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Study of Partitioned Operators and Its Applications

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Abstract

In this paper, we find new properties of Partitioned matrices, and some new relations between Positive semidefinite matrices and Hermitian block, and we give necessary and sufficient conditions for a partitioned operator matrix to have the Drazin inverse with Banachiewicz–Schur.

Mathematics Subject Classifications: 15A18; 15A60; 26C10; 30C15

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1 Introduction

The concepts of partitioned matrices and Block matrices are ubiquitous in physics and applied mathematics, appearing naturally in the description of systems with multiple discrete variables (e.g., quantum spin, quark color and flavor).

The need to calculate determinants and the multiplication of the higher matrices is very time consuming and sometime impossible since the memory in computer to store the matrices are very large.

Let H be a Hilbert space the set of all pounded linear operator on H denoted by $\mathrm{B}(H)$. Every operator $M \in \mathrm{B}(H)$ can be written in a block-form $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Let. If there exists an operator $K^D \in \mathrm{B}(H)$ such that

$$KK^D = K^D K, \qquad K^D K K^D = K, \qquad K^{k+1}K^D = K^k$$

where k the index of K, then K^D is called drazin inverse.

Let M_n^+ denote the set of positive and semidefinite $n \times n$ complex matrix and M be any positive block-matrices; that is,

$$M = \begin{bmatrix} \mathrm{A} & \mathrm{X} \\ \mathrm{X}^* & \mathrm{B} \end{bmatrix} \in M_{n+m}^+$$

where $A \in M_n^+$ and $B \in M_m^+$, a positive block-matrix can be partitioned into a small number of Hermitian blocks by adecomposition lemmas.

2 Preliminary Notes

Definition 2.1 [1] (Block matrix) *In mathematics, a block matrix or a partitioned matrix is a matrix which is interpreted as having been broken into sections called blocks or submatrices.*

A partitioned $m \times n$ *matrix is an* $m \times n$ *matrix* $A = \{a_{ij}\}$ *that has been reexpressed in the general form*

$$A_{ij} = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1C} \\ A_{21} & A_{22} & \cdots & A_{2C} \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rc} \end{bmatrix}$$

Here A_{ij} *is an* $m_i \times n_i$ *matrix* $(i = 1, ..., r\ ,\ j = 1, \ldots, c)$, *where* $m_1, \ldots, m_r$ *and* $n_1, \ldots\ , n_c$ *are positive integers such that*

$$m_1 + m_2 + \ldots + m_r = m \qquad \text{and} \qquad n_1 + n_2 + \ldots + n_c = n$$

Theorem 2.1 [3]. *Let*

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1V} \\ B_{21} & B_{22} & \cdots & B_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ B_{u1} & B_{u2} & \cdots & B_{uv} \end{bmatrix}$$

represent a partitioned $p \times q$ *matrix whose ijth block* B_{ij} *is of dimensions* $p_i \times q_i$.

B partitionedin the same way as those of A. Then

$$AB = \begin{bmatrix} F_{11} & F_{12} & \cdots & F_{1v} \\ F_{21} & F_{22} & \cdots & F_{2v} \\ \vdots & \vdots & \ddots & \vdots \\ F_{r1} & F_{r2} & \cdots & F_{rv} \end{bmatrix}$$

where $F_{ij} = \sum_{k=1}^{C} A_{ik} B_{kj} = A_{i1} B_{1j} + \cdots + A_{ic} B_{cj}$.

Definition 2.2 [3].

$$A_{ij} = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rc} \end{bmatrix}.$$

*Then A is called a **block-diagonal matrix**, and sometimes diag* $\left(A_{11}, A_{22}, ..., A_{rr}\right)$ *is written for* A *.*

Definition 2.3 [3]. *If* $A_{ij} = 0$ *for* $j < i = 1, ..., r$ *, that is if*

$$A = \begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ A_{r1} & A_{r2} & \cdots & A_{rc} \end{bmatrix},$$

*then A is called a **lower block-triangular matrix.***

Theorem 2.2 [2]. (Matrix Inversion in Block form)

Let a $m \times n$ *matrix M be partitioned into a block form:*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where the $n \times n$ *matrix A and D are invertible. Then*

$$M^{-1} = \begin{bmatrix} (A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\ -(D - CA^{-1}B)^{-1}CA^{-1} & (D - CA^{-1}B)^{-1} \end{bmatrix}.$$

Theorem 2.3 [5]. *If* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$*, where* A, B, C, D *are* $n \times n$ *matrices over F, and D is invertible , then*

$$\det \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det(AD - BD^{-1}CD).$$

