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Two-Step LM Unit Root Tests with Trend-Breaks

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Abstract

In this paper, we consider and examine the performance of two-step LM unit root tests with trend-breaks. In the first step, we jointly test for the existence and location of breaks using a maximum *F*-test. In the second step, we utilize the identified breaks and test for a unit root. A transformation procedure is adopted so that the tests with trend-breaks are invariant to nuisance parameters. We show that the two-step LM unit root tests have better properties of size and power than endogenous break unit root tests. In addition, the two-step test can be conveniently applied to allow for multiple breaks.

JEL classification: C12, C15, C22
Keywords: Unit Root Tests, Structural Breaks, Trend-shifts

1 Introduction

In this paper, we consider and examine the performance of two-step Lagrange Multiplier (LM) unit root tests with trend breaks. Our goal is to look for more reliable unit root tests when trend breaks are included in the model. As is well known, the distribution of unit root tests with breaks often depends on the

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parameter(s) describing the break(s). When the break location is known, one can use the exogenous test as in Perron (1989) and Amsler and Lee (1995). The break locations are rarely known a priori. When the break location is not known, the so-called endogenous break unit root tests have been considered.

However, Nunes, Newbold and Kuan (1997) and Lee and Strazicich (2001) raised the question on the endogenous unit root tests, regarding the dependency on the nuisance parameter(s).[4] Despite popularity, it has been reported in the literature that these tests depend on the nuisance parameter describing the break(s) under the null. As such, these tests *assume* that breaks are absent under the null. In the usual tests with breaks, the magnitude of breaks under the null should not affect the distribution of the test statistic so the same critical values can be used throughout. To achieve this outcome in endogenous tests, the critical values in the above tests are derived by assuming that the magnitude of any break under the null is zero. However, as noted in Nunes, Newbold and Kuan (1997) and Lee and Strazicich (2001), these endogenous unit root tests are not pivotal and spurious rejections occur under the null when the magnitude of breaks is not zero. As a result, when adopting these tests, rejection of the null does not imply a trend-(break) stationary time series and the possibility of a unit root with break(s) remains. Note that these endogenous tests contrast with the exogenous tests, which are invariant to the magnitude of breaks under the null. A recent survey paper by Perron (2006) also recognizes this critical drawback of assuming the absence of breaks under the null and provides a warning in this regard.[5]

In a related issue, we note that the existing endogenous unit root tests always find and include the number of breaks that are pre-specified in the model. For example, the one-break minimum or maximum unit root tests always identify and include one break. However, what if there is no break and the one-break test is adopted? Or, what if there is one break and the two-break test is adopted? In either case, one or more unnecessary breaks will be included in the unit root test and lower power will result. Whether or not a structural break exists is an empirical issue that must be determined from the data.

In practice, there is another conceptual issue in the use of exogenous versus endogenous break unit root tests. If the break point is known, then the exogenous unit root test will be preferred to increase power. In contrast, when the break point is unknown and must be estimated, endogenous tests are typically adopted to

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[4] Perhaps the most popular endogenous type unit root tests have been the minimum tests of Zivot and Andrews (1992) and Lumsdaine and Papell (1997), and the maximum tests of Perron (1997) and Vogelsang and Perron (1998).

[5] Lee, Strazicich and Meng (2012) compare properties of nine different Dickey-Fuller type endogenous break tests and issue similar warnings for these tests.

jointly identify the break and test for a unit root. However, these endogenous unit root tests have larger (absolute) critical values than those in the corresponding exogenous tests. As a result, the unit root null might be rejected for a times series when using the exogenous test, but not with the endogenous test. This begs the question, what if the break point is initially unknown, but can be accurately identified *prior* to performing the unit root test?

To address the above questions and improve upon the extant literature, in this paper we will consider a two-step procedure to identify breaks and test for a unit root. In the first step, we jointly determine *whether and where* structural breaks occur in the data, and in the second step we test for a unit root. To identify and test the significance of breaks, we adopt a maximum F (*maxF*) test. For example, in the simplest case, where no structural break should be included in the model, we want to utilize the no-break test. Then, since the location and/or existence of breaks are known following the first step, we utilize the exogenous test in the second step. This is important, since, as noted, the exogenous tests have greater power than the endogenous tests.

We additionally utilize new unit root testing procedures that are invariant to trend-breaks.[6] When there is only one break, we can find and utilize critical values at different break points. With two or more breaks, however, providing the correct critical values at different break point combinations can be inconvenient and may not be readily available to practitioners, especially in cases with trend breaks. Although this drawback is more of a practical inconvenience than a detriment to the validity of the unit root test, it would be helpful if the relevant critical values are readily available even with two or three breaks. We do this by adopting the data transformation procedure described in Park and Sung (1994). As we will demonstrate, when the lagged regressor variable is transformed using the ratio of sub-samples before or after the break, the unit root test statistic no longer depends on the location of trend-breaks. Relevant asymptotic properties are provided to justify the procedure. Moreover, the same transformation method can be conveniently applied to models with multiple breaks. This will be especially convenient when considering a larger number of possible break point combinations as the number of structural breaks increases.

Note that some existing unit root tests with multiple breaks tend to ignore the dependency on the nuisance parameter by assuming no break under the null, as in the endogenous DF-type unit root tests noted above. Such tests are simply invalid and are subject to spurious rejections. In contrast to these other tests, the two-step test with multiple breaks is a valid unit root test, since the test remains invariant to

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[6] In the presence of trend-breaks, the minimum LM unit root test depends on the location of the break(s) (Lee and Strazicich, 2003, 2004). More recently, Nunes (2004) utilizes a *t*-max LM unit root test and finds improved performance. However, the *t*-max unit root test still depends on the location of the trend-break.

nuisance parameters. As we will show, with multiple breaks the two-step unit root test depends only on the number of breaks and is valid whether the underlying series is stationary or not. Overall, we find that the proposed two-step unit root test accurately identifies breaks and has better properties of size and power than the endogenous break unit root tests.

The remainder of the paper is organized as follows. In Section 2, we examine properties of the LM unit root test when trend-breaks are known. In Section 3, two-step procedures are developed to determine the existence and location of unknown trend-breaks. In Section 4, simulation results are provided to examine the performance of the two-step unit root test. Comparisons to endogenous tests are also provided in the simulations. Section 5 summarizes and concludes. Throughout the paper, "→" indicates weak convergence as $T \to \infty$.

2 LM Unit Root Tests with Known Trend-Breaks

We consider the following data generating process (DGP) based on the unobserved component representation:

$$y_t = \delta' Z_t + e_t, e_t = \beta e_{t-1} + \epsilon_t, \tag{1}$$

where Z_t contains exogenous variables. The unit root null hypothesis is $\beta = 1$. If $Z_t = [1, t]'$, then the DGP is the same as that in the no-break test of Schmidt and Phillips (1992, hereafter SP). The level-shift only, or "crash," model can be described by $Z_t = [1, t, D_t]'$, where $D_t = 1$ for $t \geq T_B + 1$ and zero otherwise, and T_B stands for the time period of the break. The LM version of the crash model was initially examined in Amsler and Lee (1995). The trend-break, or "changing growth," model can be described by $Z_t = [1, t, DT_t^*]'$, where $DT_t^* = t - T_B$ for $t \geq T_B + 1$, and zero otherwise. Finally, when $Z_t = [1, t, D_t, DT_t^*]'$, we have the most general model with level and trend break. This general model is the most widely utilized in applied works and will be the focus of our paper.

To consider multiple breaks, we can include additional dummy variables such that:

$$Z_t = [1, t, D_{1t}, ..., D_{Rt}, DT_{1t}^*, ..., DT_{Rt}^*]', \tag{2}$$

where $D_{it}^* = 1$ for $t \geq T_{Bi} + 1$, $i = 1, ..., R$, and zero otherwise, and $DT_{it}^* = t - T_{Bi}$ for $t \geq T_{Bi} + 1$ and zero otherwise. Following the LM (score) principle, we impose the null restriction $\beta = 1$ and consider in the first step the following regression in differences:

$$\Delta y_t = \delta' \Delta Z_t + u_t, \tag{3}$$

where $\delta = [\delta_1, \delta_2, \delta_{3i}', \delta_{4i}']'$, $i = 1, ..., R$. The unit root test statistics are then obtained from the following regression:

$$\Delta y_t = \delta' \Delta Z_t + \phi \widetilde{S}_{t-1} + e_t, \tag{4}$$

where $\widetilde{S}_t$ denotes the de-trended series

$$\widetilde{S}_t = y_t - \widetilde{\Psi} - Z_t \widetilde{\delta}. \tag{5}$$

Here, $\widetilde{\delta}$ is the coefficient in the regression of Δy_t on ΔZ_t in (3), and $\widetilde{\Psi}$ is the restricted MLE of Ψ. That is, $\widetilde{\Psi} = y_1 - Z_1 \widetilde{\delta}$. Subtracting $\widetilde{\Psi}$ in (5) makes the initial value of the de-trended series begin at zero with $\widetilde{S}_1 = 0$, while letting $\widetilde{\Psi} = 0$ leads to the same result. It is important to note that in the de-trending procedure (5), the coefficient $\widetilde{\delta}$ was obtained in regression (3) using first differenced data. Thus, the de-trending parameters are estimated in the first step regression in differences. Through this channel the dependency on nuisance parameters is removed in the crash model. However, as we will discuss later, the dependency on nuisance parameters is not removed with this de-trending procedure in the model with trend breaks.

