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Mean-Variance-Skewness-Kurtosis Portfolio Optimization with Return and Liquidity

Xiaoxin W. Beardsley[1], Brian Field[2] and Mingqing Xiao[3]

Abstract

In this paper, we extend Markowitz Portfolio Theory by incorporating the mean, variance, skewness, and kurtosis of both return and liquidity into an investor's objective function. Recent studies reveal that in addition to return, liquidity is also a concern for the investor, and is best captured by not being internalized as a premium within the expected return level, but rather, as a separate factor with each corresponding moment built into the investor's utility function. We show that the addition of the first four moments of liquidity necessitates significant adjustment in optimal portfolio allocations from a mathematical point of view. Our results also affirm the notion that higher-order moments of return can significantly change optimal portfolio construction.

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[1] Department of Finance, Southern Illinois University, Carbondale, IL 62901, e-mail: xwang@business.siu.edu.
[2] B.H. Field Consulting at 4425 Moratock Lane, Clemmons, NC 27012, e-mail: brianhersterfield@gmail.com
[3] Department of Mathematics, Southern Illinois University, Carbondale, IL 62901, e-mail: mxiao@math.siu.edu

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1 Introduction

Since Harry Markowitz's 1952 seminal work "Portfolio Selection", techniques attempting to optimize portfolios have been ubiquitous in financial industry. Traditionally, risk-averse investors have considered only the first two moments of a portfolio return's distribution, namely, the mean and the variance, as measures of the portfolio's reward and risk, respectively. Subsequently, theoretical extensions aimed at addressing complexities associated with higher-order moments of return, particularly, the third and fourth moments (i.e., skewness and kurtosis), have been paid attention by some researchers (see for example, Kane (1982), Barone-Adesi (1985), Lai (1991) and Athayde and Flores (2004)). Still, specific analytical generalizations of the return skewness and kurtosis calculation have appeared only recently.

In addition to the higher moments of return, the first moment of liquidity, i.e., the level of liquidity, has been shown to affect expected return, and the second moment of liquidity, namely, liquidity co-movement, has been shown to exist across securities[4]. Even asymmetry in liquidity co-movement, that is, liquidity's skewness (third moment), has been documented by various papers, such as Chordia, Sarkar and Subrahmanyam (2005), Kempf and Mayston (2005) and

[4] See for example, Amihud and Mendelson (1986) and Brennan and Subrahmanyam (1996) for the effect of the level of liquidity on expected return. The study on liquidity co-movement is ample, see for example, Hasbrouck and Seppi (2001), Hulka and Huberman (2001), Amihud (2002), Pastor and Stambaugh (2003), Brockman, Chung, and Perignon (2006), Karolyi, Lee and Dijk (2007), and Chordia, Roll and Subrahmanyam (2008).

Hameed, Kang, and Viswanathan (2006). Since liquidity measures an investor's ability to realize a particular return, proper portfolio construction cannot be achieved without due consideration of liquidity level, liquidity commonality, liquidity skewness and even higher moments in liquidity. An investor's objective is to achieve the expected level of return with minimized risk, and to achieve this goal, the investor must trade, and to trade, (il)liquidity and its cross-security interactions naturally become a concern and cannot be ignored.

Unlike previous research that internalizes the level of (il)liquidity as a premium for expected return, we single out liquidity as a separate concern for an investor's utility function. We believe that adding liquidity to an investor's utility function as a separate consideration is more appropriate than internalizing liquidity into return premium. Though internalizing the first moment of liquidity (liquidity level) as a premium to expected return is feasible, internalizing the subsequent higher moments of liquidity may result in the loss of some important mathematical characteristics for portfolio optimization. After all, sorting out each additional return premium due to the addition of a certain moment of (il)liquidity can be a quite demanding task, while if we list each liquidity moment out in the utility function, just like the way return moments are listed, the effect from each moment on the optimal portfolio can be observed more transparently and examined more directly. The consideration for the incorporation of higher moments of return and the inclusion of moments of liquidity into portfolio optimization is necessary, not only due to the skewed nature of return distributions and the sole claim that liquidity simply matters, but rather, for more practical reasons, particularly after witnessing the financial market turmoil in 2008. This crisis, like many other crises in history, had a liquidity crisis embedded within. It was not a simple lack of liquidity in some securities, but more of a systemic liquidity crunch across the board that choked the entire market, and affected countless portfolios held by investors. Therefore, the theoretical extension to portfolio theory and its potential practical application in the industry warrants a study that incorporates moments of liquidity, not simply the level of liquidity.

This paper extends classical modern portfolio theory by including higher moments of return as well as, and perhaps more importantly, moments of liquidity. We first extend the Markowitz model theoretically by adding the 3rd and 4th moments of return and the 1st, 2nd, 3rd and 4th moments of liquidity into an investor's utility function, respectively. Thus, using first and second-order optimality conditions, we identify an optimal portfolio incorporating the portfolio's mean, variance, skewness, and kurtosis with respect to both its return and liquidity. We demonstrate the changes in portfolio allocations with respect to a two-asset portfolio as well as a three-asset portfolio. Then, using daily data on 50 pairs of S&P500 stocks in the first half of 2010, we find that not only do higher moments of return significantly change optimal portfolio construction, the addition of the first four moments of liquidity necessitates a further adjustment in portfolio allocations. Additional cross-sectional analysis shows that among the moments added, liquidity's mean, skewness and kurtosis have the most significant impact on allocation change. These findings illustrate the empirical importance of our theoretical extension to the Markowitz model. In this paper, we show that an optimal allocation can change dramatically when higher moments of return and moments of liquidity are included in an investor's utility function.

The rest of this paper is organized as follows. Section 2 reviews current literature and then extends it by discussing the importance of higher moments of return and moments of liquidity in portfolio construction. Section 3 theoretically extends the Markowitz optimization problem by including the higher return moments as well as the first four liquidity moments. Section 4 commences our empirical investigation with respect to a two-asset portfolio and later extends it to a three-asset portfolio. Section 5 provides cross-sectional analysis on the factors contributing to the importance of higher moments. Section 6 conducts a robustness check and sensitivity analysis with alternative preference parameters in the model. Section 7 offers conclusions, identifies limitations of the paper and suggests areas for future research. In addition, the theoretical derivation of the solution to the extended optimization problem shown in Section 3 is presented in the appendix of the paper.

2 Motivation and Extension to the Current Literature

2.1 The Lack of Higher Moments in Classic Markowitz Portfolio Theory

Reilly and Brown (2000) and Engels (2004) provide a thorough summary of Modern Portfolio Theory. The Markowitz model assumes a quadratic utility function, or normally-distributed returns (with zero skewness and kurtosis) where only the portfolio's expected return and variance need to be considered, that is, the higher-ordered terms of the Taylor series expansion of the utility function in terms of moments are set to be zero. Empirical evidences on return distributions have demonstrated abnormal distributions of return.[5] When the investment decision is restricted to a finite time interval, Samuelson (1970) shows that the mean-variance efficiency becomes inadequate and that the higher-order moments of return become relevant. In addition, Scott and Horvath (1980) shows that if (i) the distribution of returns for a portfolio is asymmetric, or (ii) the investor's utility is of higher-order than quadratic, then at the very least, the third and fourth moments of return must be considered.

[5] For example, Arditti (1971), Fielitz (1974), Simkowitz and Beedles (1978), and Singleton and Wingender (1986) all show that stock returns are often positively skewed. Later studies by Gibbons, Ross, and Shanken (1989),Ball and Kothari (1989), Schwert (1989), Conrad, Gultekin and Kaul (1991), Cho and Engle (2000) and Kekaert and Wu (2000) further document asymmetries in return covariances.

