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Upper Bounds for Ruin Probability in a Controlled Risk Process under Rates of Interest with Homogenous Markov Chains

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Abstract

This paper explores recursive and integral equations for ruin probability of a controlled risk process under rates of interest with homogenous Markov chains. We assume that claim and rates of interest are homogenous Markov chains, take a countable number of non – negative values. Generalized Lundberg inequalities for ruin probability of this process are derived via a recursive technique. Recursive equations for finite time ruin probability and an integral equation for ultimate ruin probability are presented, from which corresponding probability inequalities and upper bounds are obtained. An illustrative numerical example is discussed.

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Keywords: ruin probability, homogenous Markov chain, controlled risk process

1 Introduction

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The ruin problem in stochastic environments has been studied by many researchers [9], [10]. In classical risk model, the claim number process was assumed to be a Poisson process and the individual claim amounts were described as independent and identically distributed random variables. In recent years, the classical risk process has been extended to more practical and real situations. For most of the investigations treated in risk theory, it is very significant to deal with the risks that rise from monetary inflation in the insurance and finance market, and also to consider the operation uncertainties in administration of financial capital. Teugels and Sundt [16], [17] studied ruin probability under the compound Poisson risk model with the effects of constant rate. Yang [19] given both exponential and non – exponential upper bounds for ruin probabilities in a risk model with constant interest force and independent premiums and claims. Xu and Wang [18] given upper bounds for ruin probabilities in a risk model with interest force and independent premiums and claims with Markov chain interest rate. Cai [1], [2] considered the ruin probabilities in two risk models, with independent premiums and claims and used a first – order autoregressive process to model the rates of in interest. Cai and Dickson [3] built Lundberg inequalities for ruin probabilities in two discrete- time risk process with a Markov chain interest model and independent premiums and claims. P. D. Quang [11] established Lundberg inequalities using the recursive technique for ruin probabilities in two risk model with homogenous Markov chain premiums when claims and interest rates sequences are independent. P. D. Quang [12] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent interest rates and premiums when claims is a Markov chain. P. D. Quang [13] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent interest rates and Markov chain claims and Markov chain premiums. P. D. Quang [14] used martingale approach to build upper bounds for ruin probabilities in a risk model with interest force and independent claims, Markov chain premiums and Markov chain interests. P. D. Quang [15] also used recursive approach to build upper bounds for ruin probabilities in a risk model with interest force and Markov chain premiums, Markov chain claims, while the independent interest rates.

In addition, many papers studied an insurance model where the risk process can be controlled by proportional reinsurance. The performance criterion is to choose reinsurance control strategies to bound the ruin probability of a discrete-time process with a Markov chain interest. Controlling a risk process is a very active area of research, particularly in the last decade; see [4, 5, 6, 7], for instance. Nevertheless obtaining explicit optimal solutions is a difficult task in a general setting. Maikol A. Diasparra and Rosaria Romera [8] obtained generalized Lundberg inequalities for the ruin probabilities in a controlled discrete-time risk process with a Markov chain interest.

In this article, we extend the model considered by Maikol A. Diasparra and Rosaria Romera [8] to introduce homogenous Markov chain claims and homogenous Markov chain rates of interest.

2 Preliminary Notes

Let Y_n be the n – th claim payment. The random variable Z_n stands for the length of the n – th period, that is, the time between the ocurrence of the claims Y_{n-1} and Y_n. Let $\{I_n\}_{n\geq 0}$ be the interest rate process. We assume that Y_n, Z_n, I_n are defined on the probability space (Ω, A, P). We consider a discrete – time insurance risk process in with the surplus process $\{U_n\}_{n\geq 1}$ with initial surplus u can be written as

$$U_n = U_{n-1}(1+I_n) + C(b_{n-1}).Z_n - h(b_{n-1}, Y_n),\ for\, n \geq 1. \tag{2.1}$$

We make several assumptions.

Assumption 2.1. $U_o = u \geq 0$.

Assumption 2.2. $\{Y_n\}_{n\geq 0}$ is an homogeneous Markov chain, such that for any n the values of Y_n are taken from a set of non – negative numbers $G_Y = \{y_1, y_2, ..., y_n, ...\}$ with $Y_o = y_i$ and

$$p_{ij} = P\left[\omega \in \Omega : Y_{n+1}(\omega) = y_j \middle| Y_n(\omega) = y_i\right](n \in N, y_i \in G_Y, y_j \in G_Y),$$

where $0 \leq p_{ij} \leq 1, \sum_{j=1}^{+\infty} p_{ij} = 1.$

Assumption 2.3. $\{I_n\}_{n\geq 0}$ is an homogeneous Markov chain, such that for any n the values of I_n are taken from a set of non – negative numbers $G_I = \{i_1, i_2, ..., i_m, ...\}$ with $I_o = i_r$ and

$$q_{rs} = P\left[\omega \in \Omega : I_{m+1}(\omega) = i_s \middle| I_m(\omega) = i_r\right](m \in N, i_r \in G_I, i_s \in G_I),$$

where $0 \leq q_{rs} \leq 1, \sum_{s=1}^{+\infty} q_{rs} = 1.$

Assumption 2.4. $\{Z_n\}_{n\geq 0}$ is a sequence of independent and identically distributed non-negative continuous random variables with the same distributive function

$$F(z) = P(\omega \in \Omega; Z_o(\omega) \leq z).$$

Assumption 2.4. We denote by $C(b)$ the premium left for the insurer if the retention level b is chosen, where $0 < C(b) \leq c, b \in B$.

