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On the Topp-Leone Generalized Power Weibull Distribution: Properties, Applications and Regression

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Abstract

The Topp-Leone generalized power Weibull distribution, which is an extension of generalized power Weibull, is proposed and its properties explored. The failure rate of the proposed distribution exhibits increasing, reversed J, upside-down bathtub, and bathtub shapes. Some statistical properties are obtained: quantile function, moments, moment generating function, incomplete moment, mean and median deviations, mean residual life function, and Lorenz as well as Bonferroni curves. The maximum likelihood estimation approach is deployed to estimate the model parameters. Simulation studies are conducted to evaluate the performance and accuracy of the maximum likelihood estimates of the model parameters. Applications of the model to real datasets are presented. A location-scale regression model is also developed for the proposed model and its application has been demonstrated with a real dataset.

Keywords: Bonferroni, Deviation, Mimicked, Simulation, and Upside-down Bathtub.

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1. Introduction

The classical Weibull distribution by Weibull [34] is one of the examples of lifetime distributions for modeling lifetime data. The most serious drawback of this distribution is that it failed to offer a non-monotone failure rate that is common in survival and reliability studies. To improve the flexibility of its properties, numerous alternative generalized Weibull distributions have been recommended and studied in published works. Mudholkar et al. [19] introduced the exponentiated Weibull distribution. The exponentiated Weibull distribution offers a bathtub-shaped failure rate. Mudholkar et al. [20] proposed and studied a three-parameter generalized Weibull (GW) class that houses distributions with unimodal and bathtub-shaped failure rates. These special distributions in this class are analytically tractable and computationally manageable, the extended Weibull distribution by Ghitany et al. [11], A reduced new modified Weibull distribution due to Almalki [1], generalized Weibull distributions proposed and studied by Lai [17], the flexible Weibull distribution recommended by Bebbington et al. [7], McDonald generalized power Weibull distribution by Sayibu et al. [37], extended cosine generalized power Weibull by Sayibu et al. [28] and the Topp-Leone Generated Weibull distribution by Aryal et al. [4] among others.

Nikulin and Haghighi [25] studied a new generalization of the Weibull distribution by incorporating an additional shape parameter. This distribution is known as the generalized power Weibull distribution (GPW). The random variable X follows the GPW distribution, if the CDF is specified as:

$$G_{gpw}(x) = 1 - e^{\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}, \quad \gamma > 0,\ \theta > 0,\ \gamma > 0 \text{ and } x > 0 \tag{1}$$

The corresponding PDF of GPW is written succinctly as:

$$g_{gpw}(x) = \gamma\theta\lambda\left(1+\theta x^{\lambda}\right)^{\gamma-1} e^{\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]} \tag{2}$$

The survival function of GPW is as follows:

$$\overline{G}_{gpw}(x) = e^{\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]} \tag{3}$$

where θ is a scale parameter, λ and γ are shape parameters. The standard Weibull distribution is contained in GPW, when $\gamma = 1$. According to Nikulin and Haghighi [25], the failure rate function of the generalized power Weibull distribution can be constant, monotone, and non-monotone shaped.

The failure rate function of GPW is read as:

$$hr_{gpw}(x) = \gamma\theta\lambda\left(1+\theta x^{\lambda}\right)^{\gamma-1}$$

The quantile function of GPW is provided as

$$Q_X(u) = \left[\frac{\left\{1-\log\left(1-u\right)\right\}^{\frac{1}{\gamma}}-1}{\theta}\right]^{\frac{1}{\lambda}}$$

In the statistics literature, some extensions of the generalized power Weibull distribution have been studied by many researchers, namely, Selim and Badr [30] introduced the Kumaraswamy generalized power Weibull distribution Selim [29] recommended the generalized power generalized Weibull distribution, Khan [15] studied the transmuted generalized power Weibull distribution and Pena-Ramirez et al. [26] considered the exponentiated power generalized Weibull distribution.
The Topp-Leone (TL) distribution was introduced by Topp and Leone [31]. The distribution did not receive much attention until Nadarajah and Kotz [22] discovered it. Following this, Ghitany et al. [12] provided some reliability measures of the TL distribution, a discourse on kurtosis of the TL distribution was reported by Kotz and Seier [16], Vicaria et al. [33] recommended two-sided generalized Topp and Leone distribution, Al-Zahrani [2] obtained goodness of fit tests for the TL distribution. Sangsanit and Bodhisuwan [27], presented the Topp-Leone generalized exponential (TLGE) distribution as an example of the TLG distribution.
The Topp-Leone generated family (TL-G) was recommended by Sanganti and Bodhisuwan [27]. The CDF is given by:

$$F_{TL}(x) = \left[G(x)\right]^{b}\left[2-G(x)\right]^{b} = \left[1-\left(\overline{G}(x)\right)^{2}\right]^{b}, \quad x \in R, \quad b>0 \tag{4}$$

The corresponding PDF is obtained by differentiating the CDF. We get,

$$f_{TL}(x) = 2bg(x)\overline{G}(x)\left[1-\left(\overline{G}(x)\right)^{2}\right]^{b-1} \tag{5}$$

where $g(x) = G'(x)$ and $\overline{G}(x) = 1-G(x)$

Other important characteristics of lifetime data analysis are the survival and hazard functions. Those functions of the TL-G distribution are in that order

$$S_{TL}(x) = 1 - \left[G(x)\right]^b \left[2 - G(x)\right]^b = 1 - \left[1 - \left(\overline{G}(x)\right)^2\right]^b \tag{6}$$

and

$$hr_{TL}(x) = \frac{2bg(x)\overline{G}(x)\left[1 - \left(\overline{G}(x)\right)^2\right]^{b-1}}{1 - \left[1 - \left(\overline{G}(x)\right)^2\right]^b} \tag{7}$$

Furthermore, Nadarajah and Kotz [22] pointed out that the TL-G distribution offers bathtub shape of the failure rate function when $0 < b < 1$. In addition, if $b \geq 1$ the TL-G distribution has a non-increasing failure rate function. Moreover, the inverse of the cumulative distribution function of the TL-G distribution is:

$$x = F_{TL}^{-1}(u) = 1 - \sqrt{1 - u^{1/b}}$$

where u is distributed as the uniform on the interval (0,1).

In this article, a new generalization of the GPW, the Topp-Leone Generalized Power Weibull (TLGPW) is derived. The mathematical properties are studied. The motivations for deriving the TLGPW are to provide more usefulness and flexibility of the ordinary distribution and to improve its goodness-of-fit in comparison with the well-known distributions in lifetime data analysis.
The structure of the paper is unfolded as follows: Section 2 has been devoted to deriving the CDF and the associated PDF of the proposed distribution, the linear representation of the density function, the survival function, the failure rate function, cumulative hazard, and reversed hazard functions. In Section 3, the statistical properties of the proposed distribution are derived. Estimators for the parameters of the proposed model, namely, the maximum likelihood estimation technique are presented in section 4. Section 5 houses Monte Carlo simulations undertaken to evaluate the finite sample behavior of the estimators. In section 6, the empirical relevance of the TLGPW is illustrated using real datasets. A location-scale regression model and its application are covered under section 7. The concluding remarks of the article are found in section 8.

2. The Topp-Leone Generalized Power Weibull Distribution

In this section, we introduce the PDF and the CDF of TLGPW by inserting the survival function of GPW distribution in equation (3) into equation (4).

$$F_{TLgpw}(x) = \left[1 - \left(e^{\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right)^{2}\right]^{b}, \qquad \gamma > 0 \ , \theta > 0 \ , \gamma > 0 \ \text{ and } \ x > 0$$

$$F_{TLgpw}(x) = \left[1 - e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b} \tag{8}$$

The associated density function is derived by differentiating equation (6) and is given as:

$$f_{TLgpw}(x) = 2b\gamma\theta\lambda x^{\lambda-1}\left(1+\theta x^{\lambda}\right)^{\gamma-1} e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\left[1 - e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b-1} \tag{9}$$

$\gamma > 0$, $\theta > 0$, $\gamma > 0$, $b > 0$ and $x > 0$

Where b, λ, γ are shape parameters and θ is a scale parameter. A random variable X having PDF (7) is denoted as $X \sim TLGPW\left(\theta,\ \gamma,\ \lambda,\ b\right)$.

The proposed distribution reduces to the Topp-Leone Weibull when $\gamma = 1$.
Some of the shapes exhibited by the PDF of the TLGPW are shown in Figure 2. The shapes include right skewed of different kurtosis, nearly symmetric, and decreasing shapes.

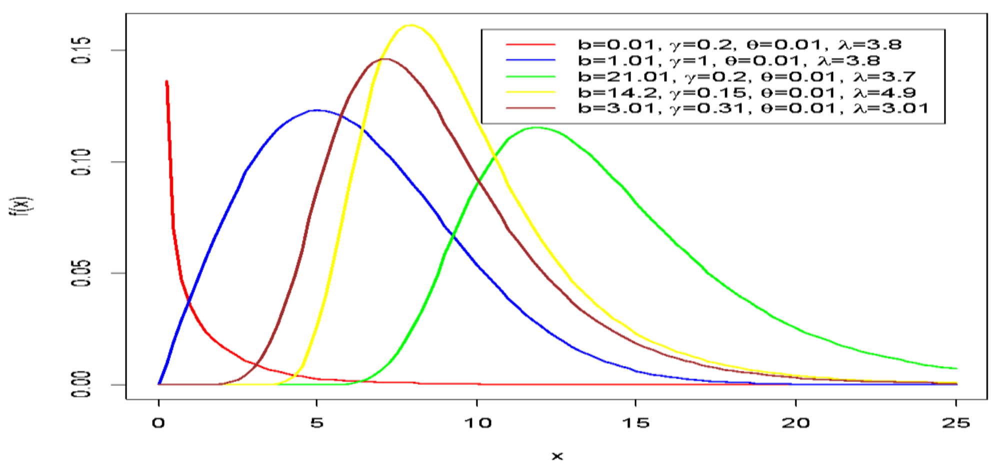

Figure 1: Plots of PDF of TLGPW for various selected values of parameters

2.1 Special models of the TLGPW

A number of essential distributions can be derived as particular cases of the TLGPW for some selected values of the model parameters (γ, λ, b). These distributions are presented in Table 1.