2.2 Incorporating Skewness and Kurtosis of Return

To date, many studies have examined the efficacy of non-normal returns and/or higher-order moments of return, and have deemed them not only important, but of critical importance in optimal portfolio construction. Earlier work such as Simonson (1972), Kane (1982), Barone-Adesi (1985), Lai (1991), and Konno, Shirakawa and Yamazaki (1993) discusses the inclusion of the third moment in portfolio optimization and indicate that improved mean-variance portfolio efficiency can be achieved by including skewness of return. Moreover, the return distribution's fourth moment, namely, kurtosis, notwithstanding the disproportionate attention pointed at skewness in the literature, has recently received increased attention. The, perhaps (mis-)appropriated attention to skewness, is due to the relatively slower development of techniques in dealing with the algebraic challenges associated with the kurtosis calculation. Recent work by Castellacci and Siclari (2003), Malevergne and Sornette (2005), Jarrow and Zhao (2006), Hong, Tu and Zhou (2007), Mitton and Vorkink (2007), Guidolin and Timmermann (2008), Martellini (2008), Wilcox and Fabozzi (2009), and Li, Qin and Kar (2010) all confirm the importance of higher moments of returns in portfolio construction, with the exception of Cremers, Kritzman and Page (2005) that documents that much of the non-normality in returns survives into the mean-variance efficient portfolios if log wealth utility is used.

In this paper, we first show the changes in optimal portfolio allocation by establishing and studying the utility function with a consideration of return skewness and kurtosis, and subsequently examine allocation changes resulting from the addition of liquidity moments.

2.3 Incorporating Liquidity---Mean, Variance, Skewness and Kurtosis

2.3.1. Liquidity Level---the First Moment of Liquidity

Most discussions on portfolio theory assume a frictionless world, and neglect the potential difficulty to actually realize returns, i.e., transaction costs (illiquidity). However many studies have documented the relationship between transaction cost and expected return. Brennan and Subrahmanyam (1996) find a significant return premium associated with both the fixed and variable costs of transacting. Amihud and Mendelson (1986) document that investors require higher returns on stocks with higher bid-ask spreads as a compensation for illiquidity. More recently, Pastor and Stambaugh (2003) find that market-wide liquidity is a state variable important for asset pricing. Liu (2006) documents a significant liquidity premium robust to the CAPM model. Steuer and Qi (2007), Garleanu (2009) find that portfolio choice depends significantly and naturally on liquidity. Huang (2009) shows that illiquidity can, theoretically, significantly affect asset returns. Applied finance research work, for example, by Borkovec, Domowitz, Kiernan, and Serbin (2010) also shows that cost-aware portfolio construction leads to different investment decisions. These theoretical and empirical studies all confirm that the mean liquidity level, or the first moment of liquidity can affect returns.

2.3.2. Liquidity Commonality---the Second Moment of Liquidity

Recent research on liquidity has also focused increasingly on the basic interactions among securities in liquidity. Empirical papers, including Hasbrouck and Seppi (2001), Hulka and Huberman (2001), Amihud (2002), Pastor and Stambaugh (2003), Brockman, Chung, and Perignon (2006), Karolyi, Lee and Dijk (2007) and Chordia, Roll and Subrahmanyam (2008) have documented the existence of cross-stock liquidity commonality. While Fabre and Frino (2004) and Sujoto, Kalev and Faff (2005) find liquidity commonality is more apparent among large stocks, Chordia, Roll and Subrahmanyam (2008) and Brockman, Chung and Perignon (2006) do not find the same pattern under their framework. But in any case, liquidity commonality seems hardly to be neglected entirely in portfolio construction.

If liquidity commonality exists, will it have any implication on asset pricing? Fernando, Herring and Subrahmanyam (2006) develop a theoretical model to explain how liquidity commonality can affect market performance. Korajczyk and Sadka (2008) show that across-measure systemic liquidity is a priced factor while within-measure systemic liquidity does not exhibit additional pricing information. However, neither of these papers builds liquidity, and its different moments, into a portfolio construction framework, which is one goal of our paper.

2.3.3. Asymmetry in Liquidity Commonality---the Third Moment in Liquidity

On top of liquidity co-movement, what if there is an asymmetry in this co-movement? Domowitz, Hansch and Wang (2005) highlight the downside risk of liquidity commonality through its asymmetry. Common liquidity deterioration usually happens more often than systemic improvement in liquidity, particularly during down markets. This asymmetry poses a new threat to efficient diversification, especially after what happened to the market during the 1998 LTCM meltdown, after Lehman Brothers filed for bankruptcy in September 2008, and the flash crash in 2010. Acharya and Pedersen (2005) find that when aggregate market liquidity falls, it falls primarily for illiquid assets - a notion often termed “flight to liquidity”. Chordia, Sarkar and Subrahmanyam (2005) show that during crisis periods, effects from liquidity commonality are maximized. Kempf and Mayston (2005) also observe that liquidity commonality is stronger in falling markets than in rising markets. Hameed, Kang, and Viswanathan (2006) document that liquidity levels and co-movement in liquidity are, indeed, higher during large negative market moves because of aggregate collateral deterioration of financial intermediaries and the concurrent forced liquidation of many asset holders, making it difficult to provide liquidity precisely when the market most demands it. They also show that the cost of supplying liquidity is highest following market downturns. All evidence here suggests that the third moment of liquidity is also embedded within the endogeneity between return and liquidity, and cannot be ignored in (optimal) portfolio construction.

2.3.4. Incorporating Moments of Liquidity into Optimal Portfolio Construction

We conjecture that, not only does the first moment of liquidity matter in portfolio optimization, so do liquidity's higher moments, just as the higher moments of return do. A traditional diversified portfolio aims to have securities with little return interactions, but their liquidity may co-move. Not only can liquidity level affect expected return level like many pervious papers have documented, but the co-movement of liquidity may not follow exactly the same pattern as the co-movement of return. Hence, although appropriately diversified in terms of return, when one security's liquidity diminishes, other securities, in turn, may experience similar deterioration, especially during periods of systemic liquidity draught. Thus, cross-sectional systematic illiquidity poses a great challenge to traders who try to enforce the diversification: the diversification benefit can evaporate significantly. This difficulty of actually realizing the benefit of diversification posed by co-movements in liquidity is thus a separate risk that needs to be minimized together with the co-movement risk of return. In addition, the asymmetry in liquidity co-movements adds a further layer of complexity to the issue. Positive skewness in liquidity, just like positive skewness in return, is desirable for investors as it represents increased systematic liquidity overweighing decreased systematic liquidity. On the contrary, negative skewness in liquidity implies bigger deterioration of systemic liquidity than its improvement, and is thus to be avoided by rational risk-averse investors. Kurtosis in liquidity, i.e., the fat tails in liquidity, represents severe outliers and variation in liquidity and, just like the fat tails in return distribution, high kurtosis in liquidity is undesirable to investors as well.

3 Main Results in Theoretical Extension

3.1 Introduction to Notation

Before proceeding, an introduction to notation is in order. Given a portfolio of *n* assets, we identify the *n*-dimensional random vector of returns. The associated return moments can be considered tensors, i.e., geometrical entities used to extend the notion of scalars. Utilizing the methodology introduced by Athayde and Flores (1997), we can identify the returns' covariance matrix as the second moment's tensor, the returns' co-skewness matrix as the third moment's tensor, and the returns' co-kurtosis matrix as the fourth moment's tensor. With the *n*-asset portfolio, these tensors can be visualized, respectively, as an (nxn) matrix (covariance), an (nxnxn) cube (co-skewness), and an (nxnxnxn) four-dimensional cube (co-kurtosis). We can then transform the respective cubes into matrices, suggesting and allowing for the application of matrix differential calculus techniques. For example, the (nxnxn) co-skewness cube is transformed into an (nxn^2) matrix and the (nxnxnxn) co-kurtosis cube is transformed into an (nxn^3) matrix. (Note that the (nx1) return vector is already in the desired matrix form.) Upon transforming the moments into tractable matrices, an important characteristic of the resulting matrices stands out, i.e., the repetition of matrix elements. In the case of a 2-asset portfolio, the co-skewness matrix will be a (2x2^2) matrix, or a (2x4) matrix, as illustrated below:

$$\begin{bmatrix} \sigma_{111} & \sigma_{112} & \sigma_{211} & \sigma_{212} \\ \sigma_{121} & \sigma_{122} & \sigma_{221} & \sigma_{222} \end{bmatrix} \tag{1}$$