The process can be controlled by reinsurance, that is, by choosing the retention level (or proportionality factor or risk exposure) $b \in B$ of a reinsurance contract for one period, where $B := [b_{min}, 1]$, $b_{min} \in (0,1]$ will be introduced below. The premium rate c is fixed.

Assumption 2.5. We denote the function $h(b,y)$ with values in $[0,y]$ specifies the fraction of the claim y paid by the insurer, and it also depends on the retention level b at the beginning of the period. Hence $y - h(b,y)$ is the part paid by the reinsurer. The retention level b =1 stands for control action no reinsurance.

In this article, we consider the case of proportional reinsurance, which means that

$$h(b,y)=b.y,\ \text{with}\ \ b\in B. \qquad (2.2)$$

Usually, the constant $b_{\min}$ in Assumption 2.4 is chosen by

$$b_{\min} := \min\{b\in(0,1];C(b)>0\}.$$

Assumption 2.6. We suppose that $\{Y_n\}_{n\geq 0}$, $\{Z_n\}_{n\geq 0}$ and $\{I_n\}_{n\geq 0}$ are independent.

Assumption 2.7. We consider Markovian control policies $\pi=\{a_n\}_{n\geq 1}$, which at each time n depend only on the current state, that is, $a_n(U_n):=b_n$ for $n\geq 0$. Abusing notation, we will indentify functions $a:X\to B,$ where $X=\square\cup\ell, B$ is the decision space.

Consider an arbitrary initial state $U_o=u\geq 0$ and a control policy $\pi=\{a_n\}_{n\geq 1}$. Then, by iteration of (2.1) and assuming (2.2), it follows that for $n\geq 1, U_n$ satisfies

$$U_n=u\prod_{l=1}^{n}(1+I_l)+\sum_{l=1}^{n}\left(C(b_{n-1})Z_l-b_{l-1}.Y_l\prod_{m=l+1}^{n}(1+I_m)\right) \qquad (2.3)$$

The ruin probability when using the policy π, given the initial surplus u, and the initial claim $Y_o=y_i$, the initial interest rate $I_o=i_r$ with Assumption 2.1 to 2.7 is defined as

$$\psi^{\pi}(u,y_i,i_r)=P^{\pi}\left(\bigcup_{k=1}^{\infty}(U_k<0)\middle|U_o=u,Y_o=y_i,I_o=i_r\right) \qquad (2.4)$$

which we can also express as

$$\psi^{\pi}(u,y_i,i_r)=P^{\pi}\left(U_k<0\ for\,some\ k\geq 1\middle|U_o=u,Y_o=y_i,I_o=i_r\right) \quad (2.5)$$

Similarly, the ruin probabilities in the finite horizon case with Assumption 2.1 to 2.7, are given by

$$\psi_n^{\pi}(u,y_i,i_r)=P^{\pi}\left(\bigcup_{k=1}^{n}(U_k<0)\middle|U_o=u,Y_o=y_i,I_o=i_r\right) \qquad (2.6)$$

Firstly, we have

$$\psi_1^{\pi}(u,y_i,i_r)\le\psi_2^{\pi}(u,y_i,i_r)\le\ldots\le\psi_n^{\pi}(u,y_i,i_r)\le\ldots, \qquad (2.7)$$

and with any $n\in N$,

$$\psi_n^{\pi}(u,y_i,i_r)\le 1. \qquad (2.8)$$

Thus, from (2.7) and (2.8), we obtain

$$\lim_{n\to\infty}\psi_n^{\pi}(u,y_i,i_r)=\psi^{\pi}(u,y_i,i_r).$$

We denote by Π the policy space. A control policy π^* is said to be optimal if for any initial $(Y_o, I_o) = (y_i, i_r)$, we have

$$\psi^{\pi^*}(u,y_i,i_r)\le\psi^{\pi}(u,y_i,i_r) \text{ for all } \pi\in\Pi.$$

3 Main Results

3.1. Integral Equation for Ruin Probability

We now construct recursive equation for finite time ruin probabilities and an integral equation

Theorem 3.1. Given model (2.1) and Assumptions 2.1 to 2.7, for n =1,2, …, we have

$$\psi_{n+1}^{\pi}(u,y_i,i_r)=\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{\int_0^{\frac{b_o y_j-u(1+i_s)}{C(b_o)}}dF(z)+\int_{\frac{b_o y_j-u(1+i_s)}{C(b_o)}}^{+\infty}\psi_n^{\pi}(u(1+i_s)-b_o y_j+C(b_o)z,y_j,i_s)dF(z)\right\}, \qquad (3.1)$$

and

$$\psi^{\pi}(u,y_i,i_r)=\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{\int_0^{\frac{b_o y_j-u(1+i_s)}{C(b_o)}}dF(z)+\int_{\frac{b_o y_j-u(1+i_s)}{C(b_o)}}^{+\infty}\psi^{\pi}(u(1+i_s)-b_o y_j+C(b_o)z,y_j,i_s)dF(z)\right\}. \qquad (3.2)$$

Where throughout this paper:

i) If $v\le 0$ then $F(v)=0$,

ii) If $v \le 0$ then $\int_{v}^{+\infty} dF(z) = \int_{0}^{+\infty} dF(z)$,

iii) If $v \le 0$ then $\int_{0}^{v} \psi^{\pi}(h(z), y_i, i_r)\,dF(z) = 0.$

Proof.