Table 1: Some nested models of TLGPW distribution

Distribution	θ	γ	λ	b	Distribution function	Author(s)
Topp-Leone Weibull	-	1	-	-	$F(x)=\left[1-e^{2\left[-\theta x^{\lambda}\right]}\right]^{b}$	Tuoyo et al. [32]
Topp-Leone generalized power exponential	-	-	1	-	$F(x)=\left[1-e^{2\left[1-(1+\theta x)^{\gamma}\right]}\right]^{b}$	New
Topp-Leone Exponential distribution	-	1	1	-	$F(x)=\left[1-e^{2[-\theta x]}\right]^{b}$	New
Topp-Leone Nadarajah-Haghighi	-	-	1	-	$F(x)=\left[1-e^{2\left[1-(1+\theta x)^{\gamma}\right]}\right]^{b}$	Yousof and Korkmaz [36]
Topp-Leone Rayleigh Distribution	-	1	2	-	$F(x)=\left[1-e^{2\left[-\theta x^{2}\right]}\right]^{b}$	New
Generalized power Weibull distribution	-	-	-	1	$F(x)=1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}$	Nikulin and Haghighi [25]
Weibull Distribution	-	1	-	1	$F=1-\exp\left\{-2\theta x^{\lambda}\right\}$	Weibull [34]
Nadarajah-Haghighi	-	-	1	1	$F(x)=1-e^{2\left[1-(1+\theta x)^{\gamma}\right]}$	Nadarajah and Haghighi [21]
Exponential	-	1	1	1	$F(x)=1-e^{-2\theta x}$	
Generalized power Rayleigh	-	-	2	1	$F(x)=1-e^{2\left[1-\left(1+\theta x^{2}\right)^{\gamma}\right]}$	New

2.2 Survival Function

The survival function of TLGPW is expressed as

$$S_{TLgpw}(x) = 1 - \left[1 - e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]} \right]^{b} \tag{10}$$

2.3 Hazard Function

The hazard function is specified by

$$hr_{TLgpw}(x) = \frac{2b\gamma\left(1+\theta x^{\lambda}\right)^{\gamma-1}\theta\lambda x^{\lambda-1} e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]} \left[1 - e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]} \right]^{b-1}}{1 - \left[1 - e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]} \right]^{b}} \tag{11}$$

The plot of the hazard function is shown in Figure 4. Some of the suitable shapes for the failure rate function include increasing, reversed J-shape, upside-down bathtub, and bathtub shapes

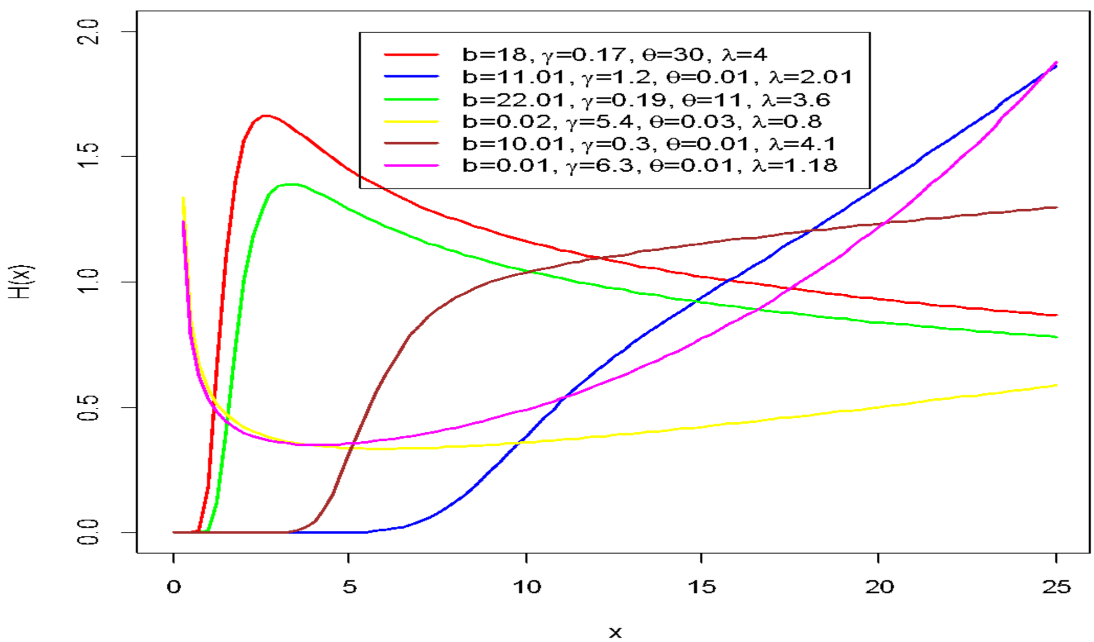

Figure 2: Plots of the failure rate of TLGPW for selected values of the parameters

2.4 Cumulative Hazard

The cumulative hazard is expressed as follows

$$H_{TLgpw}(x) = -\log\left\{1-\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b}\right\} \tag{12}$$

2.5 Reversed Hazard

The reversed hazard is defined by

$$r_{TLgpw}(x) = \frac{2b\gamma\left(1+\theta x^{\lambda}\right)^{\gamma-1}\theta\lambda x^{\lambda-1}e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b-1}}{\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b}} \tag{13}$$

2.6 Expansion for the density function

Several popular classes of distributions can be expressed as infinite or finite weighted series of their baseline distributions (Eugene et al., [10]; Jones, [13]; Cordeiro et al., [8]). This also means that the properties and inferences can be derived from the same measures of its baseline distributions. Expressing the density function in linear representation is a key determinant in deriving useful statistical properties of any new distribution. In this segment, the density function would be written in linear representation form. This expansion is derived using the generalized binomial theorem. For any real number, $r > 0$ and $|D| < 1$, the binomial expansion is

$$(1-D)^{r} = \sum_{i=0}^{\infty}(-1)^{i}\binom{r}{i}D^{i} \tag{14}$$

where

$$\binom{r}{i} = \frac{r(r-1)...(r-i+1)}{i!}$$

Using the binomial expansion (8) in equation (7), we get the PDF as power series expansion as follows

$$\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b-1} = \sum_{i=0}^{\infty}(-1)^{i}e^{2i\left[1-\left(11+\theta x^{\lambda}\right)^{\gamma}\right]} \tag{15}$$

$$f_{TLgpw}(x) = 2b\gamma\sum_{i=0}^{\infty}(-1)^{i}\left(1+\theta x^{\lambda}\right)^{\gamma-1}\theta\lambda x^{\lambda-1}e^{2(i+1)\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}$$

3. Statistical Properties

It is important to derive the statistical properties of any new distribution. The statistical properties of TLGPW distribution will be obtained in this segment. These properties are quantile functions, moments, moment-generating functions, incomplete moments, mean and median deviations, mean residual life, Lorenz and Bonferroni curves and order statistics.

3.1 Quantile Function

The quantile function is explained as the inverse of the cumulative distribution function. The quantile function of a random variable X is obtained by solving the system equation $F_{TLgpw}(x) = p$.

Thus

$$\left[1-\left(e^{\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right)^{2}\right]^{b} = p$$

$$1-\left(e^{\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right)^{2} = p^{\frac{1}{b}}$$

Rearranging gives

$$e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]} = 1-p^{\frac{1}{b}}$$

Taking the natural logarithm of both sides gives

$$2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right] = \log\left(1-p^{\frac{1}{b}}\right)$$

Simplifying further and expressing the equation in terms of x gives the quantile function as

$$Q_X(p) = \left\{\frac{1}{\theta}\left[\left\{1-\frac{1}{2}\log\left(1-p^{\frac{1}{b}}\right)\right\}^{\frac{1}{\gamma}}-1\right]\right\}^{\frac{1}{\lambda}}, \quad 0<p<1 \tag{16}$$

The median of TLGPW can be obtained by inputting $p = 0.5$ into equation (16)

$$Q_X(0.5) = \left\{\frac{1}{\theta}\left[\left\{1-\frac{1}{2}\log\left(1-0.5^{\frac{1}{b}}\right)\right\}^{\frac{1}{\gamma}}-1\right]\right\}^{\frac{1}{\lambda}}$$

Simulating from TLGPW is straightforward. Using the inverse transformation method, we consider the random variable X given by

$$X = \left\{ \frac{1}{\theta} \left[\left\{ 1 - \frac{1}{2} \log \left(1 - u^{\frac{1}{b}} \right) \right\}^{\frac{1}{\gamma}} - 1 \right] \right\}^{\frac{1}{\lambda}}$$

where u is a uniform variate on the unit interval $(0,\ 1)$.

3.2 Moments of TLGPW

The moment and moment-generating functions play important roles in analyzing any distribution functions. Although it is sometimes involving to obtain the moments in explicit forms, they can be obtained as infinite summation of gamma functions. Distinct moments can be easily computed with the aid of any standard mathematical software.