Lemma 2.1 [5]. *Let* S *be a complex block matrix of the form*

$$S = \begin{bmatrix} S_{11} & S_{12} & \cdots & S_{1N} \\ S_{21} & S_{22} & \cdots & S_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ S_{N1} & S_{N2} & \cdots & S_{NN} \end{bmatrix} \quad (1)$$

and let us define the set of block matrices $\left\{\alpha^{(0)}, \alpha^{(1)}, \cdots, \alpha^{(N-1)}\right\}$, *where* $\alpha^{(k)}$ *is an* $(N-k)\times(N-k)$ *block matrix with blocks*

$$\begin{aligned} \alpha_{ij}^{(0)} &= S_{ij} \\ \alpha_{ij}^{(k+1)} &= \alpha_{j}^{(k)} i - \alpha_{i,N-k}^{(k)} \left(\alpha_{N-k,N-k}^{(k)}\right)^{-1} \alpha_{N-k,j}^{(k)}, \ k \geq 1. \end{aligned} \quad (2)$$

Then the determinants of consecutive $\alpha^{(k)}$ *are related via*

$$det\left(\alpha^{(k)}\right) = det\left(\alpha^{(k+1)}\right) det\left(\alpha_{N-k,N-k}^{(k)}\right).$$

Theorem 2.4 [5]. *Given a complex block matrix of the form* (1)*, and the matrices* $\alpha_{ij}^{(K)}$ *defined in Eq. (2), the determinant of* S *is given by*

$$det(S) = \prod_{k=1}^{N} det\left(\alpha_{kk}^{(N-k)}\right).$$

Lemma 2.2 [7]. *There exist a nonsingular matrix Q such that J = Q−1AQ, where,*

$$Q = \begin{bmatrix} v_s & v_{s-1} & \cdots & v_1 \end{bmatrix},$$

and vi is an eigenvector corresponding to eigenvalue λ_i *.*

Lemma 2.3 [4]. *For every matrix in* M_{n+m}^{+} *partitioned into blocks, we have a decomposition*

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & B \end{bmatrix} V^*$$

for some unitaries $U, V \in M_{n+m}^{+}$.

Corollary 2.1 [4]. *For every matrix in* M_{2n}^{+} *written in blocks of the same size, we have the decomposition:*

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U \begin{bmatrix} \frac{A+B}{2} + I(X) & 0 \\ 0 & 0 \end{bmatrix} U^* + V \begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - I(X) \end{bmatrix} V^*$$

for some unitaries $U,\ V \in M_{n+m.}$

Theorem 2.5 [3]. *Let*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{and} \quad S = (A - BD^{-1}C)$$

be the generalized Schur complement of A in M. Then,

$$M^D = \begin{bmatrix} (A - BD^D C)^{-1} & -(A - BD^D C)^D BD^D \\ -S^D CA^D & S^D \end{bmatrix} \tag{3.3}$$

if and only if

$$(I - AA^D)BS^D = A^D B(I - S^D S), \qquad (I - SS^D)CA^D = S^D C(I - A^D)$$

and

$$\begin{bmatrix} A(I - A^D A) & (I - AA^D)B \\ (I - SS^D)CI - A^D A) & S(I - S^D S) \end{bmatrix}$$

is a nilpotent operator.

3 Main Results

Theorem 3.1. *Let*

$$B = \begin{bmatrix} B_{11} & 0 & \cdots & 0 \\ B_{21} & B_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ B_{c1} & B_{c2} & \cdots & B_{cc} \end{bmatrix}$$

represent an $n \times n$ *lower block-triangular matrix whose ijth block* B_{ij} *is of* *dimensions* $n_i \times n_j$ $\left(j \geq i = 1, ..., \mathrm{c}\right)$. *Then B is lower triangular if and only if each of its diagonal blocks* $B_{11}, B_{22}, ..., B_{cc}$ *is lower triangular.*