We consider $Y_1(\omega) = y_j, I_1(\omega) = i_s, (\omega \in \Omega)$ and

$$B = \left\{\omega \in \Omega : U_o(\omega) = u, Y_o(\omega) = y_i, I_o(\omega) = i_r\right\},$$

$$A_{js} = \left\{\omega \in \Omega : Y_1(\omega) = y_j, I_1(\omega) = i_s\right\},$$

$$A_1 = \left\{\omega \in \Omega : Z_1(\omega) < \frac{U_o(\omega)(1 + I_1(\omega)) - b_o Y_1(\omega)}{C(b_o)}\right\},$$

$$\overline{A}_1 = \left\{\omega \in \Omega : Z_1(\omega) \ge \frac{U_o(\omega)(1 + I_1(\omega)) - b_o Y_1(\omega)}{C(b_o)}\right\}.$$

Let $V_k = u(Y_k, Z_k) = b_o Y_k - C(b_o) Z_k$. From (2.1), we have

$$U_1 = U_o(1 + I_1) - V_1 = u(1 + I_1) - b_o Y_1 + C(b_o) Z_1$$

Therefore

$$P^{\pi}\left(\omega \in \Omega : U_1(\omega) < 0 \,\middle|\, A_1 \cap A_{js} \cap B\right) = 1$$

$$\Rightarrow P^{\pi}\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} U_k(\omega) < 0 \,\middle|\, A_1 \cap A_{js} \cap B\right) = 1. \tag{3.3}$$

In addition,

$$P^{\pi}\left(\omega \in \Omega ; U_1(\omega) < 0 \,\middle|\, \overline{A}_1 \cap A_{js} \cap B\right) = 0. \tag{3.4}$$

Let $\left\{\tilde{Y}_n\right\}_{n\ge 0}, \left\{\tilde{Z}_n\right\}_{n\ge 0}, \left\{\tilde{I}_n\right\}_{n\ge 0}$ be independent copies of $\left\{Y_n\right\}_{n\ge 0}$, $\left\{Z_n\right\}_{n\ge 0}$, $\left\{I_n\right\}_{n\ge 0}$ with $\tilde{Y}_o(\omega) = Y_1(\omega) = y_j, Z_o(\omega) = Z_1(\omega), \tilde{I}_o(\omega) = I_1(\omega) = i_s$ and $\tilde{V}_k = b_o \tilde{Y}_k - C(b_o)\tilde{Z}_k$,

$$\tilde{U}_n = \tilde{U}_o \prod_{l=1}^{n} (1+\tilde{I}_l) + \sum_{l=1}^{n} \left(C(b_{l-1})\tilde{Z}_l - b_{l-1}\tilde{Y}_l \prod_{m=l+1}^{n} (1+\tilde{I}_m) \right).$$

Thus (2.3) and (3.4) imply

$$P^{\pi}\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} U_k(\omega) < 0 \middle| \overline{A}_1 \cap A_{js} \cap B \right) = P^{\pi}\left(\omega \in \Omega : \bigcup_{k=2}^{n+1} U_k(\omega) < 0 \middle| \overline{A}_1 \cap A_{js} \cap B \right)$$

$$= P^{\pi}\left(\omega \in \Omega : \bigcup_{k=2}^{n+1} \left[U_o(\omega)(1+I_1(\omega) - b_o Y_1(\omega) + C(b_o) Z_1(\omega) \right] \prod_{m=2}^{k} (1+I_m(\omega)) \right.$$

$$\left. + \sum_{m=2}^{k} \left(C(b_{m-1}) Z_m(\omega) - b_{m-1} Y_m(\omega) \right) \prod_{p=m+1}^{k} (1+I_p(\omega)) < 0 \middle| \overline{A}_1 \cap A_{js} \cap B \right)$$

$$= P^{\pi}\left(\omega \in \Omega : \bigcup_{k=1}^{n} \left(\tilde{U}_o(\omega) \prod_{m=2}^{k} (1+\tilde{I}_m(\omega)) + \sum_{m=2}^{k} \left(C(b_{m-1}) \tilde{Z}_m(\omega) - b_{m-1} \tilde{Y}_m(\omega) \right) \prod_{p=m+1}^{k} (1+\tilde{I}_p(\omega)) < 0 \right) \middle| \overline{A}_1 \cap A_{js} \cap B \right), \quad (3.5)$$

.Now, from (2.1) implies

$$\psi_{n+1}^{\pi}(u, y_i, i_r) = P^{\pi}\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \middle| B \right)$$

$$= \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} P^{\pi}\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \middle| A_{ij} \cap B \right)$$

$$= \sum_{j=1}^{+\infty} \sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ P^{\pi}\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \middle| A_1 \cap A_{ij} \cap B \right) . P\left(A_1 \middle| A_{ij} \cap B \right) + \right.$$

$$\left. + \left\{ P^{\pi}\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \middle| \overline{A}_1 \cap A_{ij} \cap B \right) . P\left(\overline{A}_1 \middle| A_{ij} \cap B \right) \right\} \quad (3.6)$$