Denote X as a continuous random variable with density function $f(x)$, then the r^{th} non-central moments of a random variable are defined as

$$u'_r = \int_{-\infty}^{\infty} x^r f(x) dx$$

For the TLGPW family of distributions

$$u'_r = \int_{0}^{\infty} x^r f(x) dx$$

Inputting the mixture representation of the density function, we get

$$\begin{aligned} \mu'_r &= 2b\gamma \sum_{i=0}^{\infty} (-1)^i \int_0^{\infty} x^r \left(1 + \theta x^{\lambda}\right)^{\gamma - 1} \theta \lambda x^{\lambda - 1} e^{2(i+1)\left[1 - \left(1 + \theta x^{\lambda}\right)^{\gamma}\right]} dx \\ &= 2b\gamma\theta\lambda \sum_{i=0}^{\infty} (-1)^i \int_0^{\infty} x^r \left(1 + \theta x^{\lambda}\right)^{\gamma - 1} x^{\lambda - 1} e^{2(i+1)\left[1 - \left(1 + \theta x^{\lambda}\right)^{\gamma}\right]} dx \end{aligned}$$

But $e^{2(i+1)\left[1 - \left(1 + \theta x^{\lambda}\right)^{\gamma}\right]} = e^{2(i+1)} e^{-2\left(1 + \theta x^{\lambda}\right)^{\gamma}(i+1)}$, hence we get

$$\mu'_r = 2b\gamma\theta\lambda \sum_{i=0}^{\infty} (-1)^i e^{2(i+1)} \int_0^{\infty} x^r \left(1 + \theta x^{\lambda}\right)^{\gamma - 1} x^{\lambda - 1} e^{-2(i+1)\left(1 + \theta x^{\lambda}\right)^{\gamma}} dx$$

Applying Integration by substitution to simplify μ_r', the following steps are undertaken.
Let

$$y = 2(i+1)\left(1+\theta x^{\lambda}\right)^{\gamma}, \qquad x = \left[\frac{\left(\frac{y}{2(i+1)}\right)^{\frac{1}{\gamma}} - 1}{\theta}\right]^{\frac{1}{\lambda}}$$

Then as $x \to 0$, $y \to 2(i+1)$ and $x \to \infty$, $y \to \infty$

Further,
$$\frac{dy}{dx} = 2(i+1)\gamma\theta\lambda x^{\lambda-1}\left(1+\theta x^{\lambda}\right)^{\gamma-1}$$
$$dx = \frac{dy}{2(i+1)\gamma\theta\lambda x^{\lambda-1}\left(1+\theta x^{\lambda}\right)^{\gamma-1}}$$

Hence

$$\mu_r' = 2b\gamma\theta\lambda\sum_{i=0}^{\infty}(-1)^i e^{2(i+1)}\int_0^{\infty} x^r\left(1+\theta x^{\lambda}\right)^{\gamma-1} x^{\lambda-1} e^{-2(i+1)\left(1+\theta x^{\lambda}\right)^{\gamma}} \frac{dy}{2(i+1)\gamma\theta\lambda x^{\gamma-1}\left(1+\theta x^{\lambda}\right)^{\gamma-1}}$$

$$\mu_r' = \frac{1}{(i+1)} b\sum_{i=0}^{\infty}(-1)^i e^{2(i+1)}\int_{2(i+1)}^{\infty}\left\{\left(\frac{\left(\frac{y}{2(i+1)}\right)^{\frac{1}{\gamma}} - 1}{\theta}\right)^{\frac{1}{\lambda}}\right\}^r e^{-y}dy$$

$$\mu_r' = \frac{1}{(i+1)} b\sum_{i=0}^{\infty}(-1)^i e^{2(i+1)}\theta^{-\frac{r}{\lambda}}\int_{2(i+1)}^{\infty}\left(\left(\frac{y}{2(i+1)}\right)^{\frac{1}{\gamma}} - 1\right)^{\frac{r}{\lambda}} e^{-y}dy$$

Using the generalized form of binomial expansion; $(v+u)^j = \sum_{k=0}^{j}\binom{j}{k} v^{j-k}u^k$, $|v| > |u|$

$$v = \left(\frac{y}{2(i+1)}\right)^{\frac{1}{\gamma}}, \quad u = -1$$

then

$$\left[\left(\frac{y}{2(i+1)}\right)^{\frac{1}{\gamma}}-1\right]^{\frac{r}{\lambda}}=\sum_{k=0}^{\infty}(-1)^{k}\begin{pmatrix}\frac{r}{\lambda}\\k\end{pmatrix}\left(\frac{y}{2(i+1)}\right)^{\frac{1}{\gamma}\left(\frac{r}{\lambda}-k\right)}$$

$$\mu_r'=\frac{1}{(i+1)}b\sum_{i,\,k=0}^{\infty}(-1)^{i+k}e^{2(i+1)}\theta^{-\frac{r}{\lambda}}\begin{pmatrix}\frac{r}{\lambda}\\k\end{pmatrix}\int_{2(i+1)}^{\infty}\left(\frac{y}{2(i+1)}\right)^{\frac{1}{\gamma}\left(\frac{r}{\lambda}-1\right)}e^{-y}dy$$

$$\mu_r'=\frac{1}{(i+1)}b\sum_{i,\,k=0}^{\infty}(-1)^{i+k}e^{2(i+1)}\theta^{-\frac{r}{\lambda}}\begin{pmatrix}\frac{r}{\lambda}\\k\end{pmatrix}\left(\frac{1}{2(i+1)}\right)^{\left(\frac{r-\lambda k}{\gamma\lambda}\right)}\int_{2(i+1)}^{\infty}y^{\left(\frac{r-\lambda k}{\gamma\lambda}\right)}e^{-y}dy$$

$$\mu_r'=\frac{1}{(i+1)}b\sum_{i,\,k=0}^{\infty}(-1)^{i+k}e^{2(i+1)}\theta^{-\frac{r}{\lambda}}\begin{pmatrix}\frac{r}{\lambda}\\k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{r-\lambda k}{\gamma\lambda}\right)-1+1}\int_{2(i+1)}^{\infty}y^{\left(\frac{r-\lambda k}{\gamma\lambda}\right)-1+1}e^{-y}dy$$

$$\mu_r'=\frac{1}{(i+1)}b\sum_{i,\,k=0}^{\infty}(-1)^{i+k}e^{2(i+1)}\theta^{-\frac{r}{\lambda}}\begin{pmatrix}\frac{r}{\lambda}\\k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{r-\lambda(k-\gamma)}{\gamma\lambda}\right)}\int_{2(i+1)}^{\infty}y^{\left(\frac{r-\lambda(k-\gamma)}{\gamma\lambda}\right)}e^{-y}dy$$

$$\mu_r'=\frac{1}{(i+1)}b\sum_{i,\,k=0}^{\infty}(-1)^{i+k}e^{2(i+1)}\theta^{-\frac{r}{\lambda}}\begin{pmatrix}\frac{r}{\lambda}\\k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{r-\lambda(k-\gamma)}{\gamma\lambda}\right)}\Gamma\left[\left(\frac{r-\lambda(k-\gamma)}{\gamma\lambda}\right),\ 2(i+1)\right] \tag{17}$$

3.3 Moment Generating Function

The moment-generating function characterizes the distribution function. The moment generating functions are unique functions deployed to find the moments of a random variable, if it exists. If all the moments of a random variable exist then the moment-generating function of X can be written as

$$M_X(t)=E(e^{tX})\ =\int_0^{\infty}e^{tX}f(x)dx$$

Using Taylor's series of expansion

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\mu_r'$$

$$M_X(t)=\frac{1}{(i+1)}b\sum_{i,\,k,\,r=0}^{\infty}(-1)^{i+k}\frac{t^r}{r!}e^{2(i+1)}\theta^{-\frac{r}{\lambda}}\begin{pmatrix}\frac{r}{\lambda}\\k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{r-\lambda(k-\gamma)}{\gamma\lambda}\right)}\Gamma\left[\left(\frac{r-\lambda(k-\gamma)}{\gamma\lambda}\right),\ 2(i+1)\right] \tag{18}$$

3.4 Incomplete Moment

Incomplete moments can be used to explain not only the shape of a distribution of a random variable, but also play a key role in computing the mean deviation, median deviation, inequality measures, and mean residual life of the distribution of a random. The r^{th} incomplete moment is given as

$$J_r(t) = \int_0^t x^r f(x)dx$$

Using the series expansion of the TLGPW density function, we get

$$J_r(t) = 2b\gamma\theta\lambda\sum_{i=0}^{\infty}(-1)^i e^{2(i+1)}\int_0^t x^r \left(1+\theta x^\lambda\right)^{\gamma-1} x^{\lambda-1}e^{-2(i+1)\left(1+\theta x^\lambda\right)^\gamma} dx$$

Let $y = 2(i+1)\left(1+\theta x^\lambda\right)^\gamma$, $\quad x = \left[\dfrac{\left(\dfrac{y}{2(i+1)}\right)^{\frac{1}{\gamma}} - 1}{\theta}\right]^{\frac{1}{\lambda}}$

Then as $x \to 0$, $y \to 2(i+1)$ and $x \to t$, $y \to 2(i+1)\left(1+\theta t^\lambda\right)^\gamma$

Further, $$\frac{dy}{dx} = 2(i+1)\gamma\theta\lambda x^{\lambda-1}\left(1+\theta x^\lambda\right)^{\gamma-1}$$

$$dx = \frac{dy}{2(i+1)\gamma\theta\lambda x^{\lambda-1}\left(1+\theta x^\lambda\right)^{\gamma-1}}$$

$$J_r(t) = 2b\gamma\theta\lambda\sum_{i=0}^{\infty}(-1)^i e^{2(i+1)}\int_{2(i+1)}^{2(i+1)\left(1+\theta t^\lambda\right)^\gamma} x^r \left(1+\theta x^\lambda\right)^{\gamma-1} x^{\lambda-1}e^{-2(i+1)\left(1+\theta x^\lambda\right)^\gamma} \frac{dy}{2(i+1)\gamma\theta\lambda x^{\gamma-1}\left(1+\theta x^\lambda\right)^{\gamma-1}}$$

After some algebraic manipulations, the r^{th} incomplete moment is given as follows

$$J_r(t) = \frac{1}{(i+1)} b \sum_{i,k,r=0}^{\infty} (-1)^{i+k}\frac{t^r}{r!}e^{2(i+1)}\theta^{-\frac{r}{\lambda}}\begin{pmatrix}\frac{r}{\lambda}\\ k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{r-\lambda(k-\gamma)}{\gamma\lambda}\right)} \times \left[\Gamma\left(\frac{r-\lambda(k-\gamma)}{\gamma\lambda}, 2(i+1)\right) - \Gamma\left(\frac{r-\lambda(k-\gamma)}{\gamma\lambda}, 2(i+1)\left(1+\theta t^\lambda\right)^\gamma\right)\right] \tag{19}$$

3.5 Mean and Median Deviations

The first incomplete moment is a key determinant in obtaining expressions for the mean deviation from the mean as well as the mean deviation from the median. The overall spread of a random variable is explicated by these measures. The mean deviation of a random variable is specified by

$$D(\mu)=\int_0^\infty |x-\mu| f(x)dx \quad = 2\mu F(\mu)-2\int_0^\mu xf(x)dx$$

where $J_1(\mu)=\int_0^\mu xf(x)dx$ is the first incomplete ($r=1$) at $t=\mu$

$$J_1(\mu)=\frac{1}{(i+1)}b\sum_{i,k,r=0}^{\infty}(-1)^{i+k}te^{2(i+1)}\theta^{-\frac{1}{\lambda}}\begin{pmatrix}\frac{1}{\lambda}\\k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda}\right)}\times\left[\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\right)-\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\left(1+\theta\mu^{\lambda}\right)^{\gamma}\right)\right]$$