In fact, there are only four distinct elements in the above co-skewness matrix, as follows:

$$\sigma_{111},$$
$$\sigma_{112}=\sigma_{121}=\sigma_{211},$$
$$\sigma_{122}=\sigma_{212}=\sigma_{221},$$
$$\sigma_{222}. \tag{2}$$

Again, in the case of a 2-asset portfolio, the co-kurtosis matrix will be a ($2\text{x}2^3$) matrix, or a (2x8) matrix as illustrated below:

$$\begin{bmatrix}\sigma_{1111} & \sigma_{1112} & \sigma_{1121} & \sigma_{1122} & \sigma_{1211} & \sigma_{1212} & \sigma_{1221} & \sigma_{1222} \\ \sigma_{2111} & \sigma_{2112} & \sigma_{2121} & \sigma_{2122} & \sigma_{2211} & \sigma_{2212} & \sigma_{2221} & \sigma_{2222}\end{bmatrix} \tag{3}$$

in which there are only five distinct elements, as follows:

$$\sigma_{1111},$$
$$\sigma_{1112}=\sigma_{1121}=\sigma_{1211}=\sigma_{2111},$$
$$\sigma_{1122}=\sigma_{1212}=\sigma_{1221}=\sigma_{2112}=\sigma_{2211} \quad ,$$
$$\sigma_{1222}=\sigma_{2122}=\sigma_{2212}=\sigma_{2221},$$
$$\sigma_{2222}. \tag{4}$$

Given n risky assets, we must consider an n-dimensional return vector, which is associated with an n x n square co-variance matrix, of which $n(n+1)/2$ elements are distinct. Similarly, the associated skewness tensor matrix will consist of $n(n+1)(n+2)/6$ distinct elements. This repetition in the moments' tensors matrices continues as the order increases. For example, the p-th moment of a 2-asset return vector can be represented with a ($2\text{x}2^{p-1}$) matrix, consisting of 2^p elements, of which significantly less than 2^p are distinct. Knowledge of the repetition of the matrix entries can be tremendously helpful when considering the "curse of dimensionality". For example, the co-kurtosis matrix for a common portfolio of 500 assets, like the S&P 500, would consist of 62,500,000,000 entries; the S&P 500 co-kurtosis matrix would be a (500 x 500^4) matrix, or a (500 x 62,500,000,000) matrix. Even though this matrix holds a majority of repeated values, it still illustrates the intractability often inherent in larger portfolio analyses.

3.2 The Extended Framework for Optimal Portfolio Construction---Adding Higher Moments of Return and Moments of Liquidity

First, we include the higher-order moments of return by extending the traditional Markowitz quadratic utility $U^R_{MV} = \gamma_1 * Mean^R - \gamma_2 * Variance^R$ to a fourth-degree utility function with return, $U^R_{MVSK} = \gamma_1 * Mean^R - \gamma_2 * Variance^R + \gamma_3 * Skewness^R - \gamma_4 * Kurtosis^R$. Then, we incorporate liquidity and its higher moments, by further extending a risk-averse investor's utility function to the following, namely,:

$$U^{RL}_{MVSK} = \lambda_R[\gamma_1 * Mean^R - \gamma_2 * Variance^R + \gamma_3 * Skewness^R - \gamma_4 * Kurtosis^R]$$
$$+\lambda_L[\gamma_1 * Mean^L - \gamma_2 * Variance^L + \gamma_3 * Skewness^L - \gamma_4 * Kurtosis^L].$$

Utilizing matrix notation, the investor's objective is to:

Maximize

$$U^{RL}_{MVSK} = \lambda_R[\gamma_1\alpha^T M_1 - \gamma_2\alpha^T M_2\alpha + \gamma_3\alpha^T M_3(\alpha\otimes\alpha) - \gamma_4\alpha^T M_4(\alpha\otimes\alpha\otimes\alpha)]$$
$$+\lambda_L[\sigma_1\alpha^T L_1 - \sigma_2\alpha^T L_2\alpha + \sigma_3\alpha^T L_3(\alpha\otimes\alpha) - \sigma_4\alpha^T L_4(\alpha\otimes\alpha\otimes\alpha)]$$

subject to: $\bar{1}\alpha = 1$ (5)

Where M_i (i=1,2,3,4) stands for the i^{th} moment matrix of return, while L_i(i=1, 2, 3, 4) stands for the i^{th} moment matrix of liquidity.[6] $M_1(L_1)$ represents the vector, $M_2(L_2)$ represents the covariance matrix, M_3 (L_3) represents the co-skewness matrix, and $M_4(L_4)$ represents the co-kurtosis matrix, for return and liquidity

[6] Similarly, Serbin, Borkovec and Chigirinskiy (2011) include transaction costs into the optimization objective function, without the higher moments though.

respectively. Here ⊗ represents the *Kronecker Product* and α_i represents the percentage allocation to asset *i*. Note that in a general model, we do not prohibit short sales, but if short selling is forbidden, we can simply add a further restriction that requires non-negativity constraints on the allocations. One primary hurdle with the extension of the Markowitz quadratic utility U^R_{MV} to a fourth-degree utility function including skewness and kurtosis of both return and liquidity, is the kurtosis calculation. The problem formulation indicates a need for nonlinear and/or non-convex techniques in the solution of the utility function.

Here is an explanation of the extended utility function shown in Equation (5). A high expected return level and a high expected liquidity level are the reward for the investor, while further moments in both represent the uncertainty, i.e., the risk, in return and in realizing the return (liquidity). Even moments represent extreme values, disliked by investors. Positive (negative) odd moments represent good surprises overweighing bad surprises. Therefore, a risk-averse investor will prefer a portfolio with a higher expected level, a lower variance, a higher skewness, and a lower kurtosis … which can be extended to the limit in terms of additional moments. In this paper, we stop at Kurtosis. The coefficients γ_i and σ_i (i=1, 2, 3, 4) represents an investor's preference among the four moments. We will try equal preferences first with the understanding that all the preference parameters (*γ's,σ's* and *λ's*) can be adjusted to suit each investor's need, without loss of generalization. For example, if an investor favors higher moments, s/he can choose $\gamma_1 < \gamma_2 < \gamma_3 < \gamma_4$ and vice versa for an investor who favors lower moments.[7] In addition, different investors may assign different preferences between return and liquidity, represented by λ_R and λ_L. A theoretical derivation of the solution to the optimization problem shown in Equation (5) is presented in the appendix of the paper.

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[7] For robustness check, we repeat our analyses with (γ_1-γ_2-γ_3-γ_4) being (1-2-3-4) and (4-3-2-1) among the four moments and present the results in Section 5.

Incorporating higher moments of return only, Athayde and Flores (2004) compares the Markowitz solution with two special higher-moment cases, in which variance is minimized given the same expected excess return and either a given skewness or a given kurtosis in return. They find that even though the Markowitz solution may have come close to a higher-moment optimal solution, the two solutions differ most of the time. In fact, they find that apart from a zero-measure set, the Markowitz solution is *never* equal to the other higher-moment optimal solutions. Consequently, the need to consider higher-order moments is supported. Lai (1991) argues that an inefficient mean-variance portfolio may in fact be an optimal portfolio in the mean-variance-skewness context and vice-versa. Athayde and Flores (1997) extend the initial three-moment portfolio frontier investigation of Ingersoll (1975) by developing a methodology for analytically solving a three-moment portfolio optimization. Earlier works generalizing portfolio construction with the inclusion of higher-order moments of return considered only the marginal higher moments, disregarding completely the co-moments of the same order. The technique developed by Athayde and Flores (1997) allows for the appropriate consideration of co-moments in the portfolio construction which we adopt in this paper. In addition, the most significant contribution of our paper is to include the moments of liquidity into the framework and demonstrate the further change in optimal allocation caused by the inclusion of moments of liquidity.