The LM unit root test statistic is defined by:

$$\widetilde{\tau} = t\textit{-statistic for the null hypothesis } \phi = 0. \tag{6}$$

To allow for serially correlated and heterogeneously distributed innovations, we require the assumption that the innovations ϵ_t satisfy the regularity conditions of Phillips and Perron (1988, p. 336). We define two nuisance parameters $\sigma^2 = \lim T^{-1} E[(\sum_{t=1}^{T} \epsilon_t)^2]$ and $\sigma_\epsilon^2 = \lim T^{-1} E[\sum_{t=1}^{T} \epsilon_t^2]$, and let $\omega^2 = \sigma^2 / \sigma_\epsilon^2$. The innovation variance σ_ϵ^2 is estimated as the error sum of squares from regression (4). The long-run variance $\sigma^2$is estimated by choosing a truncation lag parameter l and a set of weights w_j, $j = 1, ..., l$: $\widetilde{\sigma}^2 = \widehat{\gamma}_0 + 2 \sum w_j \widehat{\gamma}_j$, where $\widehat{\gamma}_j$ is the jth sample autocovariance of the residuals from regression (4). We can modify the statistics accordingly with the correction factor $\widetilde{\omega}^2 = \widetilde{\sigma}^2 / \widetilde{\sigma}_\epsilon^2$ to correct for the effect of autocorrelated errors with $Z_\tau = \widetilde{\tau} / \widetilde{\omega}$. Alternatively, one can include the terms $\Delta \widetilde{S}_{t-j}$, $j = 1, ..., k$ in (4) to correct for serial correlation in the usual augmented type tests:

$$\Delta y_t = \delta' \Delta Z_t + \phi \widetilde{S}_{t-1} + \sum_{j=1}^{k} d_j \Delta \widetilde{S}_{t-j} + e_t. \tag{7}$$

The asymptotic distribution of the test statistics can be based on the following result, as shown in Im, Lee and Tieslau (2011).

Proposition 1 *Suppose that the data generating process implies (1) with* $\beta = 1$ *and* $Z_t = [1, t, D_{1t}, ..., D_{Rt}, DT^*_{1t}, ..., DT^*_{RT}]$ *for the model with level and trend-breaks. We define* $V_i^*(r)$*, the weak limit of the partial sum residual process* $\widetilde{S}_t$ *in (5), as follows:*

$$V_i^*(r) = \begin{cases} \sqrt{\lambda_1^*}V_1(r/\lambda_1) & \text{for } r \le \lambda_1 \\ \sqrt{\lambda_2^*}V_2[(r-\lambda_1)/(\lambda_2-\lambda_1)] & \text{for } \lambda_1 < r \le \lambda_2 \\ \vdots & \\ \sqrt{\lambda_{R+1}^*}V_{R+1}[(r-\lambda_R)/(1-\lambda_R)] & \text{for } \lambda_R < r \le 1 \end{cases}. \quad (8)$$

Then, we have

$$\widetilde{\tau} \to -\frac{1}{2}\omega \left[\sum_{i=1}^{R+1} \lambda_i^{*2} \int_0^1 \underline{V}_i(r)^2 dr \right]^{-1/2}, \quad (9)$$

where λ_i^* *denotes the fraction of sub-samples in each regime such that* $\lambda_1^* = T_{B1}/T$, $\lambda_i^* = (T_{Bi} - T_{Bi-1})/T$, $i = 2, ..., R,$ *and* $\lambda_{R+1}^* = (T - T_{BR})/T$. *Here,* $\underline{V}_i(r)$ *is the projection of the process* $V_i(r)$ *on the orthogonal complement of the space spanned by the trend break function* $dz(\lambda^*, r)$, *as defined over the interval* $r \in [0, 1]$, *where* $V_i(r) = W_i(r) - rW_i(1)$, *and* $W_i(r)$ *is a Wiener process for* $i = 1, ..., R$.[7]

The result in (9) shows that in contrast to the model with level shift in Amsler and Lee (1995), the asymptotic distribution of the test statistic with trend breaks depends on the nuisance parameters, λ_i^*. Thus, as noted in Im, Lee and Tieslau (2011), we adopt an approach similar to that in Park and Sung (1994). Then, the dependency of the test statistic on the nuisance parameter can be removed by performing the following transformation:

$$\widetilde{S}_t^* = \begin{cases} \dfrac{T}{T_{B1}}\widetilde{S}_t & \text{for } t \le T_{B1} \\ \dfrac{T}{T_{B2} - T_{B1}}\widetilde{S}_t & \text{for } T_{B1} < t \le T_{B2} \\ \vdots & \\ \dfrac{T}{T - T_{BR}}\widetilde{S}_t & \text{for } T_{BR} < t \le T \end{cases}. \quad (10)$$

We then replace $\widetilde{S}_{t-1}$ with $\widetilde{S}_{t-1}^*$ in the testing regression and change (7) to as follows:

$$\Delta y_t = \delta' \Delta Z_t + \phi \widetilde{S}_{t-1}^* + \sum_{j=1}^{k} d_j \Delta \widetilde{S}_{t-j} + e_t. \quad (11)$$

Theorem 1 *Let* $\widetilde{\tau}^*$ *be the* t *-statistic for* $\phi = 0$. *Then, the asymptotic distributions of these test statistics will be invariant to the nuisance parameter* λ□

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[7] In the above, the argument $r_i^* = (r - \lambda_i)/(\lambda_i - \lambda_{i-1})$ is defined over the range between λ_{i-1} and λ_i, which has been transformed into r defined over the range 0 to 1.

$$\widetilde{\tau}^* \rightarrow -\frac{1}{2}\left[\sum_{i=1}^{R+1}\int_0^1 \underline{V}_i(r)^2 dr\right]^{-1/2}, \tag{12}$$

where $\underline{V}_i(r)$ *is defined in Proposition* 1.

The proof is given in Im, Lee and Tieslau (2011). The above result shows that, following the transformation, the unit root test statistic $\widetilde{\tau}^*$ no longer depends on the nuisance parameter λ_i in the trend-break model, although information on λ is required to construct the test statistic. Instead, the asymptotic distribution of $\widetilde{\tau}^*$ depends only on the number of trend breaks, since the distribution is given as the sum of $R+1$ independent stochastic terms. With one trend-break ($R=1$), the distribution of $\widetilde{\tau}^*$ is the same as that of the untransformed test $\widetilde{\tau}$ using $\lambda = 1/2$, regardless of the initial location of the break(s). Similarly, with two trend-breaks ($R=2$), the distribution of $\widetilde{\tau}^*$ is the same as that of the untransformed test $\widetilde{\tau}$ using $\lambda_1 = 1/3$ and $\lambda_2 = 2/3$. In general, for the case of R multiple breaks, the same analogy holds: the distribution of $\widetilde{\tau}^*$ is the same as that of the untransformed test $\widetilde{\tau}$ using $\lambda_i = i/(R+1)$, $i = 1, ..., R$. Therefore, we do not need to simulate new critical values at all possible break point combinations. Instead, we only need critical values that correspond to the number of breaks, R. The critical values of the transformed LM tests with trend shifts are provided in Im, Lee and Tieslau (2011) for various values of T when $R = 1, ..., 4$.

As we will see, the above invariance results will prove helpful in constructing LM unit root tests with unknown trend-breaks. However, the invariance result does not mean that one can adopt an incorrect number and/or placement of breaks, even under the null. In fact, one should include the correct number of breaks and their correct placement when performing unit root tests. First is the matter of power: unit root tests will lose power under the stationary alternative hypothesis if the number and/or placement of the break(s) are incorrect. As noted, Perron (1989) showed that the usual ADF tests will be biased against rejecting the null when the stationary alternative is true and a structural break is ignored. This outcome will similarly hold for LM tests with trend-breaks. However, what about properties under the null when an incorrect model is adopted? Amsler and Lee (1995) showed that the LM unit root tests with level-shifts will be unaffected asymptotically under the null, even when an incorrect number of breaks or their incorrect location is adopted. However, this is not the case with trend-breaks. In the following, we demonstrate that LM unit root tests allowing for level-shifts will be affected even under the null if trend breaks exist but are ignored.[8]

[8] Perron (1989) initially examined the effect of ignoring breaks when the alternative hypothesis is true, but did not examine the effect of ignoring breaks when the null is true.

Proposition 2 *Suppose that the data generating process implies (1) with* $\beta = 1$ *and* $Z_t = [1, t, D_{1t}, ..., D_{Rt}, DT^*_{1t}, ..., DT^*_{RT}]$*, but the crash model with level-shifts only is adopted. We denote the resulting test statistic as* $\widetilde{\tau}^+_A$*. Then, it follows that the asymptotic distribution of tests using the mis-specified model will be as follows:*

$$\widetilde{\tau}^+_A = O_p(T^{-1/2}).$$

Proof: see the Appendix.

Proposition 2 indicates that the effect of ignoring trend-breaks in the unit root testing model is not negligible even asymptotically. Downward size distortions will occur when existing trend-breaks are ignored and only level shifts are included. We expect a similar bias to occur if the usual LM unit root test without breaks is adopted when the underlying model includes trend-breaks. The results of Proposition 2 can be easily generalized: similar size distortions are expected when the number of trend-breaks is under-estimated.