From (3.3), we have

$$P^{\pi}\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \middle| A_1 \cap A_{ij} \cap B \right) . P\left(A_1 \middle| A_{ij} \cap B \right)$$

$$= P^{\pi}\left(\omega \in \Omega : Z_1(\omega) < \frac{b_o y_j - u(1+i_s)}{C(b_o)} \right)$$

$$= \int_{0}^{\frac{b_o y_j - u(1+i_s)}{C(b_o)}} dF(z), \tag{3.7}$$

and from (3.5), we have

$$P^{\pi}\left(\omega \in \Omega : \bigcup_{k=1}^{n+1} (U_k(\omega) < 0 \middle| \overline{A}_1 \cap A_{ij} \cap B \right).P\left(\overline{A}_1 \middle| A_{ij} \cap B \right)$$

$$= P^{\pi}\left(\omega \in \Omega : \bigcup_{k=1}^{n} \left(\tilde{U}_o(\omega) \prod_{m=2}^{k} (1+\tilde{I}_m(\omega)) + \sum_{m=2}^{k} \left(C(b_{m-1})\tilde{Z}_m(\omega) - b_{m-1}\tilde{Y}_m(\omega) \right) \prod_{p=m+1}^{k} (1+\tilde{I}_p(\omega)) < 0 \right) \middle| \overline{A}_1 \cap A_{js} \cap B \right).$$

.

$$P^{\pi}\left(\overline{A}_1 \middle| A_{js} \cap B \right)$$

$$= \int_{\frac{b_o y_j - u(1+i_s)}{C(b_o)}}^{+\infty} \psi_n^{\pi}\left(u(1+i_s) - b_o y_j + C(b_o)z, y_j, i_s \right) dF(z).(3.8)\ .(3.8)$$

Combining (3.7) and (3.8), therefore (3.6) may be written

$$\psi_{n+1}^{\pi}(u, y_i, i_r) = \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} \left\{ \int_{0}^{\frac{b_o y_j - u(1+i_s)}{C(b_o)}} dF(z) + \int_{\frac{b_o y_j - u(1+i_s)}{C(b_o)}}^{+\infty} \psi_n^{\pi}(u(1+i_s) - b_o y_j + C(b_o)z, y_j, i_s) dF(z) \right\}.(3.9)$$

When n = 0, we have

$$\psi_1^{\pi}(u, y_i, i_r) = \sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij} q_{rs} F\left(\frac{b_o y_j - u(1+i_s)}{C(b_o)} \right). \tag{3.10}$$

From the dominated convergence theorem, the integral equation for $\psi^{\pi}(u, y_i, i_r)$ in Theorem 3.1 then follows immediately by letting $n \to \infty$ in (3.9).

3.2. Inequalities for Ruin Probability

We now establish inequalities for the ruin probability corresponding to (2.4) and (2.6), respectively. We first prove the following Lemma.

Lemma 3.1. Given model (2.1) and Assumptions 2.1 to 2.7, and

$$E^{\pi}\left[(b_oY_1 - C(b_o)Z_1)\middle|Y_o = y_i\right] < 0,$$

and

$$P^{\pi}\left[b_oY_1 - C(b_o)Z_1 > 0\middle|Y_o = y_i\right] > 0, \tag{3.11}$$

For any $y_i \in G_Y$, then there exists a unique positive constant R_i satisfying

$$E^{\pi}\left[e^{-R_i[C(b_o)Z_1 - b_oY_1]}\middle|Y_o = y_i\right] = 1. \tag{3.12}$$

Proof.

Let the function

$$f_i(t) = E^{\pi}\left[e^{-t[C(b_o)Z_1 - b_oY_1]}\middle|Y_1 = y_i\right] - 1,\ t \in (0; +\infty).$$

We have

$$f_i^{'}(t) = E^{\pi}\left[[b_oY_1 - C(b_o)Z_1]e^{-t[C(b_o)Z_1 - b_oY_1]}\middle|Y_1 = y_i\right],$$

$$f_i^{''}(t) = E^{\pi}\left[\left[b_oY_1 - C(b_o)Z_1\right]^2 e^{-t[C(b_o)Z_1 - b_oY_1]}\middle|Y_1 = y_i\right] \geq 0.$$

Which implies that

$$f_i(t) \text{ is a convex function with } f_i(0) = 0, \tag{3.13}$$

and

$$f_i^{'}(0) = E^{\pi}\left[b_oY_1 - C(b_o)Z_1\middle|Y_1 = y_i\right] < 0. \tag{3.14}$$

As $P^{\pi}\left[b_oY_1 - C(b_o)Z_1 > 0\middle|Y_o = y_i\right] > 0$, we can find some constant $\delta > 0$ such that

$$P^{\pi}\left[b_oY_1 - C(b_o)Z_1 > \delta > 0\middle|Y_o = y_i\right] > 0.$$

We therefore have

$$f_i(t) = E^{\pi}\left[e^{-t[C(b_o)Z_1 - b_oY_1]}\middle|Y_1 = y_i\right] - 1$$

$$\geq E^{\pi}\left\{\left[e^{-t[C(b_o)Z_1 - b_oY_1]}\middle|Y_1 = y_i\right]1_{\{b_oY_1 - C(b_o)Z_1 > \delta\}}\right\} - 1 \geq e^{\delta t} - 1,$$

implying that

$$\lim_{t \to +\infty} f_i(t) = +\infty, \tag{3.15}$$

From (3.13), (3.14) and (3.15) there exists a unique positive constant R_i satisfying (3.12).