4 Empirical Investigation

4.1 A Sample of Two-Asset Portfolio

In this section, we empirically solve the 4-moment portfolio optimization problem shown in Equation (7) with the following two stocks, AMD and WYNN from January 6, 2010 to June 30, 2010 with daily data. Daily data is chosen due to a practical presumption that liquidity is a bigger concern for active traders than long-term, buy-and-hold investors. We start with the Markowitz mean-variance utility function for mean return and variance, i.e., U^R_{MV} (when $\gamma_1=\gamma_2=1$, $\gamma_3=\gamma_4=0$), move on to a utility function for mean return, variance, and skewness, i.e., U^R_{MVS} ($\gamma_1=\gamma_2=\gamma_3=1$, $\gamma_4=0$) and a utility function for mean return, variance, skewness, and kurtosis, i.e., U^R_{MVSK} ($\gamma_1=\gamma_2=\gamma_3=\gamma_4=1$). Next, we add liquidity, measured by daily trading volume (scaled down by dividing 10^9), into the framework and maximize U^{RL}_{MV}($\gamma_1=\gamma_2=1$, $\gamma_3=\gamma_4=0$; $\sigma_1=\sigma_2=1$, $\sigma_3=\sigma_4=0$; $\lambda_R=\lambda_L=1$), U^{RL}_{MVS} ($\gamma_1=\gamma_2=\gamma_3=1$; $\gamma_4=0$; $\sigma_1=\sigma_2=\sigma_3=1$; $\sigma_4=0$; $\lambda_R=\lambda_L=1$), and U^{RL}_{MVSK}($\gamma_1=\gamma_2=\gamma_3=\gamma_4=1$; $\sigma_1=\sigma_2=\sigma_3=\sigma_4=1$; $\lambda_R=\lambda_L=1$), respectively. As said earlier, for the time being, we assign equal preference between return and liquidity so that $\lambda_R=\lambda_L=1$.

We adopt the aforementioned Kronecker product in the implementation of the Athayde and Flores (1997) technique. The addition of higher moments creates a nonlinear, non-convex objective function. For this reason, we will utilize Maple and solve it with the aid of Lagrange's method and the consideration of first- and second-order optimality conditions. It should be noted that the addition of the third and fourth moments, or any higher-order moment for that matter, will result in a polynomial utility function. Therefore, polynomial optimization techniques are appropriate, i.e., we can consider the function smooth or, in other words, once and twice differentiable which justifies the use of first- and second-order optimality conditions. The resulted optimal allocations and the corresponding utility values are presented in Table 1.

As we can see, for the baseline Markowitz portfolio optimization, Maple generates a portfolio consisting of 620% of AMD and -520% of WYNN with a maximal utility of U^R_{MV} = 0.314. The allocations vary greatly when incorporating return skewness and kurtosis to -2398% of AMD and 2498% of WYNN with a maximum utility of U^R_{MVSK}= 0.024. When including liquidity, the allocations change once again before finally arriving at 340% of AMD and -240% of WYNN with U^{RL}_{MVSK}= 0.237 for the 4-moment return and liquidity included utility function. Clearly, as the higher moments of return and moments of liquidity are added into the framework, the optimal allocations change significantly, and the corresponding optimal utility level changes as well. It is our purpose in this paper to simply document the dramatic change in allocations without investigating how and why they change in a particular way or other. Note that the maximum utility level does not necessarily increase because including these other terms into the utility function may not add to utility, but it is certainly something that needs to be considered as the optimal allocation is affected by it. Additionally, in reality, short selling a large proportion of stock requires a prohibitively high level of margin and/or leverage and is therefore not entirely realistic for most investors, especially when the optimal allocation requires the amount of short selling shown in Table 1.[8]

4.2 A Sample of Three-Asset Portfolio

To provide another empirical example, and to illustrate the difficulty in dealing with larger portfolios, we investigate a three-asset portfolio in this section, namely, AMD, HCBK, and WYNN, for the same sample period as in the last section. We will move through the empirical solutions quickly to arrive at the ultimate comparison. Note that the covariance matrix now consists of 9 components, 6 of which are distinct; the co-skewness matrix has 27 components, 10 of which are distinct; and the co-kurtosis matrix has 81 entries, 16 of which are distinct. As mentioned previously, the analysis for larger portfolios can become

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[8] We understand that an investor can put on specific restrictions suitable for his/her own margin level with regards to short selling. The theoretical derivation in the appendix has the boundary *a* and *b* set up for this purpose, therefore the results here with short sale allowed are provided without loss of generalization. In addition, the results so far correspond to equal preference among the moments; alternative preferences can be found in Section 5.

prohibitively intractable as the portfolio, and resulting matrices grow larger.

Table 2 presents the results from Maple. Markowitz portfolio optimization with optional short selling generates a portfolio consisting of -2455% of AMD, -522% of HCBK, and 3077% of WYNN with a maximal utility of U^{R}_{MV} = 0.025. Incorporating return skewness suggests that the investor allocate 100% of available capital to WYNN to generate an optimal utility of 0.002.

Considering liquidity in addition to the return vector introduces further changes into the optimal allocations. The mean-variance scenario yields optimal allocations of 686% to AMD, 81% to HCBK, and -667% to WYNN for an optimal utility of 0.443. Adding liquidity skewness changes the optimal allocation to a 100% investment in HCBK and none in the other two. Lastly, adding liquidity kurtosis suggests the following optimal portfolio mix, namely, 389% allocated to AMD, 33% allocated to HCBK, and -323% allocated to WYNN generating a utility value of 0.320. Clearly, we can see that the optimal allocations are affected by the extent to which moments of liquidity are considered, each requiring a significant rebalancing of the optimal portfolio, and thus meriting the importance of liquidity and its moments in optimal portfolio construction.

5 What Factors Cause the Most Significant Changes in Allocation?

Though Section 3 demonstrates a couple of examples where portfolio allocation changes significantly when higher moments of return and moments of liquidity are added into the optimization framework, we do realize that not all stocks will experience a significant re-allocation. In this section, we examine the issue more generally by looking at what factors can cause the most significant changes in allocation when higher moments of return and moments of liquidity are added.

To start, we sort the S&P 500 stocks based on daily return skewness, and starting from the 6th stock, pick one stock for every 10 stocks down the list, resulting in a sample of 50 stocks, from which we form 1225 pairs. For each pair, we solve for the allocation on each stock for six different Utility functions (U^R_{MV}, U^R_{MVS}, U^R_{MVSK}, U^{RL}_{MV}, U^{RL}_{MVS}, and U^{RL}_{MVSK}). Like in Table 1 and 2, the six utility functions start from the basic Markowitz return-mean-variance form and end with the full 4-moment return-liquidity form. The cross-pair average allocations when maximizing the six utility functions and the resulting five changes in average allocations are listed in Table 3. We report the absolute values of the changes to prevent increases in allocation cancelling out with decreases through averaging, as our purpose is to see how much the change is in allocation, not on whether the allocation increases or decreases, *per se*. Because we are dealing with 2-stock portfolios, we only present the results for the allocation in one stock (α_1), with the understanding that the allocation to the other stock is always going to be 1- α_1.

As we can see from Table 3, the mean allocation on the first stock ranges from 55% for U^R_{MV}, to 465% for U^R_{MVS}, to 72% for U^R_{MVSK} when higher moments of returns are added, before shifting to 57% for U^{RL}_{MV}, -145% for U^{RL}_{MVS} and 68% for U^{RL}_{MVSK} when moments of liquidity are further added. When the mean absolute change in this allocation is examined, we see that the absolute change in α_1 from the baseline Markowitz U^R_{MV} allocation is impressive: the smallest change from the baseline is 33%, occurring from the baseline Markowitz U^R_{MV} to the inclusion of the third and fourth moments in return (U^R_{MVSK}), while the largest change is 1641%, when liquidity's first three moments are added to the first three moments of return in the utility function (U^{RL}_{MVS}).