3 LM Unit Root Tests with Unknown Trend-Breaks

In practice, one rarely knows the correct number (R) and location (T_{Bi}) of structural breaks. As noted, the correct number and location of breaks is necessary to avoid a loss of power. Determining the correct number of breaks, however, has been difficult due mainly to the nuisance parameter dependency problem. Following Davies (1987), it is well known that tests for m versus $m + 1$ breaks face a nuisance parameter problem, due to the fact that the location parameter is not identified under the null of m breaks. With stationary data there are good solutions to this problem, but these solutions cannot be applied when the order of integration is one or unknown.[9] To propose a solution, we suggest adopting a two-step procedure. In the first step, we identify and test for the

[9] For example, the popular Bai and Perron (1998) procedure cannot be used to determine the number and/or location of breaks in unit root tests, since the method is valid only when the data is known to be stationary. The paper by Vogelsang (1998) is encouraging in this regard, since the suggested method to test for a trend is argued to be valid whether the order of integration is 1 or 0. But, it seems that this method cannot be directly applicable to unit root testing procedures. In order to apply the procedure to unit root tests, prior information on the order of integration is necessary, which must first be determined with a unit root test. In a recent paper, Perron and Kim (2006) propose a two-step method to identify a trend-break that is suggested to be valid in unit root tests where the order of integration is unknown. However, their method may not be directly applicable to LM tests and we do not pursue their method in the present paper. Moreover, there is a sense that information on the persistence of the data is required before the trend-break can be identified.

significance of breaks using an F-test. Then, in the second step, we utilize the exogenous LM unit root test with one break if the null of no break is rejected. Similarly, we use the conventional no break LM unit root test if the null of no break is not rejected. We can utilize the same F-test to sequentially determine the number of breaks.

There are compelling reasons why a two-step procedure is appropriate. Most important, when utilizing the endogenous break unit root tests there is a size distortion leading to under rejections of the null that will depend on the likelihood of correctly identifying the break(s).[10] For example, suppose that the magnitude of a break is small or negligible under the null or alternative. If so, using a minimum or maximum endogenous unit root test that assumes a certain number of breaks would be inappropriate. In our two-step procedure, we suggest that if the magnitude of the break (in a one-break test) is small, it is more sensible to apply a unit root test without break. In contrast, if the magnitude of the break is large then the chance of identifying the break correctly is almost certain, and it is sensible to include the break in the unit root test. As such, in cases where the magnitude of the break is large and can be estimated with certainty, it seems misleading to use the larger (in absolute value) critical values from the endogenous break unit root test.[11]

From the above, it is clear that the distribution of the endogenous break unit root test statistic will depend on the probability of correctly estimating the break location, $P(\widetilde{\lambda} = \lambda)$, such that

$$\int_0^1 \Phi[\widetilde{\tau}^*(\widetilde{\lambda})|\widetilde{\lambda} = \lambda]P(\widetilde{\lambda} = \lambda)d\lambda,$$

where $\Phi[\widetilde{\tau}^*(\widetilde{\lambda})|\widetilde{\lambda} = \lambda]$ is the asymptotic distribution of $\widetilde{\tau}^*$ conditional on $\widetilde{\lambda} = \lambda$. In essence, it is simply difficult to determine the exact distribution of the test statistic and the question remains as to which critical values to adopt. As an extreme case, suppose that $P(\widetilde{\lambda} = \lambda) = 1$. In this case, the break points are

[10] Note that this (negative) size distortion in endogenous break unit root tests is more general and differs from the spurious rejections found in the Dickey-Fuller type endogenous break tests.

[11] Perron (1997) argues convincingly that using exogenous unit root tests is appropriate in many cases when the break dates are known to relate to well recognized events, such as the Great Depression or the oil price shocks of the early 1970s. In applied works, practitioners can be confused about which test to use in such cases. For example, suppose that a break occurs and the break point is known. Any good endogenous test should be able to detect the known break point. Then, it is not clear if the exogenous or endogenous test critical values should be used if the break point was estimated and conclusions may differ in each case.

correctly estimated with certainty. This is the case when Assumption (a) is satisfied as follows:

Assumption (a)

$$\widetilde{T}_{Bi} \rightarrow T_{Bi}, \textit{ and } \widetilde{\lambda}_i = \lambda_i, \textit{ as } T \rightarrow \infty.$$

Under Assumption (a) the distribution of $\widetilde{\tau}^*$ asymptotically approaches that of the $\widetilde{\tau}$ test, which assumes known, or exogenous, breaks. This outcome will occur when the size of the break (d) is large enough. When d is small, $P(\widetilde{\lambda} = \lambda)$ departs from one, and when $d = 0$, $P(\widetilde{\lambda} = \lambda)$ has a uniform distribution. As such, using endogenous test critical values can lead to (negative) size distortions that depend on the magnitude of the break(s).

The above begs the question: how big should a break be in order to be considered as important structural change? The answer to this question is an empirical matter that must be determined by statistical inference and will depend on the magnitude of the break. To do so, we utilize an F-test in the first step to test for the existence of breaks. As the location and size of the break(s) is unknown, we estimate the break locations obtained by maximizing the F-statistic for the joint significance of the break dummy variables (we refer to this method as max$F$). The max$F$ method is essentially the same as the t-max method, which amounts to minimizing the SSR. As we will demonstrate, the max$F$ procedure determines the break point with 100% accuracy for large breaks and provides justification for using the exogenous test in our second step. At the same time that we identify the break point, we apply the F-statistic to test the null hypothesis of no break against the alternative that a break exists. We use the following F-statistic:

$$F \textit{ or } F^* = \frac{(SSR_0 - SSR_1(\lambda))/k}{SSR_1(\lambda)/(T-q)}, \tag{13}$$

where k is the number of restrictions, q is the number of regressors in (7) or (11), and SSR_0 denotes the SSR from the LM unit root test without breaks. The use of F or F^*depends on whether $SSR_1(\lambda)$ denotes the SSR from regression (7) for the untransformed tests (F), or from regression (11) for the transformed tests (F^*). A sequential method using the same F-test can be adopted to test for the existence of multiple breaks. For example, given the presence of one break, we can then similarly test for the existence of two breaks, and so on. Alternatively, we can test for the existence of two breaks against the alternative hypothesis of no break. In this case, the restricted and unrestricted models can be properly defined.

When the break location is known a priori, the distribution of the F-statistic in (13) is standard and the usual critical values can be employed. When λ is unknown, using the untransformed and transformed data respectively, we identify the break by performing the following grid search:

$$maxF = \underset{\lambda}{\text{Max}}\ F \textit{ and } maxF^* = \underset{\lambda}{\text{Max}}\ F^* \tag{14}$$

where $maxF$ ($maxF^*$) uses the estimated break location, $\widehat{\lambda} = \underset{\lambda}{\text{argmax}}\ F$ ($\widehat{\lambda} = \underset{\lambda}{\text{argmax}}\ F^*$). The asymptotic distribution of the F-test is given as follows:

Proposition 3 *Suppose that y_t is generated by the data generating process* (1) *with $\beta = 1$, there are no breaks. Then,*

$$maxF^* \to \underset{\lambda}{\text{Max}}\ \frac{1}{8}(\sigma_\epsilon^2/\sigma^2)\left[(\sum_{i=1}^{R+1}\int_0^1 \underline{V}_i(r)^2 dr)^{-1} - (\int_0^1 \underline{V}_0(r)^2 dr)^{-1}\right], \quad (15)$$

where $\underline{V}_0(r)$ is a demeaned Brownian bridge, and $\underline{V}_i(r)$ is defined in Proposition 1.

Proof. See the Appendix.

The critical values of the maxF^* test are displayed in Panel A of Table 1 for $R = 1, 2$, and were obtained against the DGP involving no breaks. For comparison, the critical values of the F-test using the untransformed regression (7) are reported in Panel B of Table 1 for $R = 1$. Note that the distribution of the F-statistic in (13) is non-standard for two reasons. First, this outcome is due to the fact that the nuisance parameter λ is present only under the alternative hypothesis and is not identified under the null hypothesis of no break. However, any nuisance parameter dependency is resolved due to the fact that maxF^* is an order statistic estimated by finding the maximum F-statistic (13) over the range of breaks $\lambda = [0.15, 0.85]$. Second, the unit root null is imposed in the DGP (1) with $\beta = 1$ in constructing test statistics for max F^*. Certainly, it is desirable to develop a test that is valid regardless of whether the data is integrated or not. This task, however, seems difficult in our case of applying to unit root tests. A troublesome issue noted in recent developments in this direction, such as in Vogelsang (1998) and elsewhere, involves the fact that there is no feasible way to develop valid tests without using a priori information on the order of integration. However, if the order of integration is known, there is no need to perform unit root tests. As a solution, one may possibly impose $\beta = 0$ in the DGP, but this also poses a difficulty. This way of dealing with the problem includes the approach suggested by Bai and Perron (1998) when testing for the presence of multiple structural changes. While these types of tests are valid with stationary data, they tend to diverge when the data is persistent or integrated. This outcome is obvious from the findings in Phillips (1986), who showed that an F-test tends to diverge spuriously in integrated series. For this reason, the Bai and Perron (1998) method cannot be adopted to test for structural changes in unit root tests. However, imposing $\beta = 1$ in the DGP is justified for our F-test and can be used to obtain valid tests under the unit root null hypothesis. Our proposed F-test has decent size and power against trend-breaks under the unit root hypothesis. How this treatment affects performance under the stationarity alternative is a question to consider. Fortunately, our simulation results in the next section show that any power loss problem is mild or negligible. We conclude that the F-test can be well applied to a stationary or nonstationary process.