Therefore, the mean deviation of TLGPW is given as follows

$$D(\mu)=2\mu F(\mu)-\frac{1}{(i+1)}2b\sum_{i,k,r=0}^{\infty}(-1)^{i+k}te^{2(i+1)}\theta^{-\frac{1}{\lambda}}\begin{pmatrix}\frac{1}{\lambda}\\k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda}\right)}\times\left[\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\right)-\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\left(1+\theta\mu^{\lambda}\right)^{\gamma}\right)\right] \tag{20}$$

The median deviation is defined as

$$D(M)=-\mu+2\int_m^\infty xf(x)dx \quad =-\mu+2\left\{J_1(m)\right\}$$

$$D(M)=-\mu+\frac{1}{(i+1)}2b\sum_{i,k,r=0}^{\infty}(-1)^{i+k}te^{2(i+1)}\theta^{-\frac{1}{\lambda}}\begin{pmatrix}\frac{1}{\lambda}\\k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda}\right)}\times\left[\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\right)-\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\left(1+\theta m^{\lambda}\right)^{\gamma}\right)\right] \tag{21}$$

3.6 Mean Residual Life

The mean residual life (MRL) function at time t, indicates the estimated additional life span for a unit alive at time t. The MRL ($t>0$) is defined as

$$\text{MRL}=E\left(X-t \mid X>t\right)=\frac{\mu_1'-J_1(t)}{S(x)}$$

where μ_1' denotes the first non-central moment.

Inserting the relevant terms, we get

$$\mathrm{MRL}=\frac{A\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\right)-B\left[\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\right)-\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\left(1+\theta t^{\lambda}\right)^{\gamma}\right)\right]}{1-\left[1-e^{2\left\{1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right\}}\right]^{b}}-t \tag{22}$$

where

$$A=\frac{1}{(i+1)}b\sum_{i,\,k=0}^{\infty}(-1)^{i+k}e^{2(i+1)}\theta^{-\frac{1}{\lambda}}\begin{pmatrix}\frac{1}{\lambda}\\ k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda}\right)}$$

and

$$B=\frac{1}{(i+1)}b\sum_{i,\,k,\,r=0}^{\infty}(-1)^{i+k}te^{2(i+1)}\theta^{-\frac{1}{\lambda}}\begin{pmatrix}\frac{1}{\lambda}\\ k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda}\right)}$$

3.7 Lorenz and Bonferroni Curves

The Lorenz and Bonferroni curves are tools deployed in the measurement of, for example, income inequalities. The Lorenz curve is defined as

$$Lq=\frac{1}{\mu}\int_0^t xf(x)dx=\frac{1}{\mu}J_1(t)$$

Substituting, we obtain the Lorenz curve of TLGPW as follows

$$Lq=\frac{1}{\mu(i+1)}b\sum_{i,\,k,\,r=0}^{\infty}(-1)^{i+k}te^{2(i+1)}\theta^{-\frac{1}{\lambda}}\begin{pmatrix}\frac{1}{\lambda}\\ k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda}\right)}\times\left[\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\right)-\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\left(1+\theta t^{\lambda}\right)^{\gamma}\right)\right] \tag{23}$$

Also, the expression for the Bonferroni curve is obtained by

$$Bq=\frac{Lq}{F(x)}$$

Inserting the relevant terms, the Bonferroni curve for the TLGPW family of distributions is given as

$$Bq=\frac{\frac{1}{\mu(i+1)}b\sum_{i,\,k,\,r=0}^{\infty}(-1)^{i+k}te^{2(i+1)}\theta^{-\frac{1}{\lambda}}\begin{pmatrix}\frac{1}{\lambda}\\ k\end{pmatrix}\left(2(i+1)\right)^{-\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda}\right)}\times\left[\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\right)-\Gamma\left(\frac{1-\lambda(k-\gamma)}{\gamma\lambda},2(i+1)\left(1+\theta t^{\lambda}\right)^{\gamma}\right)\right]}{\left[1-e^{2\left\{1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right\}}\right]^{b}} \tag{24}$$

3.8 Order Statistics

Order statistics feature prominently in distribution theory and practice. Suppose $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ be a haphazard sample of size n, then the PDF of the k^{th} order statistic is given by

$$f_{k:n}(x) = \frac{n!}{(n-k)!(k-1)!} f(x)\left[F(x)\right]^{k-1}\left[1-F(x)\right]^{n-k}$$

For the TLGPW family of distributions, the PDF is given by

$$f_{k:n}(x) = \frac{n!}{(n-k)!(k-1)!} f(x)\left\{1-\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b}\right\}^{n-k}\left\{\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b}\right\}^{k-1}$$

The PDF of the largest order statistics is obtained by inserting $k = n$

$$f_{n:n}(x) = \frac{n!}{(n-n)!(n-1)!} f(x)\left\{1-\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b}\right\}^{n-n}\left\{\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b}\right\}^{n-1}$$

$$f_{n:n}(x) = nf(x)\left\{\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b}\right\}^{n-1}$$

$$f_{n:n}(x) = 2nb\gamma\left(1+\theta x^{\lambda}\right)^{\gamma-1}\theta\lambda x^{\lambda-1}e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b-1}\left\{\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b}\right\}^{n-1} \tag{25}$$

The PDF for the smallest order statistics ($k = 1$) for the TLGPW is

$$f_{1:n}(x) = \frac{n!}{(n-1)!(1-1)!} f(x)\left[F(x)\right]^{1-1}\left[1-F(x)\right]^{n-1}$$

$$f_{1:n}(x) = nf(x)\left[1-F(x)\right]^{n-1}$$

Substituting we obtained

$$f_{1:n}(x) = 2nb\gamma\left(1+\theta x^{\lambda}\right)^{\gamma-1}\theta\lambda x^{\lambda-1}e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b-1}\left\{1-\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b}\right\}^{n-1} \tag{26}$$

4. Parameter Estimation

In this section, the unknown parameters of the TLGPW class of distributions were estimated using the maximum likelihood estimation technique.

4.1 Maximum Likelihood Estimation

Suppose $x_1, x_2, ..., x_n$ denote a random sample of complete data from the TLGPW distribution. The likelihood function is given as

$$L = \prod_{i=1}^{n} f_{TLgpw}(x) \tag{27}$$

Inserting equation (7) into equation (27), we have

$$L = \prod_{i=1}^{n} \left\{ 2b\gamma \left(1+\theta x^{\lambda}\right)^{\gamma-1} \theta\lambda x^{\lambda-1} e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]} \left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right]^{b-1} \right\}$$

The log-likelihood function for the TLGPW is given as

$$\ell = n\log(2b\gamma\theta\lambda) + (\gamma-1)\sum_{i=1}^{n}\log\left(1+\theta x^{\lambda}\right) + (\lambda-1)\sum_{i=1}^{n}\log x + 2\sum_{i=1}^{n}\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right] + (b-1)\sum_{i=0}^{n}\log\left[1-e^{2\left[1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right]}\right] \tag{28}$$

To obtain the MLE of the parameters, we maximize the score function by taking the first partial derivative of the equation (28). Thus

$$\frac{\partial \ell}{\partial b} = \frac{n}{b} + \sum_{i=1}^{n}\log\left[1-e^{2\left\{1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right\}}\right] \tag{29}$$

$$\frac{\partial \ell}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n}\log\left(1+\theta x^{\lambda}\right) - 2\sum_{i=1}^{n}\left(1+\theta x^{\lambda}\right)^{\gamma}\log\left(1+\theta x^{\lambda}\right) + 2(b-1)\sum_{i=1}^{n}\frac{\left(1+\theta x^{\lambda}\right)^{\gamma}\log\left(1+\theta x^{\lambda}\right)e^{2\left\{1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right\}}}{\left[1-e^{2\left\{1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right\}}\right]} \tag{30}$$

$$\frac{\partial \ell}{\partial \lambda}=\frac{n}{\lambda}+(\gamma-1)\sum_{i=1}^{n}\frac{\theta\lambda x^{\lambda-1}}{\left(1+\theta x^{\lambda}\right)}+\sum_{i=1}^{n}\log x-2\gamma\theta\lambda\sum_{i=1}^{n}x^{\lambda-1}\left(1+\theta x^{\lambda}\right)^{\gamma-1}+2\gamma\theta\lambda(b-1)\sum_{i=1}^{n}\frac{x^{\lambda-1}\left(1+\theta x^{\lambda}\right)^{\gamma-1}e^{2\left\{1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right\}}}{\left[1-e^{2\left\{1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right\}}\right]} \tag{31}$$

$$\frac{\partial \ell}{\partial \theta}=\frac{n}{\theta}+(\gamma-1)\sum_{i=1}^{n}\frac{x^{\lambda}}{\left(1+\theta x^{\lambda}\right)}-2\gamma\sum_{i=1}^{n}x^{\lambda}\left(1+\theta x^{\lambda}\right)^{\gamma-1}+2\gamma(b-1)\sum_{i=1}^{n}\frac{x^{\lambda}\left(1+\theta x^{\lambda}\right)^{\gamma-1}e^{2\left\{1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right\}}}{\left[1-e^{2\left\{1-\left(1+\theta x^{\lambda}\right)^{\gamma}\right\}}\right]} \tag{32}$$

The maximum likelihood estimates of b, γ, λ and θ are the simultaneous

solutions of the equations $\frac{\partial \ell}{\partial b}=0$, $\frac{\partial \ell}{\partial \gamma}=0$, $\frac{\partial \ell}{\partial \lambda}=0$ and $\frac{\partial \ell}{\partial \theta}=0$.

These equations cannot be solved analytically and statistical software can be used to solve them numerically by using iterative techniques.