Proof. Let B be a lower triangular matrix then

Diagonal of $\mathrm{B} = \mathrm{diag}\left(B_{11}, B_{22}, ..., B_{cc}\right)$ and $\mathrm{B_{ij}}$ can be written as

$$B_{ij} = \begin{bmatrix} b_{n_1+\cdots+n_{i-1}+1, n_1+\cdots+n_{j-1}+1} & \cdots & b_{n_1+\cdots+n_{i-1}, n_1+\cdots+n_j} \\ \vdots & \vdots & \vdots \\ b_{n_1+\cdots+n_i+1, n_1+\cdots+n_{j-1}+1} & \cdots & b_{n_1+\cdots+n_i, n_1+\cdots+n_j} \end{bmatrix}.$$

where $b_{ij} \in B$, since B lower triangular matrix,

thus, $b_{n_1+\cdots+n_{i-1}+1, n_1+\cdots+n_{i-1}+1}, \ldots, b_{n_1+\cdots+n_i, n_1+\cdots+n_i}$ $(i = 1, \cdots, c)$ represent diagonal of B. Thus , $B_{11}, B_{22}, ..., B_{cc}$ will be written as

$$B_{ii} = \begin{bmatrix} b_{n_1+\cdots+n_{i-1}+1, n_1+\cdots+n_{i-1}+1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ b_{n_1+\cdots+n_i+1, n_1+\cdots+n_{i-1}+1} & \cdots & b_{n_1+\cdots+n_i, n_1+\cdots+n_i} \end{bmatrix},$$

where $i = 1, \cdots, c$. Then $B_{11}, B_{22}, ..., B_{cc}$ is lower triangular.

Now, let $B_{11}, B_{22}, ..., B_{cc}$ be lower triangular, thus, B_{ii} can be written as above .Then B is lower triangular. □

Theorem 3.2. *Let*

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{11} & \mathrm{O} \\ \mathrm{O} & \mathrm{A}_{22} \end{bmatrix},$$

Where A_{11} *is* $k \times k$ *and* A_{22} is $(n-k) \times (n-k)$.

If A_{11} *and* A_{22} *are both invertible, then*

$$\mathrm{A}^{-1} = \begin{bmatrix} \mathrm{A}_{11}^{-1} & \mathrm{O} \\ \mathrm{O} & \mathrm{A}_{22}^{-1} \end{bmatrix}.$$

Proof. Since A_{11} and A_{22} are both invertible, then

$$\begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix}\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11}^{-1}A_{11} & O \\ O & A_{22}^{-1}A_{22} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = I$$

and

$$\begin{bmatrix} A_{11} & O \\ O & A_{22} \end{bmatrix}\begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} A_{11}A_{11}^{-1} & O \\ O & A_{22}A_{22}^{-1} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = I.$$

Thus, A is invertible and

$$A^{-1} = \begin{bmatrix} A_{11}^{-1} & O \\ O & A_{22}^{-1} \end{bmatrix}. \qquad \square$$

Theorem 3.3. *Let*

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A, B, C, D \in R^n$, *then,*

$$\det_R \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det_R \begin{bmatrix} -C & -D \\ A & B \end{bmatrix}.$$

Proof. Assume that $det_F\ I_n = 1$, where I_n is the $n \times n$ identity matrix, and observe that

$$\begin{bmatrix} I_n & -I_n \\ O & I_n \end{bmatrix} \begin{bmatrix} I_n & O \\ I_n & I_n \end{bmatrix} \begin{bmatrix} I_n & -I_n \\ O & I_n \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} -C & -D \\ A & B \end{bmatrix}$$

By Theorem 2.3 and since the first three matrices on the left are unitriangular. Thus,

$$\det_R \begin{bmatrix} I_n & -I_n \\ O & I_n \end{bmatrix} = \det_R \begin{bmatrix} I_n & O \\ I_n & I_n \end{bmatrix} = \det_R \begin{bmatrix} I_n & -I_n \\ O & I_n \end{bmatrix} = \det_R\ I_n = 1$$

it follows from this that

$$\det_R \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \det_R \begin{bmatrix} -C & -D \\ A & B \end{bmatrix}. \qquad \square$$

Theorem 3.4. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix}$$

be 3×3 *upper-block triangular matrix. Then*

$$\det\ A = \det\left(A_{11}A_{22}A_{33} - A_{12}A_{33}\right).$$

Proof. By Theorem 2.4

$$\det(\mathrm{A}) \;=\; \det(\alpha_{11}^{(2)})\ \det(\alpha_{22}^{(1)})\ \det(\alpha_{33}^{(0)}),$$

and by Lemma 2.1

$$\alpha_{ij}^{(0)} \;=\; A_{ij}, \qquad \alpha_{ij}^{(1)} \;=\; A_{ij} - A_{i3}A_{33}^{-1}\ A_{31}$$

$$\alpha_{ij}^{(2)} \;= \left[A_{ij} - A_{i3}A_{33}^{-1}\ A_{3j}\right] - \qquad ,$$