Table 1: Critical Values of F-Statistics with Trend-Breaks

Test	R	sig. level (%)	Sample Size				
			$T = 50$	$T = 100$	$T = 200$	$T = 500$	$T = 1000$
Panel A. *Max*F-statistic for the Significance of Structural Changes (*Transformed* Tests with Trend-breaks)							
maxF^*	1	1	14.084	13.089	12.960	13.369	13.691
		5	10.801	10.435	10.546	11.062	11.532
		10	9.397	9.245	9.466	10.045	10.486
	2	1	12.490	11.612	10.881	11.120	11.889
		5	10.456	9.591	9.546	9.988	10.371
		10	9.373	8.777	8.728	9.286	9.807
Panel B. *Max*F-statistic for the Significance of Structural Changes (*Untransformed* Tests with Trend-breaks)							
max$F$	1	1	13.683	12.654	12.352	13.132	13.166
		5	10.262	9.862	10.109	10.670	11.148
		10	8.931	8.751	8.932	9.554	10.120

The two-step procedure can be summarized as follows. First, we apply the maxF^* test to identify the break locations and test the significance of breaks. Second, if we cannot reject the null of no break, we apply the usual no-break LM unit root test of Schmidt and Phillips (1992). If the null of no break is rejected, then we know both the existence and location of the break(s) and can employ the one break (or R breaks) exogenous LM unit root tests in Amsler and Lee (1995) and Lee and Strazicich (2003). Thus, our testing procedure is as follows:

$$\widetilde{\tau}_{\text{2-step}} = \begin{cases} \widetilde{\tau}_{SP} & \text{if the null of no breaks is not rejected in the F-test} \\ \widetilde{\tau}^{*} & \text{if the null of no breaks is rejected in the F-test.} \end{cases} \tag{16}$$

Comparison with Endogenous Tests

For comparison, we also consider two popular approaches to identify a structural break. The first approach uses the minimum test statistic as suggested in Zivot and Andrew (1992). The break location is determined at the point where the t-statistic testing the unit root null hypothesis is minimized. The second approach utilizes the method suggested by Perron (1997) and Vogelsang and Perron (1998), who suggest estimating the break point where the absolute value of the t-statistic on the parameter associated with the break dummy variable is maximized. As noted in Section 1, these Dickey-Fuller type tests assume the absence of breaks under the null in order to eliminate the dependency of the test statistics on nuisance parameters. However, no such assumption is required in the LM framework, since the test statistic is free of this nuisance parameter (see Amsler and Lee, 1995, and Lee and Strazicich, 2003).

From the results in Perron (1997), we expect that using the t-max method will lead to better performance in identifying the break location. We therefore examine properties of the following two LM unit root test statistics: $\widetilde{\tau}_\tau^* = \underset{\lambda}{\text{Inf}}\widetilde{\tau}^*(\widetilde{\lambda}_\tau^*)$and $\widetilde{\tau}_{|td|}^* = \widetilde{\tau}^*(\widetilde{\lambda}_{|td|})$, where a grid search is implemented over the range of the nuisance parameter, λ. $\widetilde{\lambda}_\tau^* = \text{argmin } \widetilde{\tau}^*(\widetilde{\lambda}_\tau)$ denotes the minimum value of the unit root t-test statistic, and $\widetilde{\lambda}_{|td|} = \text{argmax } |td|$ denotes the maximum absolute value of the t- or F-statistic for the significance of the break dummy coefficients, respectively. If we impose the restriction that the coefficients of both D_{it}^* and DT_{it}^*, $i = 1, ..., R$, are jointly zero, we employ an F-statistic. For convenience we continue to denote the resulting break estimate and corresponding statistics with the subscript $|td|$. Note that the above grid search is justified since the distribution of $\widetilde{\tau}^*$ does not depend on $\tilde{\lambda}$.However, the search is implemented in the testing regression (7) or (11) by using different values of the break locations ($T_B = \lambda T$ to define the dummy variables for the transformation in (10). As a practical matter, we trim the end points and grid-search the value of λ over the range $\lambda \in [0.15 \sim 0.85]$. The asymptotic distribution is obtained accordingly from Theorem 1. We let $Q(\lambda) = -\frac{1}{2}[\sum_{i=1}^{R+1} \int_0^1 \underline{V}_i(r)^2 dr]^{-1/2}$ and can have $\widetilde{\tau}_\tau^* \to Q(\widetilde{\lambda}_\tau^*)$ and $\widetilde{\tau}_{|td|}^* \to Q(\widetilde{\lambda}_{|td|})$, where $\widetilde{\lambda}_\tau^*$ and $\widetilde{\lambda}_{|td|}$ are obtained from the search procedures above. We denote $Q(\lambda)$ in terms of λ to signify that it is the asymptotic distribution of the statistic obtained by using a certain optimal value of λ(i.e., $\widehat{T}_B = \widehat{\lambda}T$) in constructing $\widetilde{S}_t^*$ in (10), although Q is not dependent on $\tilde{\lambda}$ $\widehat{T}_B = (\widehat{T}_{B1}, \widehat{T}_{B2}, ..., \widehat{T}_{BR})$ denotes the estimated locations of the breaks. We report critical values of these endogenous tests in Panel A of Table 2 when $R = 1$ and 2. We have chosen to omit critical values for $R \geq 3$ due to the computational burden in the grid search, but in general we advise not to consider more than 2 breaks in endogenous break models since they yield much less powerful tests. As previously noted, this might appear contrary to the perception in many empirical works that allowing for more breaks leads to higher power. In fact, endogenous break unit root tests with more than two breaks lose considerable power.[12] As such, we recommend using a parsimonious number of breaks; or adopting a non-linear model if the data is subject to many breaks (see Enders and Lee, 2011). For comparison, in panel B of Table 2, we provide critical values of the untransformed endogenous tests when R = 1.

Note that $\widetilde{\lambda}$ (either $\widetilde{\lambda}^*_\tau$ or $\widetilde{\lambda}_{|td|}$) has a well-defined probability distribution over the parameter space Λ, and the true value of λ is included in the sample space of $\widetilde{\lambda}$ when the DGP implies the stationary alternative. Assuming a stationary time series, Bai (1994) shows that for the maximum likelihood estimate of λ, say $\widetilde{\lambda}^+$, $T(\widetilde{\lambda}^* - \lambda) = O_p(\frac{1}{d'd})$ and $T^{1/2-b}||d|| \to \infty$ for $b \in (0, 1/2)$, where d denotes the magnitude of structural breaks in the regression model. These results also hold in our case under the alternative hypothesis. The ML estimate $\widetilde{\lambda}^+$ is obtained as the value minimizing the sum of squared residuals in regression (11), while $\widetilde{\lambda}$ is obtained in (13) or (14). Thus, both $\widetilde{\lambda}^+$ and $\widetilde{\lambda}$ are T-consistent, but only when d is large enough. Therefore, neither $\widetilde{\lambda}^+$ nor $\widetilde{\lambda}$ is expected to be estimated correctly in finite samples if d is small.[13]

[12] As described in Section 1, many empirical papers report that the null of a unit root is rejected more often when more breaks are included. This is possible if the data is truly trend-stationary with breaks (i.e., the alternative hypothesis is true and breaks occur in the data). However, in many cases, the results of greater rejections with more breaks might also reflect spurious rejections from the presence of a unit root with break(s) as described in Section 1. Given that the power in unit root tests will be extremely low when the number of breaks is increased, we suspect that the latter explanation might be more common.

[13] The properties of $\widetilde{\lambda}^+$ or $\widetilde{\lambda}$ under the unit root null have not been fully examined in the literature, while Perron and Zhu (2005) provide a useful insight that the order of convergence is usually lower in an integrated process than in a stationary process in the specific model they consider. This issue, however, remains beyond the scope of the present paper.

Table 2: Critical Values of Endogenous LM Unit Root Test Statistics with Trend-Breaks

Test	R	sig. level (%)	$T=50$	$T=100$	$T=200$	$T=500$	$T=1000$
			Panel A. *Transformed* Endogenous Tests				
$\widetilde{\tau}^*_\tau$	1	1	-5.429	-5.106	-5.059	-4.975	-4.938
		5	-4.772	-4.598	-4.497	-4.438	-4.417
		10	-4.447	-4.301	-4.218	-4.166	-4.151
	2	1	-6.751	-6.332	-6.204	-6.125	-6.152
		5	-6.099	-5.775	-5.654	-5.643	-5.688
		10	-5.796	-5.512	-5.372	-5.433	-5.496
$\widetilde{\tau}^*_{\lvert td\rvert}$	1	1	5.004	4.760	4.696	4.591	4.530
		5	4.338	4.153	4.073	3.976	3.952
		10	3.994	3.832	3.766	3.691	3.684
	2	1	6.289	5.975	5.647	5.580	5.542
		5	5.572	5.222	5.036	5.037	5.028
		10	5.182	4.928	4.747	4.783	4.757
			Panel B. *Untransformed* Endogenous Tests				
$\widetilde{\tau}_\tau$	1	1	5.483	5.126	4.963	4.797	4.755
		5	4.730	4.507	4.322	4.284	4.270
		10	4.399	4.196	4.062	4.024	4.005
$\widetilde{\tau}_{\lvert td\rvert}$	1	1	4.938	4.744	4.575	4.386	4.449
		5	4.291	4.050	3.939	3.854	3.867
		10	3.897	3.699	3.626	3.590	3.580

Notes: R denotes the number of level and trend-breaks.