To examine what factors are the most influential in causing allocation changes, for each change in the allocations caused by the inclusion of additional moment(s) of return and/or liquidity, we regress it cross-sectionally against the additional moments of return and/or liquidity that cause the change. The independent variables are measured by the absolute percentage difference between the two stocks. A total of eight regressions are run as follows:

When return is the only concern in the utility function:

Regression 1: from U^R_{MV} to U^R_{MVS}, $\Delta\alpha_{1,p}=\beta_0+\beta_1*(|\%\Delta S^R_p|)+\varepsilon_p$, (6)

Regression 2: from U^R_{MVS} to U^R_{MVSK}, $\Delta\alpha_{1,p}=\beta_0+\beta_1*(|\%\Delta K^R_p|)+\varepsilon_p$, (7)

Regression 3: from U^R_{MV} to U^R_{MVSK}, $\Delta\alpha_{1,p}=\beta_0+ \beta_1*(|\%\Delta S^R_p|)+\beta_2*(|\%\Delta K^R_p|)+\varepsilon_p$, (8)

When the utility function adds liquidity as an additional term:

Regression 4: from U^R_{MV} to U^{RL}_{MV}, $\Delta\alpha_{1,p}=\beta_0+ \beta_1*(|\%\Delta M^L_p|)+ \beta_2*(|\%\Delta V^L_p|)+\varepsilon_p$, (9)

Regression 5: from U^{RL}_{MV} to U^{RL}_{MVS}, $\Delta\alpha_{1,p}=\beta_0+ \beta_1*(|\%\Delta S^L_p|)+\varepsilon_p$, (10)

Regression 6: from U^{RL}_{MVS} to U^{RL}_{MVSK}, $\Delta\alpha_{1,p}=\beta_0+ \beta_1(|\%\Delta K^L_p|)+\varepsilon_p$, (11)

Regression 7: from U^{RL}_{MV} to U^{RL}_{MVSK}, $\Delta\alpha_{1,p}=\beta_0+ \beta_1*(|\%\Delta S^L_p|)+ \beta_2(|\%\Delta K^L_p|)+\varepsilon_p$, (12)

The overall change:

Regression 8: from U^R_{MV} to U^{RL}_{MVSK},

$$\Delta\alpha_{1,p}=\beta_0+ \beta_1*(|\%\Delta S^R_p|)+\beta_2*(|\%\Delta K^R_p|)$$

$$+ \beta_3*(|\%\Delta M^L_p|)+ \beta_4*(|\%\Delta V^L_p|)+ \beta_5*(|\%\Delta S^L_p|)+ \beta_6(|\%\Delta K^L_p|)+\varepsilon_p, \quad (13)$$

where, p=1,2,…, 1225 pairs, $\Delta\alpha_{1,p}$ is the change in allocation on stock 1 in the p^{th} pair, and,

$$|\%\Delta S^R_p|= |(S^R_{2,p}-S^R_{1,p})/[(S^R_{1,p}+S^R_{2,p})/2]|,$$
$$|\%\Delta K^R_p|= |(K^R_{2,p}-K^R_{1,p})/[(K^R_{1,p}+K^R_{2,p})/2]|,$$
$$|\%\Delta M^L_p|= |(M^L_{2,p}-M^L_{1,p})/[(M^L_{1,p}+M^L_{2,p})/2]|,$$
$$|\%\Delta V^L_p|= |(V^L_{2,p}-V^L_{1,p})/[(V^L_{1,p}+V^L_{2,p})/2]|,$$
$$|\%\Delta S^L_p|= |(S^L_{2,p}-S^L_{1,p})/[(S^L_{1,p}+S^L_{2,p})/2]|,$$
$$|\%\Delta K^L_p|= |(K^L_{2,p}-K^L_{1,p})/[(K^L_{1,p}+K^L_{2,p})/2]|, \quad (14)$$

Where R stands for return, L stands for liquidity, M stands for Mean, V stands for Variance, S stands for Skewness and K stands for Kurtosis.

Table 4 reports the regression results. When return skewness and kurtosis are added separately into the utility function, their percentage difference between the two stocks does not significantly affect the average allocation change. However, when they are added jointly from U^{R}_{MV} to U^{R}_{MVSK}, the percentage difference in return kurtosis significantly decreases the allocation change. When liquidity mean and variance are added, the percentage difference in mean liquidity significantly increases the allocation change. When added alone, liquidity skewness significantly reduces while liquidity kurtosis significantly increases the allocation change, and the significant negative impact from liquidity skewness persists when both liquidity skewness and kurtosis are added simultaneously.

Perhaps, the result that is worth the most investigation is from the overall regression when the utility function changes from U^{R}_{MV} to U^{RL}_{MVSK}. Consistent with individual regressions, the percentage difference in return skewness and liquidity variance continue to be insignificant, while return kurtosis, liquidity mean, liquidity skewness and liquidity kurtosis continue to be significant. This result highlights the importance of incorporating liquidity, not only with respect to its first moment (i.e., mean), but also its higher moments such as skewness and kurtosis, into portfolio optimization. It is in this respect that we do not feel it is sufficient nor appropriate to simply internalize liquidity level into a premium of the expected return as in many previous research, because not only does liquidity level matter (its first moment), liquidity skewness and especially liquidity kurtosis also matters significantly. In a sense, this result demonstrates that the inclusion of higher moments of return, such as return skewness and return kurtosis, is still not enough for a complete portfolio optimization framework; the inclusion of moments of liquidity is necessary. In addition, our result demonstrates that higher moments of return, such as return kurtosis, need to be considered even if return skewness may not matter to the optimal portfolio construction. The recent increasing attention paid on to return kurtosis is a step towards this direction.

6 Robustness Check with Unequal Preference among the Moments and Between Return and Liquidity

Until now, we have assumed equal preference ($\gamma_1= \gamma_2= \gamma_3= \gamma_4=1$, and $\sigma_1=\sigma_2=\sigma_3=\sigma_4=1$) among the four moments of return and liquidity, and equal preference ($\lambda_R=\lambda_L=1$) between return and liquidity in the utility function. Different investors can have different preferences and these different preferences will cause the allocations to change as well. In this section, we first examine a case of increasing preference among the four moments with $\gamma_1=\sigma_1=1, \gamma_2=\sigma_2=2, \gamma_3=\sigma_3=3, \gamma_4=\sigma_4=4$ and a case of decreasing preference with $\gamma_1=\sigma_1=4, \gamma_2=\sigma_2=3, \gamma_3=\sigma_3=2, \gamma_4=\sigma_4=1$ while holding the preference between return and liquidity equal ($\lambda_R=\lambda_L=1$). We then examine a case of higher preference on return than on liquidity ($\lambda_R=2, \lambda_L=1$) and a case of lower preference on return than on liquidity ($\lambda_R=1, \lambda_L=2$) while holding the preference among the four moments equal ($\gamma_1=\gamma_2=\gamma_3=\gamma_4=1$, and $\sigma_1=\sigma_2=\sigma_3=\sigma_4=1$). For the same 1,225 pairs of stocks as in Section 4, we re-maximize the six utility functions with the new preferences and report the results in Table 5.