5. Monte Carlo Simulation Study

A simulation study was carried out to evaluate the performance and accuracy of the estimating technique, namely, the maximum likelihood estimation procedure. Monte Carlo experiments were done based on generated data from the TLGPW distribution. By using the inversion method, 1000 random samples of size $n=30$, 50, 80, 120, 200 and 250 from TLGPW distribution for different combinations of the parameter values for: $b=0.2$, $\gamma=3.4$, $\lambda=1.8$ and $\theta=0.6$, and $b=0.8$, $\gamma=1.4$, $\lambda=0.9$ and $\theta=0.3$ were generated. The following quantities were computed in this study:

Average bias (AB) of the MLE $\hat{\psi}$ of the parameter $\hat{\psi}=b,\ \gamma,\ \lambda,\ \theta$

$$\frac{1}{N}\sum_{i=1}^{n}(\hat{\psi}-\psi)$$

Root mean squared (RMSE) of the MLE, $\hat{\psi}$ of the parameter $\hat{\psi} = b,\ \gamma,\ \lambda,\ \theta$

$$\sqrt{\frac{1}{N}\sum_{i=1}^{n}(\hat{\psi}-\psi)^{2}}$$

Table 2 presents the average bias and RMSE values of the parameters $b,\ \gamma,\ \lambda,\ \theta$ for various sample sizes. The outcomes in Table 2 show declining AB and RMSE as the sample size increases.

Table 2: Monte Carlo Simulation results of average bias and RMSE

		I		II	
Parameter	**n**	**AB**	**RMSE**	**AB**	**RMSE**
	30	0.2928	0.6962	0.5014	0.6467
	50	0.2413	0.5129	0.4945	0.4774
	80	0.1684	0.5046	0.3796	0.5506
$\hat{b}$	120	0.1058	0.3584	0.3536	0.3662
	150	0.0742	0.2265	0.3236	0.2939
	200	0.0326	0.1310	0.2528	0.2727
	250	0.0156	0.1099	0.2287	0.2484
	30	4.5258	12.5752	1.8620	0.9389
	50	0.6759	1.0312	0.8529	0.9148
	80	0.6590	0.9215	0.7247	0.8154
γ	120	0.4295	0.8211	0.7130	0.7795
	150	0.3574	0.6699	0.6635	0.7797
	200	0.3311	0.5370	0.5876	0.6088
	250	0.3263	0.4175	0.4926	0.5712
	30	0.7455	1.2038	2.5201	5.0567
	50	0.4805	0.7455	1.0371	2.0297
	80	0.3581	0.5598	0.6704	1.0594
θ	120	0.2272	0.4152	0.3492	0.7542
	150	0.1896	0.3623	0.2339	0.4239
	200	0.1617	0.2227	0.2269	0.3228
	250	0.1565	0.1985	0.2106	0.2958
	30	0.6298	0.7620	1.1639	1.3801
	50	0.5130	0.6758	0.9944	1.1910
	80	0.4701	0.5794	0.9189	1.1854
λ	120	0.3830	0.4264	0.7506	0.8946
	150	0.3040	0.4439	0.7372	0.8757
	200	0.2048	0.3868	0.6984	0.7969
	250	0.1530	0.2879	0.6812	0.6138

6. Applications

In this section, the flexibility of the proposed model is examined and compared with some existing models. These models are odd generalized exponential Weibull (OGEW) (Codeiro et al., [8]), generalized power Weibull (GPW) (Bagdonavicius and Nikulin, [6]), new Weibull Pareto (NWP) (Nasiru and Luguterah, [23]), generalized odd inverse exponentiated exponential (GOIEE) (Yakubu et al., [35]), Weibull (Weibull, [34]), and Kumaraswamy generalized power Weibull (KGPW), (Selim and Badr, [30]), the rest are generalized odd inverse Weibull (GOIEW) (Yakubu et al., [35]), generalized odd inverse Rayleigh (GOIER) (Yakubu et al., [35]), inverse Weibull (IW) (Johnson et al., [14]), and generalized odd inverse Lomax (GOIEL) (Yakubu et al., [35]).

6.1 Single Carbon Fiber Dataset

The first dataset represents the strength measured in GPA, for single carbon fibers which were tested under tension at gauge lengths of 1, 10, 20, and 50mm. for illustration purposes, the 20mm data is considered to have a sample size of 63. The dataset can be found in Badar and Priest [5] and was also used by Selim and Badr [30]. The observations are as follows:
1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, 5.020.

6.2 Bladder Cancer Dataset

The second dataset shows the remission time (in months) of a random sample of 128 bladder cancer patients. The dataset can be found in Lee and Wang [18] and was recently used by Yakubu et al., [35] and the observations include:
0.08, 6.97, 2.46, 9.74,3.88, 15.96, 4.26, 79.05, 11.79, 8.37, 12.07, 2.09, 9.02, 3.64, 14.76, 5.32, 36.66,5.41, 1.35, 18.10, 12.02,21.73,3.48, 13.29, 5.09, 26.31, 7.39, 1.05, 7.63, 2.87, 1.46, 2.02, 2.07, 4.87, 0.40, 7.2,6, 0.81, 10.34, 2.69, 17.12, 5.62, 4.40, 3.31, 3.36, 6.94, 2.26, 9.47, 2.62, 14.83, 4.23, 46.12, 7.87, 5.85, 4.51, 6.93, 8.66, 3.57, 14.24,3.82, 34.26,5.41, 1.26, 11.64, 8.26, 6.54, 8.65, 13.11, 5.06, 25.82, 5.32, 0.90, 7.62, 2.83, 17.36, 11.98, 8.53, 12.63, 23.63, 7.09, 0.51, 7.32, 2.69 10.75, 4.33, 1.40, 19.13 12.03, 22.69, 0.20, 9.22, 2.54, 10.06, 4.18, 16.62, 5.49,3.02, 1.76 20.28, 2.23, 13.80,3.70, 14.77, 5.34, 43.01, 7.66, 4.34, 3.25, 2.02, 3.52, 25.74, 5.17 32.15, 7.59, 1.19, 11.25, 5.71, 4.50, 3.36, 4.98, 0.50, 7.28, 2.64, 10.66, 2.75, 17.14, 7.93, 6.25, 6.76.

The descriptive statistics of the two datasets are given in Table 3. It can be observed that the mean and median strengths of the carbon fibers are 3.0593 and 2.9960 respectively. Since the mean is greater than the median, it implies the set is rightly skewed and this is manifested by a skewness value of 0.65. The standard deviation is 0.6209. The dataset is less peaked as compared to the normal distribution since it shows a kurtosis value of 0.41.

On the other hand, the mean of the cancer data is 9.366 with a standard deviation value of 10.508. The median is 6.395. The data is highly skewed to the right with a skewness value of 3.33. The dataset is also highly peaked as compared to the normal distribution since it has a kurtosis value of 16.15. The details are shown in Table 3.

Table 3: Descriptive statistics of Carbon and Cancer datasets

	Mean	**Median**	**Standard Deviation**	**skewness**	**Kurtosis**
Carbon data	3.0593	2.9960	0.6209	0.65	0.41
Cancer data	9.3660	6.3950	10.5080	3.3300	16.1500

The Total Test on Time transform (TTT) plot for the carbon dataset is shown in Figure 3. The carbon dataset has an increasing failure rate since the curve is above the diagonal.

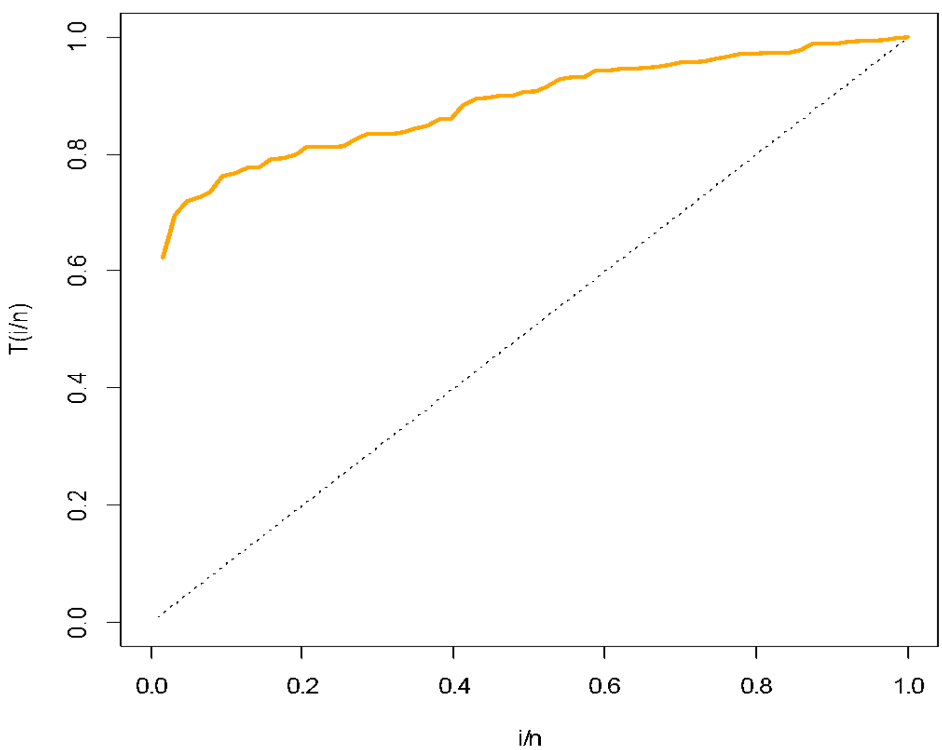

Figure 3: TTT transform plot for the carbon dataset

The TTT transform plot for the bladder cancer dataset is shown in Figures 4. The bladder cancer dataset exhibits a bathtub shape since the curve initially goes above the diagonal and then goes below it.