4 Simulation Results

In this section, we provide finite sample Monte Carlo simulation results on the LM unit root tests with trend-breaks and the two-step test. Our goal is to verify the theoretical results presented above and to examine the general performance of the tests. To perform our simulations, pseudo-iid $N(0,1)$ random numbers were generated using the Gauss procedure RNDNS with all calculations conducted using the Gauss software version 6.0. The DGP used in the simulations has the form in (1). The initial values y_0 and ϵ_0 are assumed to be random, and we assume that $\sigma^2_\epsilon = 1$. All simulation results are calculated using 20,000 replications for the tests with exogenous breaks, and 5,000 replications for the endogenous break tests. The size (frequency of rejections under the null when $\beta = 1$) and power (frequency of rejections under the alternative when $\beta = 0.9$) of the tests are evaluated using 5% critical values.

In Table 3, we report the size and power properties of the exogenous break unit root tests for different level and trend break magnitudes (d1 and d2) and locations (λ). In each case, we wish to examine how the transformed test ($\widetilde{\tau}^*$), untransformed test ($\widetilde{\tau}$), and two-step test ($\widetilde{\tau}^+_{\text{2-step}}$) behave under the null and alternative hypotheses. In particular, we wish to examine if the transformed test ($\widetilde{\tau}^*$) and the two-step test ($\widetilde{\tau}^+_{\text{2-step}}$) are invariant to the size and location of breaks. The results reported in Table 3 show that both tests have reasonably good size under the null. While they show mild size distortions in many cases, there is no clear pattern and the distortions are small. This is an encouraging finding and supports our proposition that the size properties in the transformed tests are invariant to different break locations ($\lambda = 0.3$ or 0.8) and magnitudes. Comparing the size properties of the transformed test with the untransformed test we see little difference, although the transformed test has marginally more accurate size in most cases. We also report the 5% rejection rates for the F-test of the significance of breaks. It is clear that the F-test has good power when the DGP has a unit root. While the power is lower when the magnitude of the trend-break is smaller ($d_2 =$ 0.5), the power increases quickly to 1.0 as the magnitude of the break increases. Unreported results also show that the F-test has a correct 5% rejection rate when there is no break ($d_1 = d_2 = 0$) in the DGP. As noted, the F-test assumes a unit root ($\beta = 1.0$) in the DGP. Therefore, we also want to know how the F-test performs when the DGP is a stationary process ($\beta = 0.9$). The results in the last column of Table 3 indicate that the power of the F-test in a stationary process is high and nearly identical to the power when the DGP has a unit root. This is an important property of the F-test. Most notably, this outcome differs from the reverse case found in some other tests, where the F-test assuming a stationary series exhibits spurious rejections when the DGP is non-stationary. No evidence of spurious rejections is found here in any case.

Table 3: Size and Power Properties of the Exogenous Break LM Unit Root Tests

DGP				Size ($\beta = 1.0$)				Power ($\beta = 0.9$)			
T	d_1	d_2	λ	$\widetilde{\tau}^*$	$\widetilde{\tau}$	$\widetilde{\tau}^+_{\text{2-step}}$	F-*test*	$\widetilde{\tau}^*$	$\widetilde{\tau}$	$\widetilde{\tau}^+_{\text{2-step}}$	F-*test*
100	2	0.5	0.3	0.043	0.039	**0.038**	0.760	0.138	0.137	**0.108**	0.767
			0.5	0.054	0.054	**0.047**	0.791	0.149	0.149	**0.124**	0.832
			0.8	0.040	0.034	**0.037**	0.776	0.120	0.120	**0.100**	0.792
	5	0.5	0.3	0.049	0.042	**0.049**	0.998	0.131	0.128	**0.131**	0.996
			0.5	0.053	0.053	**0.053**	0.999	0.149	0.149	**0.149**	0.998
			0.8	0.039	0.031	**0.039**	0.998	0.130	0.134	**0.129**	0.998
	2	1	0.3	0.042	0.041	**0.036**	0.926	0.134	0.132	**0.122**	0.944
			0.5	0.048	0.048	**0.045**	0.953	0.147	0.147	**0.143**	0.984
			0.8	0.043	0.030	**0.042**	0.941	0.125	0.128	**0.119**	0.957
	5	1	0.3	0.046	0.040	**0.046**	1.000	0.129	0.133	**0.129**	1.000
			0.5	0.047	0.047	**0.047**	1.000	0.148	0.148	**0.148**	1.000
			0.8	0.040	0.031	**0.040**	1.000	0.122	0.125	**0.122**	1.000
	5	1.5	0.3	0.043	0.038	**0.043**	1.000	0.137	0.127	**0.137**	1.000
			0.5	0.052	0.052	**0.052**	1.000	0.148	0.148	**0.148**	1.000
			0.8	0.043	0.033	**0.043**	1.000	0.120	0.132	**0.120**	1.000
	10	1.5	0.3	0.048	0.038	**0.048**	1.000	0.136	0.132	**0.136**	1.000
			0.5	0.051	0.051	**0.051**	1.000	0.144	0.144	**0.144**	1.000
			0.8	0.044	0.034	**0.044**	1.000	0.121	0.122	**0.121**	1.000
	5	3	0.3	0.046	0.040	**0.046**	1.000	0.137	0.134	**0.137**	1.000
			0.5	0.048	0.048	**0.048**	1.000	0.149	0.149	**0.149**	1.000
			0.8	0.041	0.029	**0.041**	1.000	0.131	0.126	**0.131**	1.000
500	2	0.5	0.3	0.045	0.044	**0.039**	0.942	0.973	0.991	**0.946**	0.973
			0.5	0.048	0.048	**0.043**	0.958	0.995	0.995	**0.987**	0.992
			0.8	0.050	0.038	**0.044**	0.927	0.874	0.996	**0.824**	0.946
	5	0.5	0.3	0.041	0.036	**0.041**	0.999	0.976	0.991	**0.976**	0.999
			0.5	0.050	0.050	**0.050**	0.999	0.996	0.996	**0.996**	1.000
			0.8	0.044	0.028	**0.044**	0.999	0.872	0.996	**0.872**	1.000
	2	1	0.3	0.049	0.041	**0.049**	0.999	0.970	0.990	**0.970**	1.000
			0.5	0.053	0.053	**0.053**	1.000	0.994	0.994	**0.994**	1.000
			0.8	0.048	0.029	**0.048**	0.998	0.880	0.994	**0.879**	0.999
	5	1	0.3	0.044	0.040	**0.044**	1.000	0.975	0.993	**0.975**	1.000
			0.5	0.048	0.048	**0.048**	1.000	0.994	0.994	**0.994**	1.000
			0.8	0.048	0.033	**0.048**	1.000	0.870	0.997	**0.870**	1.000

5	1.5	0.3	0.051	0.042	**0.051**	1.000	0.972	0.992	**0.972**	1.000
		0.5	0.045	0.045	**0.045**	1.000	0.995	0.995	**0.995**	1.000
		0.8	0.050	0.031	**0.050**	1.000	0.873	0.998	**0.873**	1.000
10	1.5	0.3	0.050	0.045	**0.050**	1.000	0.972	0.990	**0.972**	1.000
		0.5	0.048	0.048	**0.048**	1.000	0.995	0.995	**0.995**	1.000
		0.8	0.051	0.033	**0.051**	1.000	0.870	0.994	**0.870**	1.000
5	3	0.3	0.050	0.041	**0.050**	1.000	0.974	0.992	**0.974**	1.000
		0.5	0.048	0.048	**0.048**	1.000	0.994	0.994	**0.994**	1.000
		0.8	0.054	0.034	**0.054**	1.000	0.876	0.997	**0.876**	1.000

In Tables 4 and 5, we repeat the experiments of Table 3 to examine the size and power of the endogenous break unit root tests and compare them to the two-step test. In each case, we examine both the transformed and untransformed versions of the tests. We begin by examining the size properties in Table 4. As expected, neither the transformed or untransformed minimum nor maximum LM unit root test suffers from a general problem of over-rejections. However, both the transformed and untransformed tests have negative size distortions (i.e., under-rejections), which are more pronounced for the t-max test and remain as the sample size increases to $T = 500$. Following the discussion in Section 3, this outcome motivates our two-step procedure, where the two-step test utilizes the estimated break from the $\max F^*$ test. The two-step transformed test shows virtually no size distortions, and is invariant to both the size and location of the trend-break. This again demonstrates that the same critical values can be utilized to test for a unit root regardless of the size and/or location of the break. We next examine the power properties in Table 5. The results in Table 5 show that the two-step tests are more powerful than either the minimum ($\widetilde{\tau}^*_\tau$) or maximum ($\widetilde{\tau}^*_{|td|}$) endogenous break unit root tests in sample size T = 100. As in Table 4, the power of the F-test to identify the break is good and increases quickly to 1.0 as the magnitude of the break increases. Overall, the properties of the two-step test in the endogenous case are good and similar to those in the exogenous case.[14]

We next compare the accuracy of determining the break point using the transformed minimum, maximum, and two-step test. The results are displayed in Table 6. In each case, we report the relative frequencies of estimating the correct

[14] In unreported results, we also examined simulations to compare the properties of similar ADF-type two-step transformed tests. While the size properties of the ADF version of the two-step test are similar to the LM version, the power of the LM two-step test is greater in all cases. These results are available from the authors upon request.