Now consider

$$R_o = \inf\left\{R_i > 0 : E^{\pi}\left(e^{-R_o[C(b_o)Z_1 - b_oY_1]} \middle| Y_o = y_i\right) \le 1, y_i \in G_Y\right\}.$$

Remark 3.1. $E^{\pi}\left[e^{-R_o[C(b_o)Z_1 - b_oY_1]} \middle| Y_o = y_i\right] \le 1.$

Using Lemma 3.1 and Theorem 3.1, we have a probability inequality for $\psi^{\pi}(u, y_i, i_r)$ by an inductive approach as follows.

Theorem 3.2. Given model (2.1) and Assumptions 2.1 to 2.7, under the conditions of Lemma 3.1 and $R_o > 0$, we have that

$$\psi^{\pi}(u, y_i, i_r) \le \beta E^{\pi}\left[e^{-R_o u(1+I_1)} \middle| I_o = i_r\right] \tag{3.16}$$

For any u > 0, $y_i \in G_Y$ and $i_r \in G_I$, where

$$\beta^{-1} = \inf_{t>0} \frac{e^{R_oC(b_o)t}\int_0^t e^{-R_oC(b_o)z}dF(z)}{F(t)}, 0 \le \beta \le 1.$$

Proof.

a) if $\inf_{t>0} \dfrac{e^{R_oC(b_o)t}\int_0^t e^{-R_oC(b_o)z}dF(z)}{F(t)} < +\infty.$

Firstly, we have

$$\beta^{-1} = \inf_{t>0} \frac{e^{R_oC(b_o)t}\int_0^t e^{-R_oC(b_o)z}dF(z)}{F(t)} = \inf_{t>0} \frac{\int_0^t e^{R_oC(b_o)(t-z)}dF(z)}{F(t)} \ge \inf_{t>0} \frac{\int_0^t dF(z)}{F(t)} = 1.$$

Implying that $0 < \beta \le 1$.

For any v > 0, we also have

$$F(v)=\left[\frac{e^{R_oC(b_o)v}\int_0^v e^{-R_oC(b_o)z}dF(z)}{F(v)}\right]^{-1}.e^{R_oC(b_o)v}.\int_0^v e^{-R_oC(b_o)z}dF(z)$$

$$\le \beta.e^{R_oC(b_o)v}.\int_0^v e^{-R_oC(b_o)z}dF(z)\le \beta.e^{R_oC(b_o)v}.\int_0^{+\infty} e^{-R_oC(b_o)z}dF(z)=\beta.e^{R_oC(b_o)v}.E^{\pi}\left(e^{-R_oC(b_o)Z_1}\right).(3.17)$$

Let

$$K_1=\left\{(j,s):j\in\{1,2,...\},s\in\{1,2,...,\}:\frac{b_oy_j-u(1+i_s)}{C(b_o)}\le 0\right\},$$

$$K_2=\left\{(j,s):j\in\{1,2,...\},s\in\{1,2,...,\}:\frac{b_oy_j-u(1+i_s)}{C(b_o)}>\right\}$$

From (3.10), we have

$$\psi_1^{\pi}(u,y_i,i_r)=\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}F\left(\frac{b_oy_j-u(1+i_s)}{C(b_o)}\right)$$

$$=\sum_{j}\sum_{\substack{s\\(j,s)\in K_1}}p_{ij}q_{rs}F\left(\frac{b_oy_j-u(1+i_s)}{C(b_o)}\right)+\sum_{j}\sum_{\substack{s\\(j,s)\in K_2}}p_{ij}q_{rs}F\left(\frac{b_oy_j-u(1+i_s)}{C(b_o)}\right).$$

Using (3.17), we have

$$\sum_{j}\sum_{\substack{s\\(j,s)\in K_2}}p_{ij}q_{rs}F\left(\frac{b_oy_j-u(1+i_s)}{C(b_o)}\right)\le\beta\sum_{j}\sum_{\substack{s\\(j,s)\in K_2}}p_{ij}q_{rs}e^{R_oC(b_o)\frac{b_oy_j-u(1+i_s)}{C(b_o)}}E^{\pi}\left(e^{-R_oC(b_o)Z_1}\right)$$

$$=\beta\sum_{j}\sum_{\substack{s\\(j,s)\in K_2}}p_{ij}q_{rs}e^{R_o\left[b_oy_j-u(1+i_s)\right]}E^{\pi}\left(e^{-R_oC(b_o)Z_1}\right). \qquad (3.18)$$

In addition, we also have $F\left(\frac{b_oy_j-u(1+i_s)}{C(b_o)}\right)=0$ if $(j,s)\in K_1$. Therefore

$$\sum_{j}\sum_{\substack{s\\(j,s)\in K_1}} p_{ij}q_{rs}F\left(\frac{b_oy_j-u(1+i_s)}{C(b_o)}\right)=0$$

$$\le\beta\sum_{j}\sum_{\substack{s\\(j,s)\in K_1}} p_{ij}q_{rs}e^{R_o\left[b_oy_j-u(1+i_s)\right]}E^{\pi}\left(e^{-R_oC(b_o)Z_1}\right) \qquad (3.19)$$