$$\left[A_{i2} - A_{i3}A_{33}^{-1}\ A_{32}\right]\ \left[A_{22} - A_{23}A_{33}^{-1}\ A_{32}\right]^{-1}\ \left[A_{2j} - A_{23}A_{33}^{-1}\ A_{3j}\right].$$

$$\begin{aligned} det(A) \;=\; & det\Big(\left[A_{11} - A_{13}A_{33}^{-1}\ A_{31}\right] \\ & -\left[A_{12} - A_{13}A_{33}^{-1}\ A_{32}\right]\left[A_{22} - A_{23}A_{33}^{-1}\ A_{32}\right]^{-1}\left[A_{21} - A_{23}A_{33}^{-1}\ A_{31}\right]\Big) \\ & \times\ det\left(A_{22} - A_{23}A_{33}^{-1}\ A_{32}\right)det\left(A_{33}\right). \\ & = det\left(A_{11} - A_{12}A_{22}\right)\times det\left(A_{22}\right)det\left(A_{33}\right) \\ & = det\left(A_{11}A_{22}A_{33} - A_{12}A_{33}\right). \end{aligned}$$

□

Theorem 3.5. *Let B any n×m matrix, and* $\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}$ *are non singular, then*

$$\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}^{-1} = \begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}$$

Proof.

$$\begin{bmatrix} I & 0 \\ B & I \end{bmatrix}\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}$$

and

$$\begin{bmatrix} I & 0 \\ -B & I \end{bmatrix}\begin{bmatrix} I & 0 \\ B & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}. \qquad \square$$

Theorem 3.6. *Let*

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

and A_{11} , A_{12} , A_{21} , A_{22} *are square matrix. If* A_{22} *is nonsingular, then*

$$BAB^T = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix}.$$

Proof. By Theorem 2.1

$$BA = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{22} & A_{22} \end{bmatrix}$$

implies that

$$BAB^T = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{22} & A_{22} \end{bmatrix}\begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} = \begin{bmatrix} A_{11} - A_{12}A_{22}^{-1}A_{21} & 0 \\ 0 & A_{22} \end{bmatrix} \quad \square$$

Theorem 3.7. *For every matrix in* M_{2n}^{+} *written in blocks of the same size, we have the decomposition:*

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U\begin{bmatrix} \frac{A+B}{2} + R(X) & 0 \\ 0 & 0 \end{bmatrix}U^* + V\begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2} - R(X) \end{bmatrix}V^*$$

for some unitaries U, V $\in M_{2n}$.

Proof. Let $J = \frac{1}{\sqrt{2}}\begin{bmatrix} I & I \\ -I & I \end{bmatrix}$, where *I* is the identity of M_n , *J* is a unitary matrix, and we have:

$$J\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}J^* = \underbrace{\begin{bmatrix} \frac{A+B}{2}+R(X) & \frac{B-A}{2}+\frac{X-X^*}{2} \\ \frac{B-A}{2}-\frac{X-X^*}{2} & \frac{A+B}{2}-R(X) \end{bmatrix}}_{C}$$

Since $R(X)=\frac{X+X^*}{2}$

$$J^*J\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}J^*J = J^*\underbrace{\begin{bmatrix} \frac{A+B}{2}+R(X) & \frac{B-A}{2}+\frac{X-X^*}{2} \\ \frac{B-A}{2}-\frac{X-X^*}{2} & \frac{A+B}{2}-R(X) \end{bmatrix}}_{C}J$$

Factorize C as a square of positive matrices, that is

$$M=\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}=J^*Q^2J$$

Now, decompose Q^2 as in Lemma3.1.1, then

$$M=J^*(T^*T+S^*S)J=J^*(T^*T)J+J^*(S^*S)J\ ,$$

where

$$TT^*=\begin{bmatrix} \frac{A+B}{2}+R(X) & 0 \\ 0 & 0 \end{bmatrix} \quad \textit{and} \quad SS^*=\begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2}-R(X) \end{bmatrix}. \qquad \square$$

Corollary 3.1 *For every matrix in M_{2n}^+ written in blocks of the same size, we have*

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix}\le\frac{1}{2}\left\{U\begin{bmatrix} A+B+\left|X+X^*\right| & 0 \\ 0 & 0 \end{bmatrix}U^*+V\begin{bmatrix} 0 & 0 \\ 0 & A+B+\left|X+X^*\right| \end{bmatrix}V^*\right\}$$