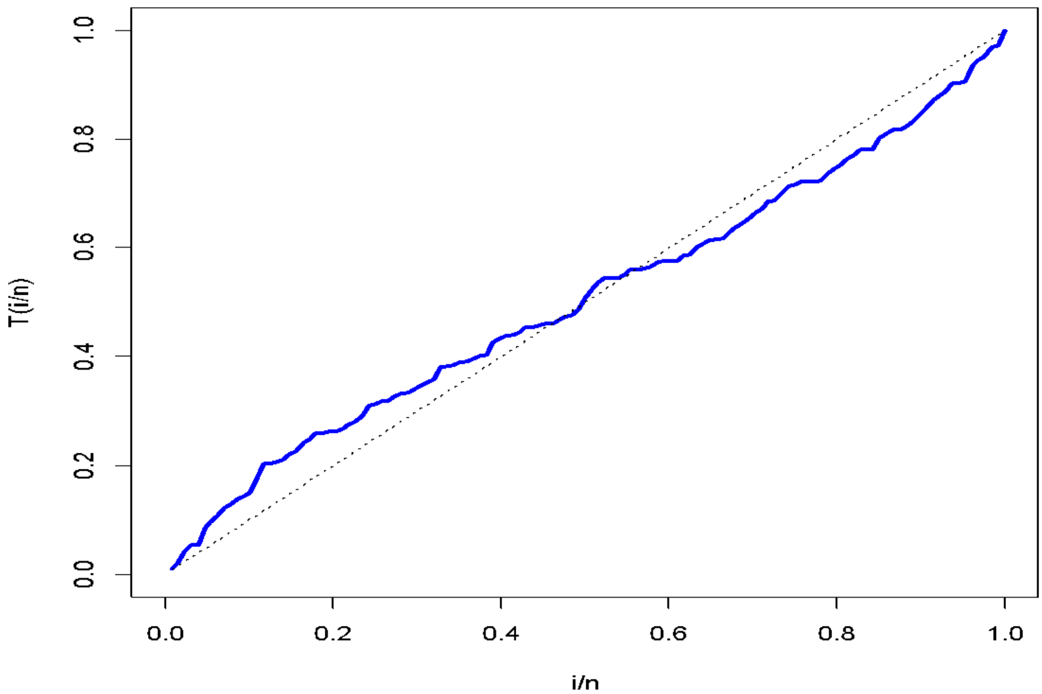

Figure 4: TTT transform plot for the bladder cancer dataset

Table 4 shows the parameter estimate and their corresponding standard errors in brackets for the carbon dataset.

Table 4: Parameters estimate for the carbon dataset

Distribution	$\hat{b}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$
TLGPW	5.5668 (2.0086)	0.2608 (0.2357)	0.0433 (0.0328)	5.3909 (3.4404)
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$
OGEW	24.3971 (0.7571)	0.0575 (0.3595)	0.3808 (0.5270)	2.7271 (5.6418)
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\beta}$	
GPW	5.9558 (1.3943)	0.5341 (0.1828)	0.0025 (0.0029)	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	
NWP	5.0494 (0.4557)	0.8766 (306.1422)	3.2294 (223.3524)	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	
GOIEE	0.5232 (0.0651)	0.7194 (0.4392)	0.1328 (0.0830)	
	$\hat{\alpha}$	$\hat{\gamma}$		
Weibull	5.0494 (0.4557)	0.3017 (0.0080)		

Table 5 shows the parameter estimate and their corresponding standard errors in brackets for the bladder cancer dataset.

Table 5: Parameters estimate for the bladder cancer dataset

Distribution	$\hat{b}$	$\hat{\gamma}$	$\hat{\theta}$	$\hat{\lambda}$
TLGPW	0.5116 (0.3232)	0.1994 (0.1020)	0.0093 (0.0271)	2.5572 (1.2758)
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\theta}$
GOIEW	0.5624 (0.2289)	1.4505 (1.1441)	0.1927 (0.1857)	0.8556 (0.2191)
	$\hat{\lambda}$	$\hat{\theta}$	$\hat{\beta}$	$\hat{\gamma}$
GOIEL	0.4963 (0.0630)	1.6500 (0.7664)	0.0291 (0.0277)	5.9006 (3.7475
	$\hat{\alpha}$	$\hat{\beta}$		
IW	2.4310 (0.2193)	0.7521 (0.0424)		
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	
GOIEE	0.5232 (0.0651)	0.7194 (0.4392)	0.1328 (0.0830)	
	$\hat{\alpha}$	$\hat{\gamma}$		
GOIER	0.5919 (0.0800)	2.4962 (1.1523)	0.1551 (0.0561)	

The performance of the fitted distributions is compared using log-likelihood ($\hat{\ell}$) Akaike information criteria (AIC), corrected Akaike information criteria (AICc), Bayesian information criteria (BIC), and Kolmogorov-Smirnov (K-S) goodness-of-fit measure. In general, the higher the values of the log-likelihood and smaller values of the AIC, AICc, BIC, and K-S of a particular model, the better the fit of the model to the dataset under consideration.

The log-likelihood, goodness of fit statistics, and information criteria of the fitted distributions have been examined and the results are presented in Tables 6 and 7 for the two datasets. The proposed model which is in bold has fitted the two datasets better than the other competing models according to the criteria given above.

Table 6: Log-likelihood and information criteria statistics for the carbon dataset

Distribution	ℓ	AIC	AICC	BIC	K-S	P-Value
TLGPW	**-56.29**	**120.5728**	**121.2624**	**129.9444**	**0.0838**	**0.7680**
KGPW	-56.35	127.914	128.114	133.4157	0.0842	0.7237
NWP	-61.96	129.9140	130.3207	136.3434	0.0876	0.7192
GPW	-59.92	125.8376	126.2444	132.2670	0.0984	0.5750
OGEW	-56.61	121.2176	121.9073	129.7901	0.0860	0.7402
Weibull	-61.96	127.9140	128.1140	132.2002	0.0876	0.7192

Table 7: Log-likelihood and information criteria statistics for the bladder cancer dataset

Distribution	ℓ	AIC	AICC	BIC	K-S	P-Value
TLGPW	**-409.36**	**826.7206**	**827.0458**	**838.1287**	**0.0316**	**0.9995**
GOIEW	-414.57	837.1456	837.4708	848.5535	0.0842	0.3237
GOIEL	-413.00	835.3095	835.6347	846.7176	0.0904	0.2838
IW	-444.00	892.0015	892.0975	897.7056	0.1408	0.0125
GOIER	-418.18	842.3567	842.5503	850.9128	0.1186	0.0548
GOIEE	-430.91	867.8152	868.0087	876.3713	0.1253	0.0359

Also, the variance-covariance matrices for the parameter estimates of the TPGPW for the carbon and bladder cancer datasets are respectively given as follows;

$$T^{-1}\begin{pmatrix} 4.0344 & -0.4697 & -0.0585 & 6.9103 \\ -0.4697 & 0.0555 & 0.0064 & -0.8044 \\ -0.0585 & 0.0064 & 0.0011 & -0.1003 \\ 6.9103 & -0.8044 & -0.1003 & 11.8363 \end{pmatrix}$$

The variances of the maximum likelihood estimates of the parameters of the TPGPW for the carbon data are: $\operatorname{var}\left(\hat{b}\right)=4.0344$, $\operatorname{var}\left(\hat{\gamma}\right)=0.0555$, $\operatorname{var}\left(\hat{\theta}\right)=0.0011$, and $\operatorname{var}\left(\hat{\lambda}\right)=11.8363$.

The 95% confidence intervals for the parameters b,γ,θ and λ of the TLGPW are estimated and presented respectively as follows: (1.6299,9.5037), (0, 0.7228), (0, 0.1076), and (0, 12.1341).

Furthermore, the variance-covariance matrix for the parameter estimates of the TLGPW for the bladder cancer data is given as follows;

$$J^{-1}\begin{pmatrix} 0.1044 & 0.0318 & 0.0087 & -0.3988 \\ 0.0318 & 0.0120 & 0.0027 & -0.0134 \\ 0.0087 & 0.0027 & 0.0007 & -0.0339 \\ -0.3988 & -0.0134 & -0.0339 & 1.6276 \end{pmatrix}$$

The variances of the maximum likelihood estimates of the parameters of the TLGPW for the bladder cancer data are: $\operatorname{var}\left(\hat{b}\right)=0.1044$, $\operatorname{var}\left(\hat{\gamma}\right)=0.0120$, $\operatorname{var}\left(\hat{\theta}\right)=0.0007$, and $\operatorname{var}\left(\hat{\lambda}\right)=1.6276$.

The 95% confidence intervals for the parameters b,γ,δ and λ of the TLGPW are estimated and presented respectively as follows: (0, 1.1451), (0, 0.3993), (0, 0.0624), and (0.5663, 5.0578).
The density plots for the single carbon fiber and bladder cancer datasets are presented in Figures 5 and 6 respectively. In both Figures 5 and 6, the TLGPW mimicked the empirical density and cumulative density better than the other competing models.