Table 4: Size Properties of the Endogenous Break LM Unit Root Tests
(5% Rejection Rates)

DGP					*Transformed Tests*				*Untransformed Tests*			
[illegible]	T	d_1	d_2	λ	$\widetilde{\tau}^*_\tau$	$\widetilde{\tau}^*_{\lvert td \rvert}$	$\widetilde{\tau}^*_{\text{2-step}}$	F-*test*	$\widetilde{\tau}_\tau$	$\widetilde{\tau}_{\lvert td \rvert}$	$\widetilde{\tau}_{\text{2-step}}$	F-*test*
1	100	2	0.5	0.3	0.026	0.033	**0.049**	0.297	0.028	0.030	0.046	0.325
				0.5	0.025	0.030	**0.053**	0.359	0.034	0.032	0.050	0.408
				0.8	0.033	0.034	**0.049**	0.264	0.026	0.028	0.030	0.238
		5	0.5	0.3	0.038	0.021	**0.052**	0.814	0.040	0.023	0.045	0.829
				0.5	0.032	0.020	**0.051**	0.825	0.044	0.028	0.056	0.849
				0.8	0.059	0.021	**0.049**	0.829	0.047	0.022	0.042	0.824
		2	1	0.3	0.016	0.018	**0.044**	0.824	0.017	0.020	0.042	0.865
				0.5	0.021	0.019	**0.050**	0.893	0.026	0.027	0.054	0.922
				0.8	0.022	0.020	**0.053**	0.731	0.016	0.020	0.034	0.712
		5	1	0.3	0.023	0.018	**0.050**	0.975	0.028	0.024	0.047	0.989
				0.5	0.026	0.018	**0.050**	0.985	0.034	0.029	0.054	0.991
				0.8	0.077	0.025	**0.057**	0.986	0.026	0.018	0.033	0.983
		5	1.5	0.3	0.021	0.016	**0.045**	0.998	0.034	0.020	0.042	1.000
				0.5	0.028	0.020	**0.049**	1.000	0.044	0.026	0.056	1.000
				0.8	0.106	0.028	**0.062**	0.998	0.022	0.018	0.033	0.999
		10	1.5	0.3	0.021	0.016	**0.048**	1.000	0.047	0.019	0.036	1.000
				0.5	0.033	0.019	**0.051**	1.000	0.063	0.024	0.051	1.000
				0.8	0.224	0.015	**0.044**	1.000	0.048	0.015	0.032	1.000
		5	3	0.3	0.024	0.018	**0.046**	1.000	0.044	0.020	0.041	1.000
				0.5	0.039	0.018	**0.048**	1.000	0.050	0.023	0.046	1.000
				0.8	0.203	0.033	**0.066**	1.000	0.032	0.016	0.036	1.000
1	500	2	0.5	0.3	0.012	0.018	**0.045**	0.939	0.017	0.022	0.035	0.946
				0.5	0.013	0.017	**0.045**	0.970	0.023	0.026	0.045	0.969
				0.8	0.016	0.021	**0.050**	0.746	0.014	0.020	0.024	0.758
		5	0.5	0.3	0.027	0.023	**0.046**	0.816	0.036	0.030	0.047	0.849
				0.5	0.027	0.023	**0.052**	0.832	0.046	0.034	0.055	0.847
				0.8	0.035	0.019	**0.043**	0.821	0.039	0.024	0.035	0.829
		2	1	0.3	0.010	0.016	**0.041**	1.000	0.016	0.024	0.038	1.000
				0.5	0.011	0.017	**0.042**	1.000	0.019	0.023	0.044	1.000
				0.8	0.011	0.017	**0.042**	0.997	0.011	0.016	0.027	1.000
		5	1	0.3	0.014	0.016	**0.041**	1.000	0.015	0.019	0.033	1.000
				0.5	0.017	0.018	**0.044**	1.000	0.028	0.029	0.048	1.000
				0.8	0.027	0.022	**0.050**	1.000	0.015	0.019	0.031	1.000
		5	1.5	0.3	0.014	0.020	**0.046**	1.000	0.021	0.025	0.041	1.000
				0.5	0.014	0.016	**0.038**	1.000	0.023	0.024	0.040	1.000
				0.8	0.032	0.025	**0.055**	1.000	0.013	0.018	0.029	1.000
		10	1.5	0.3	0.023	0.020	**0.048**	1.000	0.037	0.024	0.038	1.000
				0.5	0.027	0.018	**0.045**	1.000	0.044	0.031	0.048	1.000
				0.8	0.134	0.024	**0.057**	1.000	0.023	0.018	0.029	1.000
		5	3	0.3	0.014	0.019	**0.047**	1.000	0.025	0.023	0.041	1.000

				0.5	0.015	0.018	**0.040**	1.000	0.024	0.028	0.047	1.000
				0.8	0.044	0.031	**0.072**	1.000	0.014	0.021	0.033	1.000

Table 5: Power Properties of the Endogenous Break LM Unit Root Tests

($\beta = 0.9$, 5% Rejection Rates)

DGP					*Transformed Tests*				*Untransformed Tests*			
R	T	d_1	d_2	λ	$\widetilde{\tau}^*_\tau$	$\widetilde{\tau}^*_{\lvert td\rvert}$	$\widetilde{\tau}^*_{2\text{-step}}$	F-*test*	$\widetilde{\tau}_\tau$	$\widetilde{\tau}_{\lvert td\rvert}$	$\widetilde{\tau}_{2\text{-step}}$	F-*test*
1	100	2	0.5	0.3	0.070	0.073	**0.089**	0.245	0.074	0.076	0.089	0.283
				0.5	0.071	0.072	**0.104**	0.320	0.096	0.092	0.115	0.380
				0.8	0.091	0.080	**0.094**	0.239	0.071	0.079	0.060	0.181
		5	0.5	0.3	0.087	0.066	**0.143**	0.760	0.086	0.078	0.145	0.796
				0.5	0.086	0.071	**0.140**	0.787	0.092	0.080	0.139	0.828
				0.8	0.132	0.066	**0.137**	0.821	0.104	0.076	0.123	0.800
		2	1	0.3	0.054	0.063	**0.136**	0.887	0.061	0.069	0.129	0.923
				0.5	0.056	0.067	**0.142**	0.968	0.071	0.081	0.138	0.981
				0.8	0.074	0.072	**0.148**	0.813	0.057	0.076	0.118	0.822
		5	1	0.3	0.067	0.060	**0.130**	0.980	0.085	0.075	0.141	0.993
				0.5	0.069	0.069	**0.148**	0.996	0.095	0.079	0.148	0.999
				0.8	0.167	0.077	**0.152**	0.993	0.085	0.076	0.132	0.995
		5	1.5	0.3	0.068	0.060	**0.132**	1.000	0.095	0.077	0.134	1.000
				0.5	0.085	0.065	**0.140**	1.000	0.112	0.093	0.155	1.000
				0.8	0.209	0.079	**0.157**	1.000	0.087	0.070	0.127	1.000
		10	1.5	0.3	0.045	0.054	**0.136**	1.000	0.100	0.080	0.146	1.000
				0.5	0.082	0.063	**0.144**	1.000	0.122	0.077	0.142	1.000
				0.8	0.326	0.049	**0.121**	1.000	0.145	0.071	0.123	1.000
		5	3	0.3	0.071	0.052	**0.126**	1.000	0.131	0.070	0.126	1.000
				0.5	0.104	0.065	**0.140**	1.000	0.147	0.084	0.142	1.000
				0.8	0.382	0.091	**0.173**	1.000	0.108	0.071	0.131	1.000
1	500	2	0.5	0.3	0.915	0.798	**0.882**	1.000	0.991	0.905	0.937	1.000
				0.5	0.984	0.869	**0.931**	1.000	0.995	0.897	0.926	1.000
				0.8	0.788	0.662	**0.764**	0.942	0.993	0.865	0.894	0.996
		5	0.5	0.3	0.933	0.893	**0.856**	0.881	0.996	0.961	0.897	0.907
				0.5	0.991	0.962	**0.913**	0.920	0.994	0.970	0.929	0.938
				0.8	0.872	0.769	**0.796**	0.899	0.997	0.985	0.880	0.884
		2	1	0.3	0.895	0.869	**0.939**	1.000	0.988	0.958	0.977	1.000
				0.5	0.976	0.925	**0.966**	1.000	0.988	0.953	0.972	1.000
				0.8	0.770	0.654	**0.775**	1.000	0.992	0.904	0.928	1.000
		5	1	0.3	0.907	0.890	**0.955**	1.000	0.996	0.950	0.969	1.000
				0.5	0.987	0.936	**0.972**	1.000	0.995	0.951	0.971	1.000
				0.8	0.859	0.719	**0.822**	1.000	0.995	0.975	0.983	1.000
		5	1.5	0.3	0.907	0.888	**0.949**	1.000	0.992	0.927	0.953	1.000