Like the results in equal preferences reported in Table 3, results in Table 5 show that the average allocation to the first stock of each pair changes tremendously. The mean α_1 increases from 55% for U^R_{MV} to 348% for U^R_{MVS}, before dropping to 68% for U^R_{MVSK} when the preference among the four moments is increasing and from 61% to 876% to 85% when the preference is decreasing. With liquidity added, the average α_1 decreases from 54% to -78% before rising to 62% for increasing preferences, and changes from 63% to -335% to 73% for decreasing preferences. When holding the four preferences among the four moments equal but increasing and decreasing the preference between return and liquidity, we also observe changes in the mean α_1, moving from 59% to 1404% to 68% for higher preference on liquidity than return, while going from 56% to 663% to 73% for higher preference on return than liquidity. These results show that with alternative preference parameters, optimal allocation continues to change when higher moments of return and moments of liquidity are added into the portfolio optimization framework.

7 Conclusions

In this paper, we examine the generalization of the Markowitz mean-variance portfolio theory with the inclusion of the 3rd and 4th moments of return and the 1st, 2nd, 3rd and 4th moments of liquidity. With new analytical techniques available for the higher moment analyses, we are able to compare and contrast the optimal portfolio constructions with varying moments included. We confirm the notion that the addition of return skewness and kurtosis can affect the optimal portfolio construction, as previously presented by significant research. More important, the addition of the first four moments of liquidity, that is, liquidity mean, liquidity variance, liquidity skewness and liquidity kurtosis, further adjusts the allocation in the optimal portfolio.

Cross-sectionally, among the newly included moments, mean liquidity and liquidity skewness and kurtosis are the most influential in changing portfolio allocations, while return skewness is, surprisingly, not statistically significant. This finding illustrates the importance of not stopping at the 3rd moment of return (return skewness) when considering portfolio optimization. In addition, the preference parameters between return and liquidity concerns in the utility function, together with the preference parameters among the four moments of return and liquidity, continue to cause changes in optimal allocations.

We realize that the above conclusions are limited in certain ways. In this paper, we are simply documenting that the optimal allocation of securities can change when higher moments of return and moments of liquidity are added, with no attention being paid on why the allocations change in a certain way, which by itself, is a very interesting question that warrants further study. The non-linear optimization problems encountered in this paper can be examined with the use of analytical nonlinear programming techniques, which, in some instances, require the use of gradients and hessian matrices, often causing some consternation due to the additional complexity. These non-linear algorithms may aid in the development of efficient analyses of the portfolio problem when higher moments are considered. It may be possible for even higher moments (higher than the four moments we address here in this paper) of return and liquidity to impact portfolio choice; this is an area not yet developed deeply. Another potential area of new research lies in the discovery of a tractable methodology for higher moment portfolio construction with respect to extremely large portfolios, which would be a significant contribution and would likely find immediate implementation in the investment industry.

Appendix

Theoretical Derivation

In this section, we consider a two-asset optimization problem incorporating mean, variance, skewness, and kurtosis of return as well as liquidity. The corresponding optimization problem is as follows (assuming equal preference between return and liquidity, that is λ_R=λ_L=1 from Equation (5)):

$$\text{Maximize} \quad \begin{aligned} U^{RL}{}_{MVSK} &= \gamma_1\alpha^T M_1 - \gamma_2\alpha^T M_2\alpha + \gamma_3\alpha^T M_3\alpha\otimes\alpha - \gamma_4\alpha^T M_4\alpha\otimes\alpha\otimes\alpha \\ &+ \sigma_1\alpha^T L_1 - \sigma_2\alpha^T L_2\alpha + \sigma_3\alpha^T L_3\alpha\otimes\alpha - \sigma_4\alpha^T L_4\alpha\otimes\alpha\otimes\alpha \end{aligned}$$

$$\text{subject to} \quad \sum_{i=1}^{2} \alpha_i = 1,\ a \le \alpha_i \le b. \qquad \text{(A.1)}$$

Being consistent with the previous section, we denote

$$M_1 = \begin{bmatrix} m_1 \\ m_2 \end{bmatrix}, M_2 = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}, M_3 = \begin{bmatrix} \sigma_{111} & \sigma_{112} & \sigma_{211} & \sigma_{212} \\ \sigma_{121} & \sigma_{122} & \sigma_{221} & \sigma_{222} \end{bmatrix}$$

$$, \qquad \text{(A.2)}$$

$$M_4 = \begin{bmatrix} \sigma_{1111} & \sigma_{1112} & \sigma_{1121} & \sigma_{1122} & \sigma_{1211} & \sigma_{1212} & \sigma_{1221} & \sigma_{1222} \\ \sigma_{2111} & \sigma_{1112} & \sigma_{2121} & \sigma_{2122} & \sigma_{2211} & \sigma_{2212} & \sigma_{2221} & \sigma_{2222} \end{bmatrix}$$

and

$\sigma_{12} = \sigma_{21}$, $\sigma_{112} = \sigma_{121} = \sigma_{211}$, $\sigma_{122} = \sigma_{212} = \sigma_{221}$, and

$\sigma_{1112} = \sigma_{1121} = \sigma_{1211} = \sigma_{2111}$, $\sigma_{1122} = \sigma_{1212} = \sigma_{2112} = \sigma_{2211}$, (A.3)

$\sigma_{1222} = \sigma_{2122} = \sigma_{2212} = \sigma_{2221}$.

Similarly, we write

$$L_1 = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}, L_2 = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}, L_3 = \begin{bmatrix} w_{111} & w_{112} & w_{211} & w_{212} \\ w_{121} & w_{122} & w_{221} & w_{222} \end{bmatrix}$$

$$\text{(A.4)}$$

$$L_4 = \begin{bmatrix} w_{1111} & w_{1112} & w_{1121} & w_{1122} & w_{1211} & w_{1212} & w_{1221} & w_{1222} \\ w_{2111} & w_{1112} & w_{2121} & w_{2122} & w_{2211} & w_{2212} & w_{2221} & w_{2222} \end{bmatrix}$$

with

$w_{12}=w_{21}$, $w_{112}=w_{121}=w_{211}$, $w_{122}=w_{212}=w_{221}$, and

$w_{1112}=w_{1121}=w_{1211}=w_{2111}$, $w_{1122}=w_{1212}=w_{2112}=w_{2211}$, (A.5)

$w_{1222}=w_{2122}=w_{2212}=w_{2221}$.

If we let $\alpha_1=x$, then $\alpha_2=1-x$. For convenience of further calculations, we set

$$\begin{aligned} f &= \gamma_1\alpha^T M_1-\gamma_2\alpha^T M_2\alpha+\gamma_3\alpha^T M_3\alpha\otimes\alpha-\gamma_4\alpha^T M_4\alpha\otimes\alpha\otimes\alpha \\ g &= \sigma_1\alpha^T L_1-\sigma_2\alpha^T L_2\alpha+\sigma_3\alpha^T L_3\alpha\otimes\alpha-\sigma_4\alpha^T L_4\alpha\otimes\alpha\otimes\alpha \end{aligned} \tag{A.6}$$

then we have,

$$\begin{aligned} f(x) &= \gamma_1\left(m_1x+m_2(1-x)\right)-\gamma_2\left(\sigma_{11}x^2+2\sigma_{12}x(1-x)+\sigma_{22}(1-x)^2\right) \\ &+\gamma_3\left(\sigma_{111}x^3+3\sigma_{112}x^2(1-x)+3\sigma_{212}x(1-x)^2+\sigma_{222}(1-x)^3\right) \\ &-\gamma_4\left(\sigma_{1111}x^4+4\sigma_{1112}x^3(1-x)+6\sigma_{1122}x^2(1-x)^2+4\sigma_{1222}x(1-x)^3+\sigma_{2222}(1-x)^4\right) \end{aligned}$$

$$\begin{aligned} g(x) &= \sigma_1\left(l_1x+l_2(1-x)\right)-\sigma_2\left(w_{11}x^2+2w_{12}x(1-x)+w_{12}(1-x)^2\right) \\ &+\sigma_3\left(w_{111}x^3+3w_{112}x^2(1-x)+3w_{212}x(1-x)^2+w_{222}(1-x)^3\right) \\ &-\sigma_4\left(w_{1111}x^4+4w_{1112}x^3(1-x)+6w_{1122}x^2(1-x)^2+4w_{1222}x(1-x)^3+w_{2222}(1-x)^4\right) \end{aligned}$$