Combining (3.18) and (3.19), we imply

$$\psi_1^{\pi}(u,y_i,i_r)\le\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}e^{R_o\left[b_oy_j-u(1+i_s)\right]}E^{\pi}\left(e^{-R_oC(b_o)Z_1}\right)$$

$$=\beta E^{\pi}\left[e^{R_o[b_oY_1-u(1+I_1)]}\middle|Y_o=y_i,I_o=i_r\right].E^{\pi}\left[e^{-R_oC(b_o)Z_1}\right]$$

$$=\beta E^{\pi}\left[e^{-R_o[C(b_o)Z_1-b_oY_1]}\middle|Y_o=y_i\right].E^{\pi}\left[e^{-R_ou(1+I_1)}\middle|I_o=i_r\right]\le\beta E^{\pi}\left[e^{-R_ou(1+I_1)}\middle|I_o=i_r\right],(3.20)$$

.Under an inductive hypothesis, we assume

$$\psi_n^{\pi}(u,y_i,i_r)\le\beta E^{\pi}\left[e^{-R_ou(1+I_1)}\middle|I_o=i_r\right]. \qquad (3.21)$$

So inequality (3.30) implies (3.21) holds with n =1. We have

$$\psi_n^{\pi}(u(1+i_s)-b_oy_j+C(b_o)z,y_j,i_s)\le\beta E^{\pi}\left[e^{-R_o\left[u(1+i_s)-b_oy_j+C(b_o)z\right](1+I_1)}\middle|I_o=i_r\right]$$

For $y_i\in G_Y$ and $i_r\in G_I$, $u(1+i_s)-b_oy_j+C(b_o)z\ge 0, I_1\ge 0$ then

$$\psi_n^{\pi}(u(1+i_s)-b_oy_j+C(b_o)z,y_j,i_s)\le\beta E^{\pi}\left[e^{-R_o\left[u(1+i_s)-b_oy_j+C(b_o)z\right]}\middle|I_o=i_r\right]$$

$$\le\beta e^{-R_o\left[u(1+i_s)-b_oy_j+C(b_o)z\right]}. \qquad (3.22)$$

So from Lemma 3.1, (3.9), (3.17) and (3.22), we obtain

$$\psi_{n+1}^{\pi}(u,y_i,i_r)=\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty}p_{ij}q_{rs}\left\{\int_0^{\frac{b_oy_j-u(1+i_s)}{C(b_o)}}dF(z)+\int_{\frac{b_oy_j-u(1+i_s)}{C(b_o)}}^{+\infty}\psi_n^{\pi}(u(1+i_s)-b_oy_j+C(b_o)z,y_j,i_s)dF(z)\right\}$$

$$=\sum_{j}\sum_{\substack{s\\(j,s)\in K_1}} p_{ij}q_{rs}\left\{\int_{0}^{\frac{b_o y_j-u(1+i_s)}{C(b_o)}} dF(z)+\int_{\frac{b_o y_j-u(1+i_s)}{C(b_o)}}^{+\infty}\psi_n^{\pi}(u(1+i_s)-b_o y_j+C(b_o)z,y_j,i_s)dF(z)\right\}$$

$$+\sum_{j}\sum_{\substack{s\\(j,s)\in K_2}} p_{ij}q_{rs}\left\{\int_{0}^{\frac{b_o y_j-u(1+i_s)}{C(b_o)}} dF(z)+\int_{\frac{b_o y_j-u(1+i_s)}{C(b_o)}}^{+\infty}\psi_n^{\pi}(u(1+i_s)-b_o y_j+C(b_o)z,y_j,i_s)dF(z)\right\}.(3.23)$$

.Because $(j,s)\in K_1:\dfrac{b_o y_j-u(1+i_s)}{C(b_o)}\le 0$ then

$$F\left(\frac{b_o y_j-u(1+i_s)}{C(b_o)}\right)=0,$$

$$\int_{\frac{b_o y_j-u(1+i_s)}{C(b_o)}}^{+\infty}\psi_n^{\pi}(u(1+i_s)-b_o y_j+C(b_o)z,y_j,i_s)dF(z)=\int_{0}^{+\infty}\psi_n^{\pi}(u(1+i_s)-b_o y_j+C(b_o)z,y_j,i_s)dF(z).$$

Combining with (3.22), we have

$$=\sum_{j}\sum_{\substack{s\\(j,s)\in K_1}} p_{ij}q_{rs}\left\{\int_{0}^{\frac{b_o y_j-u(1+i_s)}{C(b_o)}} dF(z)+\int_{\frac{b_o y_j-u(1+i_s)}{C(b_o)}}^{+\infty}\psi_n^{\pi}(u(1+i_s)-b_o y_j+C(b_o)z,y_j,i_s)dF(z)\right\}$$

$$=\sum_{j}\sum_{\substack{s\\(j,s)\in K_1}} p_{ij}q_{rs}\int_{0}^{+\infty}\psi_n^{\pi}(u(1+i_s)-b_o y_j+C(b_o),y_j,i_s)dF(z)\,.$$

$$\le\beta\sum_{j}\sum_{\substack{s\\(j,s)\in K_1}} p_{ij}q_{rs}\int_{0}^{+\infty}e^{-R_o\left[u(1+i_s)-b_o y_j+C(b_o)z\right]}dF(z)\,. \qquad (3.24)$$

