal{B}$, $B$ is isomorphic one of $\Gamma$, and every $t$-subset is included an only block, denoted by $S(t, \Gamma, v)$. If replacing $\Gamma$ by $\Omega$, then we use $S(t, K, v)$ express $S(t, \Omega, v)$, where $K$ is an positive integer set. The number of element from $B$ come from $K$.*

Base on this definition, we could get a Hamiltonian decomposition of complete 3-uniform hypergraph $K_{10}^3$ as followed:

012,345,678,123,234,456,567,789,089,019
029,249,479,457,357,356,168,136,018,028
013,035,058,568,468,467,247,279,129,139
124,146,169,679,579,578,358,038,023,024
235,257,027,078,068,689,469,149,134,135
346,368,138,189,179,079,057,025,245,246
034,348,389,589,159,125,127,267,067,046
145,459,049,069,026,236,238,378,178,157
256,056,015,017,137,347,349,489,289,268
367,167,126,128,248,458,045,059,039,379
478,278,237,239,359,569,156,016,014,048
036,369,269,259,258,158,147,148,047,037

This is a Hamiltonian decomposition of complete 3-uniform hypergraph $K_{10}^3$, every line is a Hamiltonian cycle, every line is isomorphic the others and every one is 1-(10,3,3) design of $X$.

Complete 3-uniform hypergraph $K_{10}^3$ also can decompose into 10 Hamiltonian cycles and 4 cycles of length 5 as followed:

012,345,678,123,234,456,567,789,089,019
036,369,269,259,258,158,147,148,047,037
078,578,568,356,346,134,124,129,029,079
189,689,679,467,457,245,235,023,013,018
056,156,126,127,278,378,348,349,049,059
167,267,237,238,389,489,459,045,015,016
038,138,168,169,469,479,247,257,025,035
149,249,279,027,057,058,358,368,136,146
067,367,236,239,289,589,458,145,014,017
178,478,347,034,039,069,569,256,125,128

024,246,468,068,028
135,357,579,179,139
048,248,268,026,046
159,359,379,137,157

Every one of the first 10 lines is a Hamiltonian cycle, every one of last 4 lines is a cycle of length 5. Then we get a decomposition of complete 3-uniform hypergraph $K_{10}^3$ into Hamiltonian cycles and cycles of length 5 and 10, which is corresponding to the Definition 2.3 for $K = \{5, 10\}$.

**Acknowledgements.** This work was supported by National Nature Science Fund (11161032), Inner Mongolia University for Nationalities Project (NMD1123, NMD1104) and Institution of Discrete Mathematics of Inner Mongolia University for Nationalities.

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