A direct calculation yields that the derivative of $f(x)$ is given by

$$f'(x)=a_0+a_1x+a_2x^2+a_3x^3,$$

where

$$\begin{aligned} a_0 &= \gamma_1(m_1-m_2)-2\gamma_2(\sigma_{12}-\sigma_{22})+3\gamma_3(\sigma_{212}-\sigma_{222})-4\gamma_4(\sigma_{1222}-\sigma_{2222}) \\ a_1 &= -2\gamma_2(\sigma_{11}-2\sigma_{12}+\sigma_{22})+6\gamma_3(\sigma_{112}-2\sigma_{212}+\sigma_{222})-12\gamma_4(\sigma_{1122}-2\sigma_{1222}+\sigma_{2222}) \\ a_2 &= 3\gamma_3(\sigma_{111}-3\sigma_{112}+3\sigma_{212}-\sigma_{222})-12\gamma_4(\sigma_{1112}-3\sigma_{1122}+3\sigma_{1222}-\sigma_{2222}) \\ a_3 &= -4\gamma_4(\sigma_{1111}-4\sigma_{1112}+6\sigma_{1122}-4\sigma_{1222}+\sigma_{2222}) \end{aligned}$$

and by

$$g'(x)=b_0+b_1x+b_2x^2+b_3x^3,$$

where

$$b_0 = \sigma_1 (l_1 - l_2) - 2\sigma_2 (w_{12} - w_{22}) + 3\sigma_3 (w_{212} - w_{222}) - 4\sigma_4 (w_{1222} - w_{2222})$$
$$b_1 = -2\sigma_2 (w_{11} - 2w_{12} + w_{22}) + 6\sigma_3 (w_{112} - 2w_{212} + w_{222}) - 12\sigma_4 (w_{1122} - 2w_{1222} + w_{2222})$$
$$b_2 = 3\sigma_3 (w_{111} - 3w_{112} + 3w_{212} - w_{222}) - 12\sigma_4 (w_{1112} - 3w_{1122} + 3w_{1222} - w_{2222})$$
$$b_3 = -4\sigma_4 (w_{1111} - 4w_{1112} + 6w_{1122} - 4w_{1222} + w_{2222})$$

Notice that the critical points of $U^{RL}{}_{MVSK} = f(x) + g(x)$ satisfy $f'(x) + g'(x) = 0$. Thus we derive the following theorem:

Theorem 2.1 We consider the following optimization problem:

Maximize
$$U^{RL}{}_{MVSK} = \gamma_1 \alpha^T M_1 - \gamma_2 \alpha^T M_2 \alpha + \gamma_3 \alpha^T M_3 \alpha \otimes \alpha - \gamma_4 \alpha^T M_4 \alpha \otimes \alpha \otimes \alpha + \sigma_1 \alpha^T L_1 - \sigma_2 \alpha^T L_2 \alpha + \sigma_3 \alpha^T L_3 \alpha \otimes \alpha - \sigma_4 \alpha^T L_4 \alpha \otimes \alpha \otimes \alpha$$

subject to $\sum_{i=1}^{2} \alpha_i = 1, a \le \alpha_i \le b.$ (A.7)

Then the maximum occurs either at $\alpha_1 = x$ (if $x \in (a,b)$), a root of the following third-order polynomial equation

$$a_0 + b_0 + (a_1 + b_1)x + (a_2 + b_2)x^2 + (a_3 + b_3)x^3 = 0 \quad \text{(A.8)}$$

or, at the endpoint

$$\alpha_1 = a \text{ or } \alpha_1 = b. \quad \text{(A.9)}$$

If we incorporate only the mean, variance, and skewness, but not the kurtosis, we may then let $\gamma_4 = \sigma_4 = 0$. In this case, we have $a_3 = b_3 = 0$ and the above polynomial becomes quadratic for which we can identify, explicitly, the solutions with the following expression:

$$x = \frac{-(a_1 + b_1) \pm \sqrt{(a_1 + b_1)^2 - 4(a_0 + b_0)(a_2 + b_2)}}{2(a_2 + b_2)} \quad \text{(A.10)}$$

Corollary 2.2 We consider the following optimization problem:

Maximize

$$U^{RL}{}_{MVS} = \gamma_1 \alpha^T M_1 - \gamma_2 \alpha^T M_2 \alpha + \gamma_3 \alpha^T M_3 \alpha \otimes \alpha + \gamma_1 \alpha^T L_1 - \gamma_2 \alpha^T L_2 \alpha + \gamma_3 \alpha^T L_3 \alpha \otimes \alpha$$

subject to $\sum_{i=1}^{2} \alpha_i = 1, a \le \alpha_i \le b$.

Then the maximum occurs either at $\alpha_1 = x$ (if $x \in (a,b)$) given by (A.10) or at the endpoint $\alpha_1 = a$ or $\alpha_1 = b$.

The above derivation is for the optimization problem consisting of two assets. For more than two assets, the essential of the approach remains, however, the symbolic calculations would quickly become massive and the high computational cost would be a serious issue.

Table 1 reports the optimal weights on a two-stock portfolio when utility functions with different objectives are maximized. The most inclusive utility function is:

$U^{RL}_{MVSK} = \lambda_R(\gamma_1 {*} Mean^R - \gamma_2 {*} Variance^R + \gamma_3 {*} Skewness^R - \gamma_4 {*} Kurtosis^R)$
$\quad + \lambda_L(\sigma_1 {*} Mean^L - \sigma_2 {*} Variance^L + \sigma_3 {*} Skewness^L - \sigma_4 {*} Kurtosis^L)$,

from which, all the other utility functions are parts of. In a utility function, *R* stands for return, and *L* stands for liquidity. M/V/S/K stands for the $1^{st}/2^{nd}/3^{rd}/4^{th}$ moment (Mean, Variance, Skewness, Kurtosis).
α_i is the weight on the i^{th} stock. Daily return and volume (measure of liquidity, scaled down by dividing 10^9) data for AMD and WYNN is from January 6, 2010 to June 30, 2010.

Table 1: Optimal portfolio construction for a pair of stocks--AMD and WYNN

	Utility	λ preference	γ and σ preference	α₁	α₂	utility
Return only	U^R_{MV}	-	1-1-0-0	620%	-520%	0.314
	U^R_{MVS}	-	1-1-1-0	0%	100%	0.002
	U^R_{MVSK}	-	1-1-1-1	-2398%	2498%	0.024
Adding liquidity	U^{RL}_{MV}	1-1	1-1-0-0	940%	-840%	0.314

U^{RL}_{MVS}	1-1	1-1-1-0	782%	-682%	0.383
U^{RL}_{MVSK}	1-1	1-1-1-1	340%	-240%	0.237

Table 2 reports the optimal weights on a three-stock portfolio when utility functions with different objectives are maximized. The most inclusive utility function is:

$U^{RL}_{MVSK} = \lambda_R(\gamma_1 * Mean^R - \gamma_2 * Variance^R + \gamma_3 * Skewness^R - \gamma_4 * Kurtosis^R)$
$+\lambda_L(\sigma_1 * Mean^L - \sigma_2 * Variance^L + \sigma_3 * Skewness^L - \sigma_4 * Kurtosis^L)$, from which, all the other utility functions are parts of. In a utility function, *R* stands for return, and *L* stands for liquidity. M/V/S/K stands for the 1st/2nd/ 3rd/4thmoment (Mean, Variance, Skewness, Kurtosis). α_i is the weight on the i^{th} stock. Daily return and volume (measure of liquidity, scaled down by dividing 10^9) data for AMD, HCBK, and WYNN is from January 6, 2010 to June 30, 2010.