Using (3.17) and (3.24), we have

$$\sum_{j}\sum_{\substack{s\\(j,s)\in K_2}} p_{ij}q_{rs}\left\{F\left(\frac{b_o y_j - u(1+i_s)}{C(b_o)}\right) + \int_{\frac{b_o y_j - u(1+i_s)}{C(b_o)}}^{+\infty} \psi_n^{\pi}(u(1+i_s) - b_o y_j + C(b_o)z, y_j, i_s)dF(z)\right\}$$

$$\leq \beta\sum_{j}\sum_{\substack{s\\(j,s)\in K_2}} p_{ij}q_{rs}\left\{\int_{0}^{\frac{b_o y_j - u(1+i_s)}{C(b_o)}} e^{-R_o\left[u(1+i_s)-b_o y_j+C(b_o)z\right]}dF(z) + \int_{\frac{b_o y_j - u(1+i_s)}{C(b_o)}}^{+\infty} e^{-R_o\left[u(1+i_s)-b_o y_j+C(b_o)z\right]}dF(z)\right\}$$

$$= \beta\sum_{j}\sum_{\substack{s\\(j,s)\in K_2}} p_{ij}q_{rs}\int_{0}^{+\infty} e^{-R_o\left[u(1+i_s)-b_o y_j+C(b_o)z\right]}dF(z)\,. \tag{3.25}$$

From (3.24) and (3.35), we obtain

$$\psi_{n+1}^{\pi}(u, y_i, i_s) \leq \beta\sum_{j=1}^{+\infty}\sum_{s=1}^{+\infty} p_{ij}q_{rs}\int_{0}^{+\infty} e^{-R_o\left[u(1+i_s)-b_o y_j+C(b_o)z\right]}dF(z)$$

$$= \beta E^{\pi}\left[e^{R_o[b_o Y_1 - u(1+I_1)]}\middle| Y_o = y_i, I_o = i_r\right].E^{\pi}\left[e^{-R_o C(b_o)Z_1}\right]$$

$$= \beta E^{\pi}\left[e^{-R_o[C(b_o)Z_1 - b_o Y_1]}\middle| Y_o = y_i\right].E^{\pi}\left[e^{-R_o u(1+I_1)}\middle| I_o = i_r\right] \leq \beta E^{\pi}\left[e^{-R_o u(1+I_1)}\middle| I_o = i_r\right]$$

.

Consequently

$$\psi_{n+1}^{\pi}(u, y_i, i_r) \leq \beta E^{\pi}\left[e^{-R_o u(1+I_1)}\middle| I_o = i_r\right],$$

Such that inequality (3.21) holds for any n = 1, 2, 3, ... and inequality (3.16) follows by letting $n \to \infty$ in inequality (3.21).

b) If $\inf\limits_{t>0} \dfrac{e^{R_o C(b_o)t}\int_0^t e^{-R_o C(b_o)z}dF(z)}{F(t)} = +\infty \Leftrightarrow \beta = 0.$

With any $\varepsilon > 0$: $\dfrac{e^{R_o C(b_o)t}\int_0^t e^{-R_o C(b_o)z}dF(z)}{F(t)} \geq \varepsilon$ and

$$F(v) \leq \frac{1}{\varepsilon}e^{R_o C(b_o)v}\int_0^v e^{-R_o C(b_o)z}dF(z).$$

We also prove similar such that a), we obtain

$$\psi_n^{\pi}(u, y_i, i_r) \le \frac{1}{\varepsilon} E^{\pi}\left[e^{-R_o u(1+I_1)} \middle| I_o = i_r \right]. \tag{3.26}$$

Let $n \to +\infty$ in inequality (4.16), we imply

$$\psi^{\pi}(u, y_i, i_r) \le \frac{1}{\varepsilon} E^{\pi}\left[e^{-R_o u(1+I_1)} \middle| I_o = i_r \right]. \tag{3.27}$$

Let $\varepsilon = n\,(n \in N^*)$ then (3.27) becomes

$$\psi^{\pi}(u, y_i, i_r) \le \frac{1}{n} E^{\pi}\left[e^{-R_o u(1+I_1)} \middle| I_o = i_r \right]. \tag{3.28}$$

letting $n \to \infty$ in inequality (3.28), we have

$$\psi^{\pi}(u, y_i, i_r) \le 0 = \beta E^{\pi}\left[e^{-R_o u(1+I_1)} \middle| I_o = i_r \right].$$

Thus, inequality (3.16) holds when $\beta = 0$. □

Remark 3.2. Let $A(u, i_r) = \beta E^{\pi}\left[e^{-R_o u(1+I_1)} \middle| I_o = i_r \right]$. From $I_1 \ge 0, \beta \le 1$, we have

$$A(u, i_r) = \beta E^{\pi}\left[e^{-R_o u(1+I_1)} \middle| I_o = i_r \right] \le \beta e^{-R_o u} \le e^{-R_o u}.$$

So an upper bound for the ruin probability from inequality (3.16) is better than $e^{-R_o u}$.

4 Numerical Example

In this section we give a numerical example to illustrate the bounds of $\psi^{\pi}(u, y_i, i_r)$ derived in Section 3.