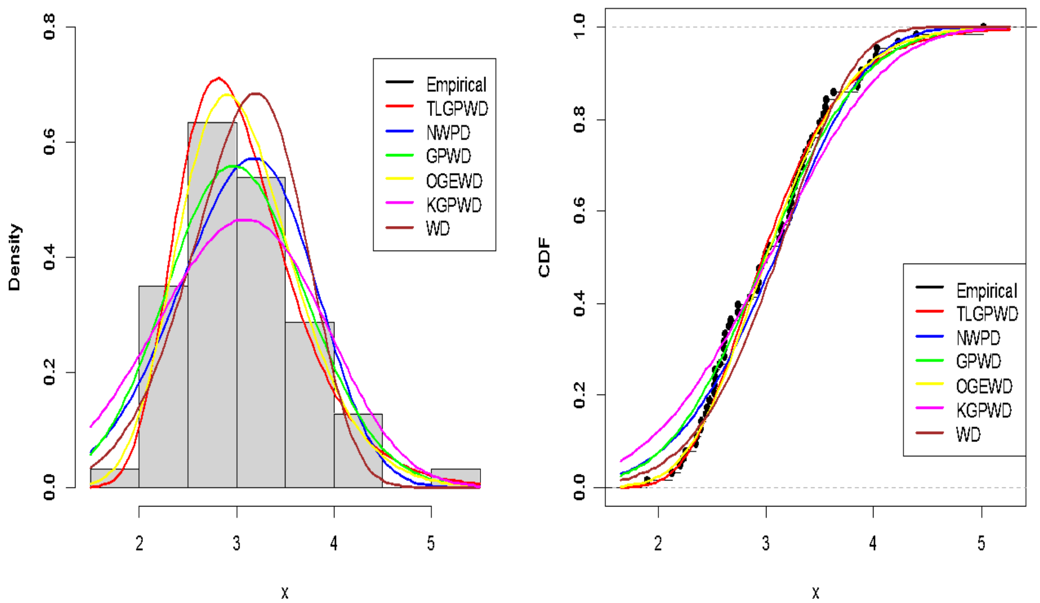

Figure 5: PDF and CDF plots for the carbon data

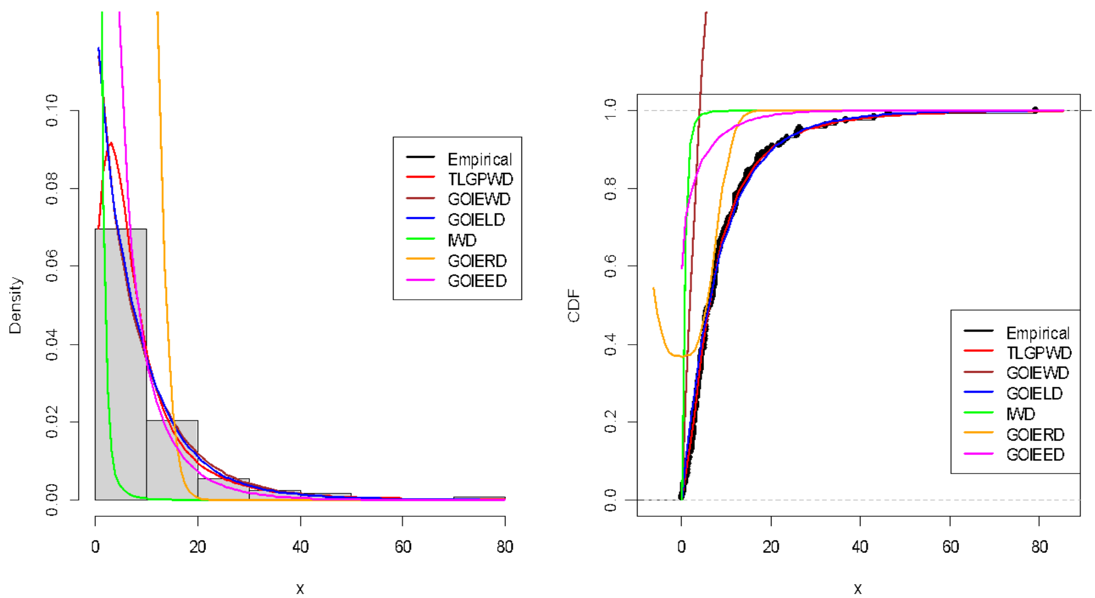

Figure 6: PDF and CDF plots of the bladder cancer data

Figure 7 shows the probability-probability (P-P) plots for the carbon dataset of the TLGPW and the other competing models. The plots indicate that the TLGPW fits the carbon dataset better than the competing models since it has almost all of its points along the diagonal line.

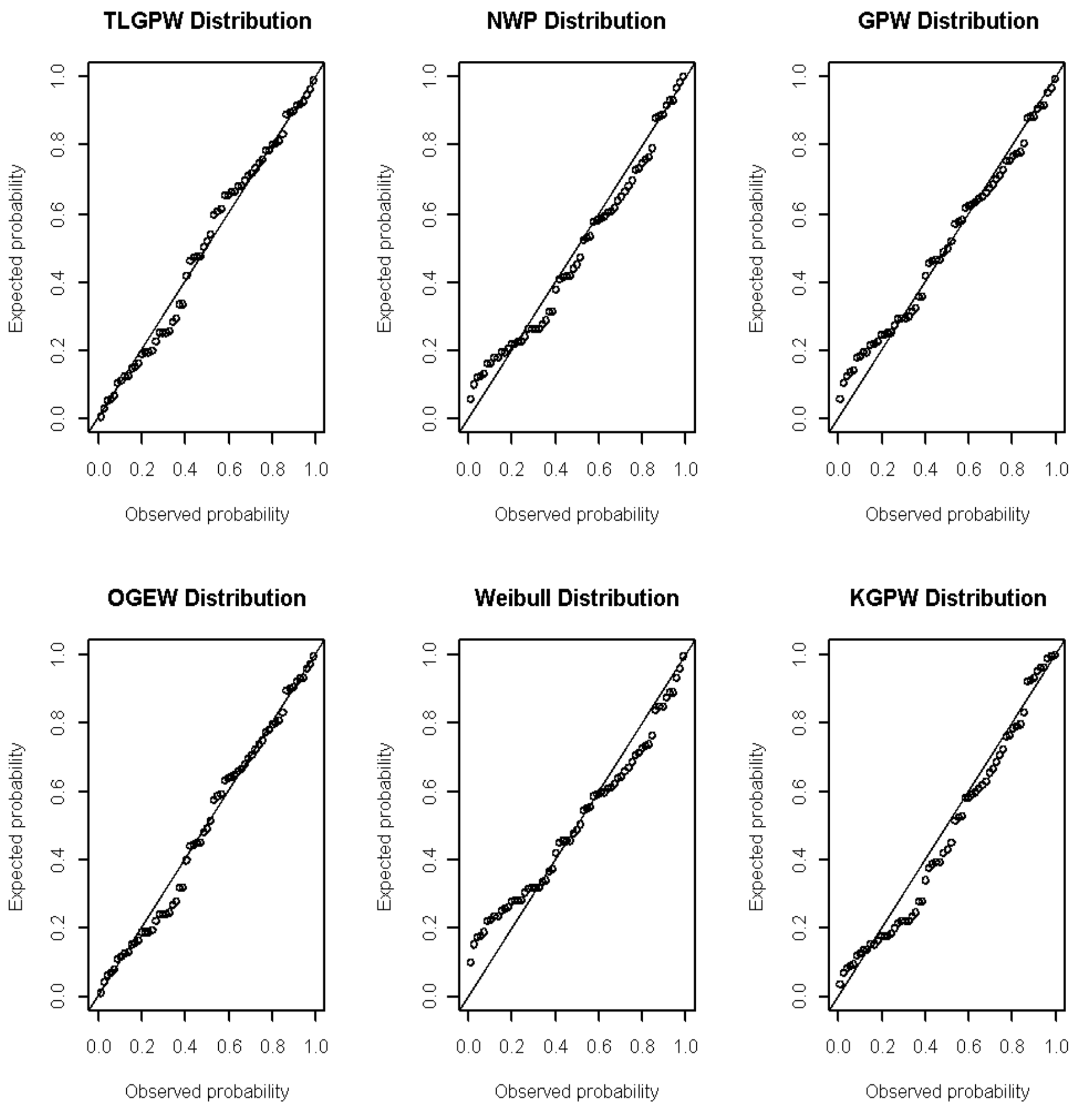

Figure 7: P-P plots of the competing models for the carbon data

Figure 8 shows P-P plots for the bladder cancer dataset of the TLGPW and the other competing models. The plots show that the TLGPW fits the bladder cancer dataset better than the competing models since it has almost all of its points along the diagonal line.

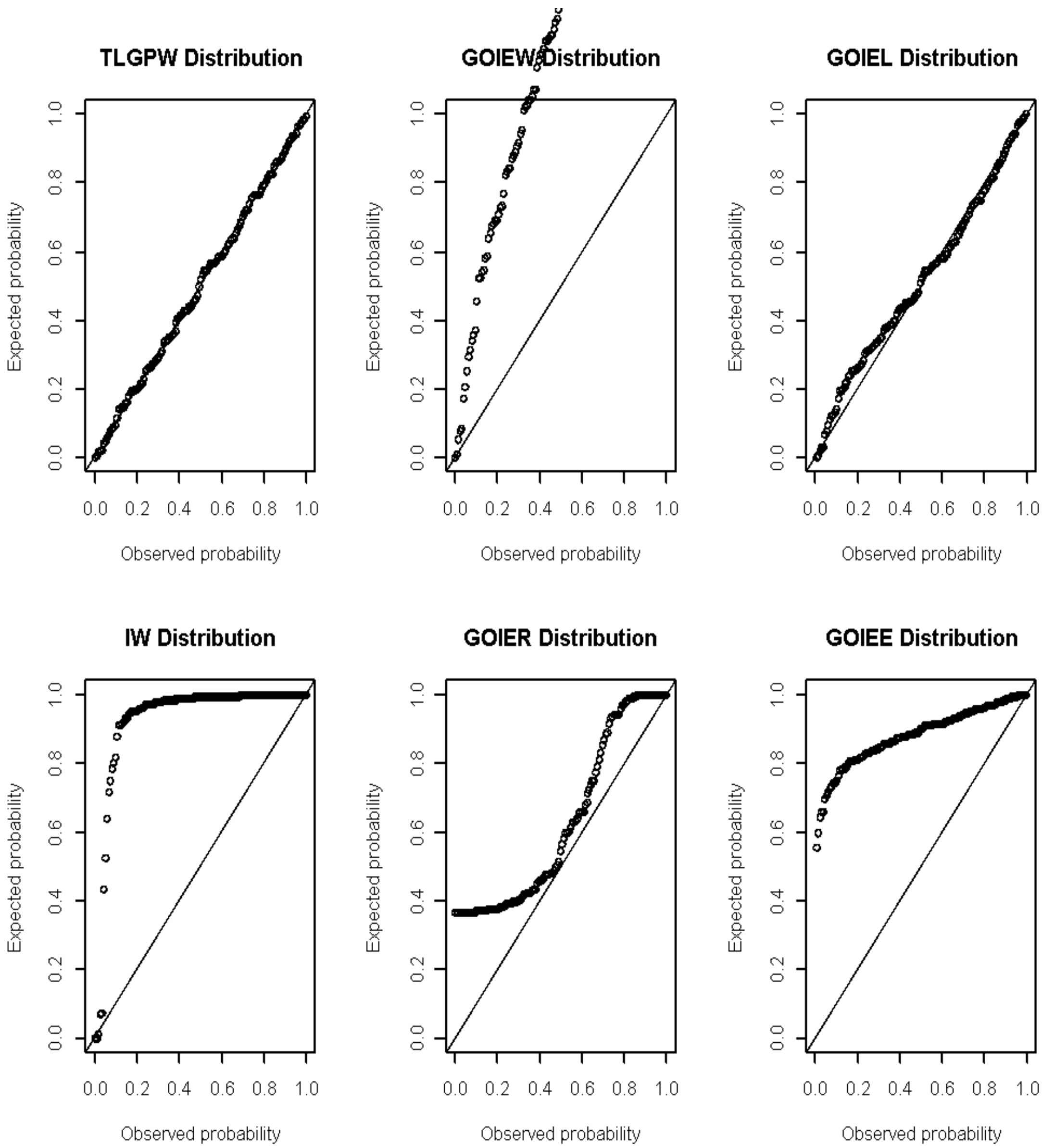

Figure 8: P-P plots of the competing models for the bladder cancer data

7. Log TLGPW Location-Scale (LTLGPW) Regression model

This section covers the log TLGPW location-scale regression model. Given that the random variable X follows the TLGPW, then the random variable $y=\log(X)$ follows the log TLGPW (LTLGPW). Assuming that $\theta=e^{-\mu/\sigma}$ and $\lambda=\frac{1}{\sigma}$, then the density function of the LTLGPW regression model is given by

$$f(y;\xi)=\frac{2b\gamma\exp(z)(\tau)^{\gamma-1}\exp\left(2\left(1-(\tau)^{\gamma}\right)\right)\left[1-\exp\left(2\left(1-(\tau)^{\gamma}\right)\right)\right]^{b-1}}{\sigma} \tag{33}$$