Proof. By the fact $R(X)\le|R(X)|$ and Theorem 3.7

$$\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} = U\begin{bmatrix} \frac{A+B}{2}+R(X) & 0 \\ 0 & 0 \end{bmatrix}U^* + V\begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2}-R(X) \end{bmatrix}V^*$$

$$\leq U\begin{bmatrix} \frac{A+B}{2}+|R(X)| & 0 \\ 0 & 0 \end{bmatrix}U^* + V\begin{bmatrix} 0 & 0 \\ 0 & \frac{A+B}{2}+|R(X)| \end{bmatrix}V^*$$

□

Theorem 3.8. *Let* $A \in \mathbb{C}^{n\times n}$ *then There exist a nonsingular matrix Q such that*

$$A^D = Q\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}$$

Proof. *L*et *A* be of index K then , by Lemma 2.2. There exist a nonsingular matrix Q such that $J = Q^{-1}AQ$, where $Q = \begin{bmatrix} v_s & v_{s-1} & \cdots & v_1 \end{bmatrix}$ and v_i is an eigenvector corresponding to eigenvalue of *A* implies that $A = QJ\,Q^{-1}$.

Now, let

$$M = Q\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}Q^{-1}.$$

Then

$$A^K MA = A \ , MAM = M \ , AM = MA$$

which implies that *M* is drizn inverse of *A*. □

Theorem 3.9. *Let M be a given matrix of form (3) and S = D - CA^DB be the generalized Schur complement of A in M. Then*

$$M^D = \begin{bmatrix} (A-BD^DC)^{-1} & -(A-BD^DC)^D BD^D \\ -S^DCA^D & S^D \end{bmatrix}$$

if

$$C(I-A^DA) = (I-SS^D)C = 0 \ \text{and} \ (I-AA^D)BS^D = A^DB(I-S^DS).$$

Proof.

$$A(I-A^DA) \quad \text{and} \quad (I-S^DS)C$$

are nilpotent operators and since

$$C(I-A^DA)=(I-SS^D)C=0\,, \quad (I-SS^D)CI-A^DA)$$

are nilpotent operators, this implies

$$\begin{bmatrix} A(I-A^DA) & (I-AA^D)B \\ (I-SS^D)CI-A^DA) & S(I-S^DS) \end{bmatrix}$$

are nilpotent operators. Then by Theorem 2.5

$$M^D=\begin{bmatrix} (A-BD^DC)^{-1} & -(A-BD^DC)^D BD^D \\ -S^DCA^D & S^D \end{bmatrix}$$ □

Theorem 3.10. *Let A be a nonsingular matrix, then*

$$\det\begin{bmatrix} B & A \\ A^T & 1 \end{bmatrix}=\det\begin{bmatrix} B & A \\ A^T & 1 \end{bmatrix}=\det\left(B\right)\left(1-A^TB^{-1}A\right).$$

Proof. By Theorem 2.4

$$\det\begin{bmatrix} B & A \\ A^T & 1 \end{bmatrix}=\det\left(B\right)\left(1-A^TB^{-1}A\right)$$

and

$$\det\begin{bmatrix} B & A \\ A^T & 1 \end{bmatrix}=\det\left(B\right)\left(1-A^TB^{-1}A\right)$$

□

Theorem 3.11. *Let A be a square matrix and a block-diagonal with the form*

$$A=\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{bmatrix}$$

where A_{ii} is a square matrix. Then A is nonsingular if and only if $A_{11}, A_{22}, \ldots, A_{rc}$ are nonsingular.

Proof. Let A is nonsingular, Then A^{-1} exist but

$$\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{bmatrix}\begin{bmatrix} A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{-1}_{rr} \end{bmatrix}$$

$$=\begin{bmatrix} A_{11}A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr}A^{-1}_{rr} \end{bmatrix}=\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}$$

$$\begin{bmatrix} A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{-1}_{rr} \end{bmatrix}\begin{bmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_{rr} \end{bmatrix}$$

$$=\begin{bmatrix} A_{11}^{-1}A_{11} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1}A_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{-1}_{rr}A_{rr} \end{bmatrix}=\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \end{bmatrix}$$

Then inverse of A is

$$\begin{bmatrix} A_{11}^{-1} & 0 & \cdots & 0 \\ 0 & A_{22}^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A^{-1}_{rr} \end{bmatrix}$$

This implies $A_{11}^{-1}, A_{22}^{-1}, \ldots, A^{-1}_{rc}$ exists then $A_{11}, A_{22}, \ldots, A_{rc}$ nonsingular. Now, let $A_{11}, A_{22}, \ldots, A_{rc}$ are nonsingular, then A^{-1} exists and A nonsingular.

□

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