Let $\{Z_n\}_{n\ge 0}$ be a sequence of independent and identically distributed non-negative continuous random variables with the same distributive function $F(z) = 1 - e^{-0,25z}\,(z \ge 0)$

Let $\{Y_n\}_{n\geq 0}$ be a homogeneous Markov chain such that for any n, Y_n take values $G_Y = \{1,3\}$ with Y_1 having a distribution:

Y_1	1	3
P	0,4	0,6

and matrix $P = \lfloor p_{ij} \rfloor_{2x2}$ is given by

$$P = \begin{bmatrix} 0,3 & 0,7 \\ 0,2 & 0,8 \end{bmatrix}$$

Let $\{Y_n\}_{n\geq 0}$ be a homogeneous Markov chain such that for any n, I_n take value in $G_Y = \{0,1; 0,15\}$ with I_1 having a distribution:

I_1	0,1	0,15
P	0,35	0,65

and matrix $Q = \lfloor q_{ij} \rfloor_{2x2}$ is given by

$$Q = \begin{bmatrix} 0,25 & 0,75 \\ 0,6 & 0,4 \end{bmatrix}$$

Then, we have

$$E\left(Y_1 \middle| Y_o = 1\right) = 1.0,3 + 3.0,7 = 2,4; E\left(Y_1 \middle| Y_o = 3\right) = 1.0,2 + 3.0,8 = 2,6; E(X_1) = \frac{1}{0,25} = 4.$$

We chose $\pi = \{a_n\}_{n\geq 0}$ với a_n = 1 nên b_o = 1, C(b_o)= 1, therefore

$$E(Y_1 | Y_o = y_i) < (E(Z_1), y_i \in G_Y, \qquad (4.1)$$

In the other hand,

$$P\left(Y_1 - X_1 > 0 \middle| Y_o = 1\right) > 0, P\left(Y_1 - X_1 > 0 \middle| Y_o = 3\right) > 0. \qquad (4.2)$$

Combining (4.1), (4.2) imply that Lemma 2.1 holds.

Next, we solve equation (3.12).

Firstly, we have

$$E\left[e^{R_i(Y_1 - Z_1)} \middle| Y_o = y_i\right] = E\left[e^{R_i Y_1} \middle| Y_o = y_i\right].E\left[e^{-R_i Z_1}\right](i = 1,2).$$

where

$$E\left[e^{-R_1Z_1}\right] = 0,25\int_0^{+\infty} e^{-(R_i+0,25)x}dx = \frac{0,25}{R_i+0,25}\,(i=1,2).$$

and

$$E\left[e^{R_1Y_1}\middle|Y_o=1\right] = e^{R_1}.P\left[Y_1=1\middle|Y_o=1\right] + e^{3R_1}.P\left[Y_1=3\middle|Y_o=1\right]$$
$$=0,3e^{R_1}+0,7e^{3R_1}$$

$$E\left[e^{R_2Y_1}\middle|Y_o=3\right] = e^{R_2}.P\left[Y_1=1\middle|Y_o=3\right] + e^{3R_2}.P\left[Y_1=3\middle|Y_o=3\right]$$
$$=0,2e^{R_2}+0,8e^{3R_2}$$

Respective equation (3.12) for R$_1$, R$_2$ by

$$0,3e^{R_1}+0,7e^{3R_1}=4R_1+1 \tag{4.3}$$

$$0,2e^{R_2}+0,8e^{3R_2}=4R_2+1 \tag{4.4}$$

Using Maple, we find respective root of (3.12) for R$_1$, R$_2$, by

$$R_1\approx 0,33878; R_2\approx 0,28124$$

Hence, $R_o=\min\{R_1,R_2\}=0,28124$.

We can apply the result of Theorem 3.2 for $\psi^{\pi}(u,y_i,i_r)$

$$\psi^{\pi}(u,y_i,i_r)\le E^{\pi}\left[e^{-R_ou(1+I_1)}\middle|I_o=i_r\right]=g(u,i_r),i_r\in G_I. \tag{4.5}$$

where

$$g(u;0,1)=E\left[e^{-R_ou(1+I_1)}\middle|I_o=0,1\right]$$
$$=e^{-1,1R_ou}.P\left[I_1=0,1\middle|I_o=0,1\right]+e^{-1,15R_ou}.P\left[I_1=0,15\middle|I_o=0,1\right]$$

$$=0,25e^{-1,1R_ou}+0,75e^{-1,15R_ou}$$

$$g(u;0,15)=E\left[e^{-R_ou(1+I_1)}\middle|I_o=0,15\right]$$
$$=e^{-1,1R_ou}.P\left[I_1=0,1\middle|I_o=0,15\right]+e^{-1,15R_ou}.P\left[I_1=0,15\middle|I_o=0,15\right]$$

$$= 0,6e^{-1,R_0 u} + 0,4e^{-R_0 u}$$

Table 4.1 shows values upper bounds $g(u,i_r)$ of $\psi^{\pi}(u,y_i,i_r)$ for a range of value of u

Table 4.1. Upper bounds $g(u,i_r)$ of $\psi^{\pi}(u,y_i,i_r)$ with $\pi = \{a_n\}_{n\geq 0}$: $a_n = 1$

u	$g(u;0,1)$	$g(u;0,15)$
1	0.726228	0.729814
2	0.527426	0.532654
3	0.38306	0.388775
4	0.27822	0.283774
5	0.202082	0.207141
6	0.146785	0.15121
7	0.106624	0.110387
8	0.077454	0.080588
9	0.056266	0.058836
10	0.040876	0.042958
15	0.008276	0.008919
20	0.001677	0.001854

5 Conclusion

Theorem 3.2 provide recursive equations for $\psi_n^{\pi}(u,y_i,i_r)$ and an integral equation for $\psi^{\pi}(u,y_i,i_r)$, by using a recursive technique. Using Lemma 3.1 and Theorem 3.2, we obtain a probability inequality for $\psi^{\pi}(u,y_i,i_r)$ by an inductive approach. An illustrative numerical example is discussed.

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