		0.5	0.987	0.940	**0.977**	1.000	0.994	0.957	0.975	1.000
		0.8	0.860	0.713	**0.828**	1.000	0.997	0.980	0.987	1.000
10	1.5	0.3	0.907	0.907	**0.964**	1.000	1.000	0.975	0.984	1.000
		0.5	0.993	0.962	**0.984**	1.000	0.997	0.975	0.986	1.000
		0.8	0.960	0.745	**0.861**	1.000	0.998	0.989	0.995	1.000
5	3	0.3	0.906	0.869	**0.947**	1.000	0.996	0.928	0.953	1.000
		0.5	0.985	0.948	**0.975**	1.000	0.993	0.961	0.979	1.000
		0.8	0.892	0.734	**0.841**	1.000	0.991	0.983	0.990	1.000

break location (T_B) and nearby locations within $T_B \pm 1$, $T_B \pm 2$, $T_B \pm 3$, and $T_B \pm 5$. Throughout, we assume that the break location parameter is given as 0.5, but other values of λdo not significantly affect the results. Note that frequencies of the two-step test can be under-evaluated when the break size is small or insignificant, since the no-break LM test would be suggested in such cases. It is clear that the t- max method can identify the break more accurately than the minimum t-test.

Table 6: Estimated Break Locations from the Transformed Endogenous LM Unit Root Tests

DGP				*Test*	Null ($\beta = 1.0$)					Alternative ($\beta = 0.9$)				
T	d_1	d_2	λ		T_B	± 1	± 2	± 3	± 5	T_B	± 1	± 2	± 3	± 5
100	2	0.5	0.5	$\tilde{\tau}^*_\tau$	0.041	0.151	0.211	0.265	0.363	0.059	0.193	0.271	0.346	0.464
				$\tilde{\tau}^*_{\lvert td \rvert}$	0.280	0.355	0.403	0.446	0.517	0.235	0.345	0.416	0.479	0.567
				2-step	0.142	0.179	0.202	0.217	0.244	0.116	0.161	0.187	0.207	0.235
	5	0.5	0.5	$\tilde{\tau}^*_\tau$	0.101	0.245	0.295	0.324	0.368	0.176	0.321	0.381	0.412	0.453
				$\tilde{\tau}^*_{\lvert td \rvert}$	0.960	0.962	0.963	0.965	0.966	0.958	0.961	0.963	0.965	0.968
				2-step	0.817	0.817	0.818	0.818	0.819	0.780	0.781	0.781	0.781	0.782
	2	1	0.5	$\tilde{\tau}^*_\tau$	0.052	0.270	0.393	0.501	0.657	0.081	0.326	0.464	0.578	0.761
				$\tilde{\tau}^*_{\lvert td \rvert}$	0.360	0.512	0.590	0.643	0.730	0.308	0.556	0.673	0.749	0.843
				2-step	0.343	0.483	0.557	0.605	0.684	0.305	0.550	0.665	0.738	0.828
	5	1	0.5	$\tilde{\tau}^*_\tau$	0.066	0.308	0.402	0.484	0.632	0.114	0.341	0.437	0.523	0.685
				$\tilde{\tau}^*_{\lvert td \rvert}$	0.928	0.956	0.966	0.970	0.977	0.921	0.961	0.978	0.986	0.991
				2-step	0.919	0.947	0.957	0.960	0.967	0.919	0.958	0.975	0.984	0.988
	5	1.5	0.5	$\tilde{\tau}^*_\tau$	0.039	0.290	0.409	0.535	0.769	0.069	0.289	0.419	0.565	0.817
				$\tilde{\tau}^*_{\lvert td \rvert}$	0.883	0.949	0.967	0.971	0.980	0.868	0.960	0.985	0.990	0.995
				2-step	0.883	0.949	0.967	0.971	0.980	0.868	0.960	0.985	0.990	0.995
	10	1.5	0.5	$\tilde{\tau}^*_\tau$	0.076	0.226	0.280	0.365	0.605	0.133	0.238	0.282	0.368	0.635
				$\tilde{\tau}^*_{\lvert td \rvert}$	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000
				2-step	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000	1.000

This result confirms the findings of Perron (1997) and Nunes (2004) for the Dickey-Fuller and LM type tests, respectively. Also, note that performance of the two-step test to identify the break is almost identical to that of the t-max test. As the magnitude of the break increases, the frequency of correctly estimating the break point approaches 1.0 in the t-max and two-step test, the same as in the exogenous case where the break point is known a priori.

5 Concluding Remarks

In this paper, we propose new two-step LM unit root tests that are invariant to level and trend-breaks. In contrast to existing tests, the suggested two-step tests using transformed data have the invariance property that the asymptotic distribution is free of the nuisance parameter even in the presence of trend-breaks. The proposed tests depend only on the number of structural breaks, while they are invariant to the size and location of the breaks. In the first step, we determine whether and where the break(s) exist by using a maximum F-test. This differs from the endogenous break unit root tests that implicitly assume or impose a specific number of breaks. In the second step, we utilize the information from the first step and test the null hypothesis of a unit root. The two-step unit root test can be conveniently applied to allow for multiple breaks, and remains free of nuisance parameters whether the series is stationary or not. Overall, the two-step test accurately identifies the number and location of break(s) and has better properties of size and power than the endogenous break unit root tests.

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Appendix

Proofs of Proposition 1 and Theorem 1 are provided in Im, Lee and Tieslau (2011). Here we prove Propositions 2 and 3 using the results in Proposition 1.

Proof of Proposition 2

We want to show the effect of mis-specifying the model under the null. Specifically, we examine the effect of using a "crash model" with level-shifts when the DGP implies the model with trend-breaks. We consider under the null of a unit root:

DGP: $$\Delta y_t = \delta_2 + \sum_{j=1}^{R} \delta_{3j} B_{jt} + \sum_{j=1}^{R} \delta_{4j} D_{jt} + \epsilon_t,$$

Estimation: $$\Delta y_t = \delta_2 + \sum_{j=1}^{R} \delta_{3j} B_{jt} + u_t.$$

Notice that including the one point dummy variables has the effect of "dummying-out" the corresponding observation. After omitting the corresponding observations, we get:

$$\begin{aligned}
\widetilde{\delta}_2 &= \frac{1}{T-1}\sum_{t=2}^{T} \Delta y_t - \frac{1}{T-1}\sum_{i=1}^{R} \Delta y_{T_{Bj}+1} \\
&= \frac{1}{T-1}\sum_{t=2}^{T}\left[\delta_2 + \sum_{j=1}^{R}\delta_{3j}B_{jt} + \sum_{j=1}^{R}\delta_{4j}D_{jt} + \epsilon_t\right] \\
&\quad - \frac{1}{T-1}\sum_{t=2}^{T}\left[\delta_2 + \sum_{j=1}^{R}\delta_{3j}B_{T_{Bj}+1} + \sum_{j=1}^{R}\delta_{4j}D_{T_{Bj}+1} + \epsilon_{T_B+1}\right] \qquad \text{(A.1)} \\
&= \delta_2 + \frac{1}{T-1}\sum_{j=1}^{R}\delta_{3j} + \frac{1}{T-1}\sum_{j=1}^{R}\delta_{4j}(T - T_{Bj}) + \bar{\epsilon} \\
&\quad - \frac{1}{T-1}\left[\delta_2 + \sum_{j=1}^{R}\delta_{3j} + \sum_{j=1}^{R}\delta_{4j} + \epsilon_{T_B+1}\right],
\end{aligned}$$

and

$$
\begin{aligned}
\widetilde{\delta}_{3j} &= \Delta y_{T_{Bj}+1} - \widetilde{\delta}_2 \\
&= [\delta_2 + \delta_{3j} + \delta_{4j} + \epsilon_{T_B+1}] - \widetilde{\delta}_2 \\
&= \delta_2 + \delta_{3j} B_{T_{Bj}+1} + \delta_{4j} D_{T_{Bj}+1} + \epsilon_{T_B+1} - \delta_2 - \frac{1}{T-1}\delta_{3j} - \frac{1}{T-1}\delta_{4j}(T - T_{Bj}) \\
&\quad - \bar{\epsilon} + \frac{1}{T-1}[\delta_2 + \delta_{3j} + \delta_{4j} + \epsilon_{T_B+1}] \\
&= \delta_{3j} + \delta_{4j}\left[1 - \frac{1}{T-1}(T - T_{Bj})\right] - \bar{\epsilon} + \frac{1}{T-1}\left[\delta_2 + \delta_{3j} + \delta_{4j} + \epsilon_{T_B+1}\right].
\end{aligned} \tag{A.2}
$$