Where $\tau=1+\exp(z)$, $\xi=b,\gamma,\sigma,\mu$, $z=\frac{y-\mu}{\sigma}$, $b>0$ and $\gamma>0$ are shape parameters, $\sigma>0$ is a scale parameter and $\mu\in\mathbb{R}$ is a location parameter. The density of the LTLGPW regression model is defined for $y\in\mathbb{R}$.

The CDF corresponding to equation (33) is given by

$$F(y;b,\gamma,\sigma,\mu)=\left(1-\exp\left(1-\left(1+\exp\left(\tfrac{y-\mu}{\sigma}\right)\right)^{\gamma}\right)\right)^{b},\ y\in\mathbb{R} \tag{34}$$

The survival function of the LTLGPW regression model is given by

$$S(y;b,\gamma,\sigma,\mu)=1-\left(1-\exp\left(1-\left(1+\exp\left(\tfrac{y-\mu}{\sigma}\right)\right)^{\gamma}\right)\right)^{b},\ y\in\mathbb{R} \tag{35}$$

Using the density function in equation (33), the LTLGPW location-scale regression model is defined with the following regression feature.

$$y_i=q_i^{\tau}\phi+\sigma z_i,\ i=1,2,3,\ldots,n$$

where $\mu=q_i^{\tau}\phi$ is the location parameter depending on a set of covariates, $\phi=(\phi_0,\phi_1,\ldots,\phi_k)'$ are the regression parameters, k represents the number of covariates, $q_i=(q_{i1},q_{i2},\ldots,q_{ik})'$ represents the covariates and z_i is the random error term which follows the density function as defined in equation (33). The parameters of the location-scale regression model are estimated using the maximum likelihood estimation technique.

The log-likelihood function of the LTLGPW regression model is

$$\ell = n\log\left(\frac{2b\gamma}{\sigma}\right)+\sum_{i=1}^{n} z_i+(\gamma-1)\sum_{i=1}^{n}\log\left(1+\exp\left(z_i\right)\right)+2\left(1-\left(1+\exp\left(z_i\right)\right)^{\gamma}\right)$$
$$+(b-1)\sum_{i=1}^{n}\log 2\left(1-\exp\left(1-\left(1+\exp\left(z_i\right)\right)^{\gamma}\right)\right), y\in \mathbb{R} \tag{36}$$

where $z_i=\left(y_i-q_i^{\tau}\phi\right)/\sigma$ and n is the number of observations. The estimates of the parameters are obtained by maximizing the log-likelihood function in equation (36). The adequacy of the LTLGPW regression model is examined using the Cox-Snell residuals (Cox and Snell, [9]).

The Cox-Snell residuals of the LTLGPW regression model is $\hat{r}_i=-\log\left(S\left(y_i/\hat{b},\hat{\gamma},\hat{\mu},\hat{\sigma}\right)\right), i=1,2,...,n$, where $S\left(y_i/\hat{b},\hat{\gamma},\hat{\mu},\hat{\sigma}\right)$ is defined in equation (35). If the LTLGPW regression model fits the given data well, its Cox-Snell residuals are expected to follow the standard exponential distribution.

The application of the LTLGPW regression model is demonstrated in this section by modeling the relationship between long term interest rates (LTIR) of the Organization for Economic Co-operation Development (OECD) countries $\left(x_i\right)$, and foreign direct investment (FDINT), (ϕ). The data can be sourced from previous studies such as (Altun and Cordeiro, [3]) and (Nasiru et al., [24]).

2.640, 0.596, 0.680, 2.190, 4.560, 2.140, 0.410, 0.530, 0.750, 0.280, 4.390, 3.390, 5.190, 0.800, 2.160, 2.640, 0.060, 2.549, 0.930, 0.310, 0.540, 7.750, 0.470, 2.810, 1.760, 3.170, 1.760, 1.010, 0.990, 1.318, 0.550, 0.040, 1.374 and 2.890.

30.78, 57.87, 121.52, 90.17, 45.39, 11.08, 55.92, 51.54, 56.31, 43.34, 11.64, 20.85, 21.99, 276.22, 28.81, 27.56, 30.60, 21.02, 5.93, 7.24, 380.10, 15.76, 305.44, 8.94, 48.05, 5.41, 23.68, 3.56, 14.53, 41.90, 71.70, 162.75, 61.86 and 40.43.

The data is modeled by the regression equation;

$$y_i=q_0+q_1\phi+\sigma z_i, i=1,2,3,...,34,$$

where $y_i=\log\left(x_i\right)$ follows the LTLGPW distribution. The performance of the LTLGPW regression model was compared with log Extended Cosine generalized power Weibull (LECGPW) location-scale regression model (Sayibu and Luguterah, [28]). The parameters estimate of the LTLGPW and LECGPW regression models are shown in Tables 8. From the goodness of fit statistics shown in Table 8, it can be established that the LTLGPW regression model has performed better in fitting the dataset than the LECGPW according to the criteria defined above. FDINT negatively affects both models. This means that FDINT decreases LTIR.

Table 8: MLE parameters estimate for the investment dataset

Model	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$	$\hat{\sigma}$	$\hat{q}_0$	$\hat{q}_1$
LTLGPW		0.8219 (0.6587)	0.8029 (0.7564)	0.3226 (0.2170)	0.5617 (0.3749)	-0.0020 (0.0007)
LECGPW	55.5663 (0.0214)	43.2107 (0.0258)	4.7078 (0.2280)	8.4465 (0.8858)	4.2486 (0.0525)	-0.0023 (0.0012)

The location scale regression model for the LTLGPW is therefore obtained as

$$y_i = 0.5617 - 0.002\phi$$

Table 9: Goodness of fit statistics of the regression models

Model	$\hat{\ell}$	AIC	AICc	BIC	K-S	P-value
LTLGPW	**-18.4300**	**46.8648**	**49.0077**	**54.4966**	**0.1185**	**0.6820**
LECGPW	-26.6900	65.3783	68.4894	74.5364	0.1648	0.2818

* Bold means best based on the selection criteria

The adequacy of the regression models is investigated using the Cox-Snell residuals of the regression models. From the P-P plots in Figure 9, the LTLGPW location-scale regression model performs better than the LECGPW regression model.

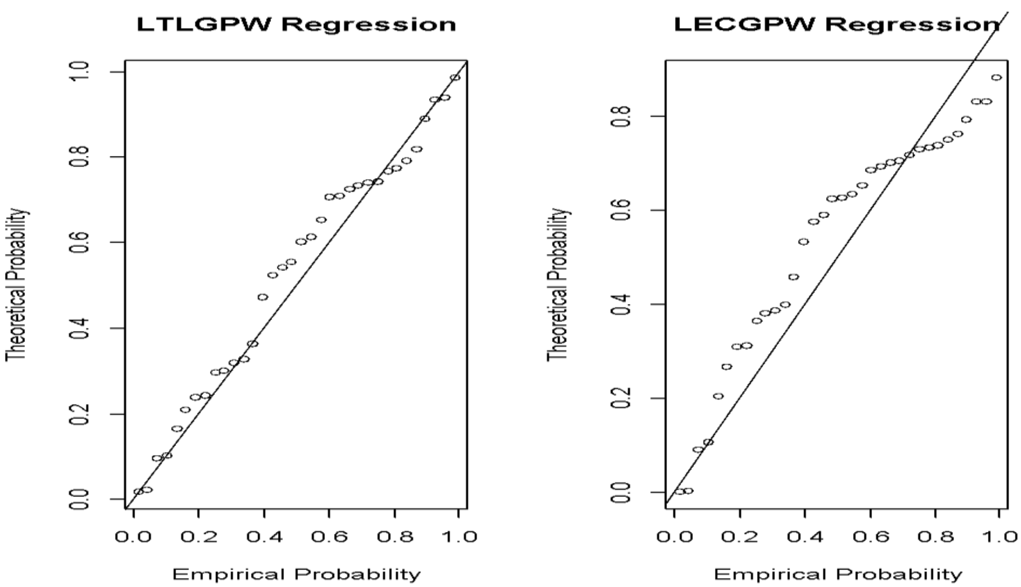

Figure 9: P-P plots for the LTLGPW and LECGPW regression models

8. Conclusions

A new distribution known as the Topp-Leone generalized power Weibull distribution is introduced and studied. The TLGPW distribution contains numerous well-known distributions as particular cases and new distributions. The TLGPW distribution wields a failure rate function with flexible behavior. Some closed-form statistical properties are derived, namely, quantile function, moments, moment generating functions, incomplete moments, mean and median deviations, mean residual life function, and Lorenz and Bonferroni curves. The maximum likelihood estimation procedure is adopted in estimating the parameters of the TLGPW distribution. Simulation studies were undertaken to evaluate the performance and accuracy of the maximum likelihood estimates. The TLGPW distribution is fitted to two real datasets to illustrate the empirical relevance. A log location-scale regression model is also developed for the proposed model. The application of the proposed regression model is demonstrated with a real dataset. The proposed location-scale regression model performed better than its competing model.

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Declaration of Competing Interest
The authors declare no conflict of interest

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