Table 2: Optimal portfolio construction for a trio of stocks--AMD, HCBK, and WYNN

	Utility	λ preference	γ and σ preference	α_1	α_2	utility
Return only	U^R_{MV}	-	1-1-0-0	620%	-520%	0.314
	U^R_{MVS}	-	1-1-1-0	0%	100%	0.002
	U^R_{MVSK}	-	1-1-1-1	-2398%	2498%	0.024
Adding liquidity	U^{RL}_{MV}	1-1	1-1-0-0	940%	-840%	0.314
	U^{RL}_{MVS}	1-1	1-1-1-0	782%	-682%	0.383
	U^{RL}_{MVSK}	1-1	1-1-1-1	340%	-240%	0.237

Table 3 reports the average optimal weights on 1225 pairs of two-stock portfolio when utility functions with different objectives are maximized. The 1225 pairs are from 50 stocks selected from the S&P 500 stocks based on the ranking of daily return skewness: starting from stock #6, and select one stock for every 10 stocks down the rank. The most inclusive utility function is:

$U^{RL}_{MVSK} = \lambda_R(\gamma_1 * Mean^R - \gamma_2 * Variance^R + \gamma_3 * Skewness^R - \gamma_4 * Kurtosis^R)$
$+\lambda_L(\sigma_1 * Mean^L - \sigma_2 * Variance^L + \sigma_3 * Skewness^L - \sigma_4 * Kurtosis^L)$, from which, all the other utility functions are parts of. In a utility function, *R* stands for return, and *L* stands for liquidity. M/V/S/K stands for the 1st/2nd/ 3rd/4th moment (Mean, Variance, Skewness, Kurtosis). α_1 is the weight on the *1st* stock. Daily return and volume (measure of liquidity, scaled down by dividing 10^9) data for the S&P 500 stocks is from January 6, 2010 to June 30, 2010. The results assume equal preferences among the moments and between return and liquidity. Average $|\Delta\alpha_1|$ from Markowitz U^R_{MV} records the cross-pair average of the absolute change in α_1 when the Utility function changes from the base-line Markowitz U^R_{MV}.

Table 3: Allocations averaged over 1225 pairs of stocks--equal preferences

	U^R_{MV}	U^R_{MVS}	U^R_{MVSK}	U^{RL}_{MV}	U^{RL}_{MVS}	U^{RL}_{MVSK}
Mean α_1	55%	465%	72%	57%	-145%	68%
Mean$|\Delta\alpha_1|$ from Markowitz U^R_{MV}		558%	33%	114%	1641%	137%

In Table 4, the dependent variable is the change in allocation on the first stock in the 1225 pairs when utility functions with different objectives are maximized. The independent variables are the absolute percentage difference in moments of return and/or liquidity between the two stocks in the pair: $|\%\Delta X^R_p| = |(X^R_{2,p} - X^R_{1,p})/[(X^R_{1,p} + X^R_{2,p})/2]|$, where X= M/V/S/K, which stands for the 1st/2nd/3rd/4th moment (Mean, Variance, Skewness, Kurtosis). *R* stands for return, and *L* stands for liquidity. The 1225 pairs are from 50 stocks selected from the S&P 500 stocks based on the ranking of daily return skewness: starting from stock #6, and select one stock for every 10 stocks down the rank. The most inclusive utility function is:

$U^{RL}_{MVSK} = \lambda_R(\gamma_1 * Mean^R - \gamma_2 * Variance^R + \gamma_3 * Skewness^R - \gamma_4 * Kurtosis^R)$

$+ \lambda_L(\sigma_1 * Mean^L - \sigma_2 * Variance^L + \sigma_3 * Skewness^L - \sigma_4 * Kurtosis^L)$, from which, all the other utility functions are parts of. Daily return and volume (measure of liquidity, scaled down by dividing 10^9) data for the S&P 500 stocks is from January 6, 2010 to June 30, 2010. The results assume equal preferences among the moments and between return and liquidity.

Table 4: Regressions of change in allocation on moments of return and/or liquidity

Dependent=$\Delta\alpha_1$	n=1225	intercept	$\|\%\Delta S^R{}_p\|$	$\|\%\Delta K^R{}_p\|$	$\|\%\Delta M^L{}_p\|$	$\|\%\Delta V^L{}_p\|$	$\|\%\Delta S^L{}_p\|$	$\|\%\Delta K^L{}_p\|$	adj R^2
$U^R{}_{MV}$ to $U^R{}_{MVS}$	coefficient	0.2410	0.0011						-0.0008
	t value	0.9039	0.1509						
$U^R{}_{MVS}$ to $U^R{}_{MVSK}$	coefficient	-0.1242		-0.2880					0.0008
	t value	-0.4534		-1.3943					
$U^R{}_{MV}$ to $U^R{}_{MVSK}$	coefficient	0.1150	-0.0005	-0.2422					0.0370
	t value	2.4985	-0.3704	-6.9774					
$U^R{}_{MV}$ to $U^{RL}{}_{MV}$	coefficient	1.9043			14.6400	-1.3910			0.0729
	t value	1.4652			3.6337	-0.4827			
$U^{RL}{}_{MV}$ to $U^{RL}{}_{MVS}$	coefficient	-2.3404					-6.3671		0.0736
	t value	-2.2942					-9.9134		
$U^{RL}{}_{MVS}$ to $U^{RL}{}_{MVSK}$	coefficient	-0.3111						2.0951	0.4255
	t value	-2.6463						30.1233	
$U^{RL}{}_{MV}$ to $U^{RL}{}_{MVSK}$	coefficient	-2.5310					***-13.2319***	8.5451	0.0346
	t value	-2.5279					-1.8655	1.2839	
$U^R{}_{MV}$ to $U^{RL}{}_{MVSK}$	coefficient	-0.8971	0.0167	-5.3781	13.9845	-5.7418	***-16.2495***	17.8984	0.0746
	t value	-1.0598	0.7209	-6.6998	3.8919	-0.9617	-1.8367	2.8386	

Note: coefficients in bold (italic) are statistically significant at 1% (10%) level.

Table 5 reports the average optimal weights on 1225 pairs of two-stock portfolio when utility functions with different objectives are maximized. The 1225 pairs are from 50 stocks selected from the S&P 500 stocks based on the ranking of daily return skewness: starting from stock #6, and select one stock for every 10 stocks down the rank. The most inclusive utility function is:

$U^{RL}_{MVSK} = \lambda_R(\gamma_1 * Mean^R - \gamma_2 * Variance^R + \gamma_3 * Skewness^R - \gamma_4 * Kurtosis^R)$
$+ \lambda_L(\sigma_1 * Mean^L - \sigma_2 * Variance^L + \sigma_3 * Skewness^L - \sigma_4 * Kurtosis^L)$, from which, all the other utility functions are parts of. In a utility function, *R* stands for return, and *L* stands for liquidity. M/V/S/K stands for the 1st/2nd/ 3rd/4th moment (Mean, Variance, Skewness, Kurtosis). α_i is the weight on the i^{th} stock. Daily return and volume (measure of liquidity, scaled down by dividing 10^9) data for the S&P 500 stocks is from January 6, 2010 to June 30, 2010. The results assume either increasing or decreasing preferences among the four moments or increasing or decreasing preferences between return and liquidity.

Table 5: Allocations averaged over 1225 pairs of stocks--unequal preferences

	Utility	λ preference	γ and σ preference	α_1	α_2	utility
Return only	U^R_{MV}	-	1-1-0-0	620%	-520%	0.314
	U^R_{MVS}	-	1-1-1-0	0%	100%	0.002
	U^R_{MVSK}	-	1-1-1-1	-2398%	2498%	0.024
Adding liquidity	U^{RL}_{MV}	1-1	1-1-0-0	940%	-840%	0.314
	U^{RL}_{MVS}	1-1	1-1-1-0	782%	-682%	0.383
	U^{RL}_{MVSK}	1-1	1-1-1-1	340%	-240%	0.237

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