Then, from $\widetilde{S}_t = y_t - y_1 - \widetilde{d}_2(t-1) - \sum_{j=1}^{R}\left[\widetilde{d}_{3j} D_{jt} + \widetilde{d}_{4j} DT_{jt}\right]$, we can show that

$$
\begin{aligned}
T^{-1/2}\widetilde{S}_{[rT]} &= T^{-1/2} S_{[rT]} - T^{-1/2}(\widetilde{\delta}_2 - \delta_2)(t-1) - T^{-1/2}\sum_{j=1}^{R}(\widetilde{\delta}_{3j} - \delta_{3j}) D_{tj} \\
&\qquad + T^{-1/2}\sum_{j=1}^{R}\delta_{4j} DT_{tj}.
\end{aligned} \tag{A.3}
$$

The first term in (A.3) follows $\sigma W(r)$, which is a standard result. The second term in (A.3) follows:

$$
\begin{aligned}
T^{-1/2}(\widetilde{\delta}_2 - \delta_2)(t-1) &= ([rT]-1)T^{-1}T^{1/2}(\widetilde{\delta}_2 - \delta_2) \\
&= ([rT]-1)T^{-1}T^{1/2}\left[\frac{1}{T-1}\sum_{j=1}^{R}\delta_{4j}(T - T_{Bj}) + \bar{\epsilon} - \frac{1}{T-1}(\delta_2 + \sum_{j=1}^{R}\delta_{4j} + \epsilon_{T_B+1})\right]
\end{aligned}
$$

Since $\frac{1}{T-1}(T - T_{Bj}) \to \lambda_j^*$, we note that the first term of the last expression diverges; it is $O_p(\sqrt{T})$. Then,

$$
T^{-1}(\widetilde{\delta}_2 - \delta_2)(t-1) \to \sigma \sum_{j=1}^{R}\delta_{4j}\lambda_j^*. \tag{A.4}
$$

It can be shown that the third term in (A.3) is $o_p(1)$. However, the last term in (A.3) is $O_p(\sqrt{T})$. This term diverges as T increases, unless $\delta_{4j} = 0$. Then, since $\sum_{j=1}^{R}\delta_{4j} DT_{jt} = \sum_{j=1}^{R}\delta_{4j}([rT] - T_{Bj})$, we obtain

$$
T^{-1}\sum_{j=1}^{R}\delta_{4j} DT_{jt} \to \sigma r \sum_{j=1}^{R}\delta_{4j}. \tag{A.5}
$$

Therefore, combining the results of (A.4) and (A.5), we have:

$$T^{-1}\widetilde{S}_{[rT]} \to \sigma\left[\sum_{j=1}^{R}\delta_{4j}\lambda_j^* + r\sum_{j=1}^{R}\delta_{4j}\right]. \tag{A.6}$$

Then, we have the result: $T^{-3}\widetilde{S}_1' M_{\Delta Z}\widetilde{S}_1 = O_p(1)$. We can see that not allowing for trend-breaks will cause a bias in $\widetilde{\tau}_A$ toward not rejecting the null hypothesis. This will result in downward size distortions, since:

$$T\widetilde{\phi} = O_p(T^{-1}) \; and \; \widetilde{\tau}_A = O_p(T^{-1/2}).$$

This completes the proof. □

Proof of Proposition 3

We now obtain the asymptotic distribution of the F-test in (13) that is based on the testing regression for the LM type statistic. First, we examine SSR_0, which is obtained from the (restricted) regression:

$$\Delta y_t = \widehat{\phi}\widehat{S}_{t-1} + \widehat{d}_0 + \widehat{u}_t,$$

where $\widehat{S}_{t-1} = (y_t - y_1) - \widehat{\gamma}(t-1)$ and $\widehat{\gamma} = (1/T)\sum_{t=1}^{T}\Delta y_t$, and where $\widehat{\phi}$ and $\widehat{d}_0$ are the OLS estimates in this regression. Then, when the unit root assumption is imposed, we get:

$$SSR_0 = \sum_{t=2}^{T}\widehat{u}_t^2 = \sum_{t=2}^{T}(\epsilon_t - \widehat{d}_0 - \widehat{\phi}\widehat{S}_{t-1})^2.$$

This can be expressed as

$$\begin{aligned}&\sum_{t=2}^{T}[(\epsilon_t - \overline{\epsilon}) - \widehat{\phi}(\widehat{S}_{t-1} - \overline{S}_1)]^2 \\ &= \sum_{t=2}^{T}(\epsilon_t - \overline{\epsilon})^2 + \widehat{\phi}^2\sum_{t=2}^{T}(\widehat{S}_{t-1} - \overline{S}_1)^2 - 2\widehat{\phi}\sum_{t=2}^{T}(\widehat{S}_{t-1} - \overline{S}_1)(\epsilon_t - \epsilon),\end{aligned} \tag{A.7}$$

where $\overline{\epsilon} = \frac{1}{T}\sum_{t=1}^{T}\epsilon_t$, and $\overline{S}_1 = \frac{1}{T-1}\sum_{t=2}^{T}\widehat{S}_{t-1}$. The first term in (A.7) can be cancelled with the same term that appears in SSR_1. The second term in the above expression can be written as

$$\begin{aligned}&(T\widehat{\phi})^2 T^{-2}\sum_{t=2}^{T}(\widehat{S}_{t-1} - \overline{S}_1)^2 \to [-\frac{1}{2}(\sigma_\epsilon^2/\sigma^2)(\int_0^1 \underline{V}_0(r)^2 dr)^{-1}]^2(\sigma^2\int_0^1 \underline{V}_0(r)^2 dr) \\ &= \frac{1}{4}(\sigma_\epsilon^4/\sigma^2)(\int_0^1 \underline{V}_0(r)^2 dr)^{-1},\end{aligned} \tag{A.8}$$

where $\underline{V}_0(r)$ is the demeaned Brownian bridge. The third term in (A.8) is shown to follow

$$- 2(T\widehat{\phi})(1/T)\sum_{t=2}^{T}(\widehat{S}_{t-1} - \overline{S}_1)(\epsilon_t - \overline{\epsilon}) \to -2\left[-\frac{1}{2}(\sigma_\epsilon^2/\sigma^2)(\int_0^1 \underline{V}_0(r)^2 dr)^{-1}\right](-\frac{1}{2}\sigma_\epsilon^2)$$
$$= -\frac{1}{2}(\sigma_\epsilon^4/\sigma^2)(\int_0^1 \underline{V}_0(r)^2 dr)^{-1}. \tag{A.9}$$

Next, SSR_1 is similarly obtained from the unrestricted regression (7) or (11). When $R = 1$, for simplicity, we have

$$\Delta y_t = \widetilde{\phi}\widetilde{S}_{t-1} + \widetilde{d}_0 + \widetilde{d}_1 \Delta D_t + \widetilde{d}_2 \Delta DT_t^* + \widetilde{u}_t,$$

where $\widetilde{S}_{t-1}$ is defined in (5), and where $\widetilde{d}_0, \widetilde{d}_1$, and $\widetilde{d}_2$ are the OLS estimates in regression (7) or (11). Then, under the null of a unit root:

$$SSR_1 = \sum_{t=2}^{T}\widetilde{u}_t^2 = \sum_{t=2}^{T}(\epsilon_t - \widetilde{d}_0 - \widetilde{d}_1\Delta D_t + \widetilde{d}_2\Delta DT_t^* - \widetilde{\phi}\widetilde{S}_{t-1})^2.$$

This can be expressed as

$$\sum_{t=2}^{T}(\underline{\epsilon}_t - \widetilde{\phi}\underline{\widetilde{S}}_{t-1})^2$$
$$= \sum_{t=2}^{T}\underline{\epsilon}_t^2 + \widetilde{\phi}^2\sum_{t=2}^{T}\underline{\widetilde{S}}_{t-1}^2 - 2\widetilde{\phi}\sum_{t=2}^{T}\underline{\widetilde{S}}_{t-1}\underline{\epsilon}_t, \tag{A.10}$$

where $\underline{\epsilon}_t$ is the element of $\underline{\epsilon} = M_{\Delta Z}\epsilon$ with $M_{\Delta Z} = I - \Delta Z(\Delta Z'\Delta Z)^{-1}\Delta Z'$; and where $\underline{\widetilde{S}}_{t-1}$ is the element of $\underline{\widetilde{S}}_1 = M_{\Delta Z}\widetilde{S}_1$ with $\widetilde{S}_1 = (\widetilde{S}_1, ..., \widetilde{S}_{T-1})'$. The first term in (A.10) can be cancelled with the similar term in SSR_0 under the null in the absence of non-linear terms. The second and third terms in (A.10) follow the same asymptotic distributions as in (A.8) and (A.9), except that $\underline{V}_0(r)$ is replaced with $\underline{V}(r)$ as defined in Proposition 1. Finally, the denominator of the F-statistic is given by

$$\frac{1}{T-q}\sum_{t=2}^{T}\widetilde{u}_t^2 + \frac{1}{T-q}O_p(1) \to \sigma_\epsilon^2, \tag{A.11}$$

where the $O_p(1)$ terms of the above expression are the same as in (A.10). The asymptotic distribution of the F-statistic is given by collecting terms in (A.8) through (A.11) using multiple breaks

$$F(k) \to \frac{1}{8}(\sigma_\epsilon^2/\sigma^2)\left[(\sum_{i=1}^{R+1}\int_0^1 \underline{V}_i(r)^2 dr)^{-1} - (\int_0^1 \underline{V}_0(r)^2 dr)^{-1}\right]. \square$$