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Autoregressive Process Parameters Estimation under Non-Classical Error Model

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Abstract

Error in measuring time varying data setting is one important source of bias in estimating of time series modeling parameters. When the measurement error model is non-classic, this raises the question whether the different measurement error model strategy might differently affect the estimation of the time series modeling parameters. In this article, we investigate this in Autoregressive (AR) model parameters estimation under the non-classical measurement error model. We compare the parameters estimation of the AR model under the classical and non-classical error models. We perform analytically this on the AR model of order p. Further, we confirm this through simulation study specifically on the AR model of order 1.

Mathematics Subject Classification: 62M10

Keywords: Non-classical measurement error model, Autoregressive process, Parameter estimation

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1 Introduction

Similar to other types of data, time series data can be prone to measurement error (e.g., measuring of population abundances or density, air pollution or temperatures levels, disease rates, medical indices) [1, 2]. That is a variable of interest is observed with some measurement error and modeled as an unobserved component. Since various error models are already known to be classical measurement error model [3, 4], (a measurement error is classical if it is independent of unobserved variable), while in many situation measurement error might follows the model as follows,

$$y_t = \gamma_0 + \gamma_1 x_t + u_t \qquad\qquad t = 1, ..., T \tag{1}$$

where the observed error prone variable y_t is assumed to be related to the true unobserved x_t through the above model [2, 3]. This error model is referred to as the non-classical measurement error model or error calibration model [3, 4]. Model (1) is the classical measurement error model when $\gamma_0 = 0, \gamma_1 = 1$. This implies that $E(Y_t|X_t = x_t) = \gamma_0 + \gamma_1 x_t$, unlike the classical error model, suggesting that Y_t is a possibly biased measure of X_t. Here, it is assumed that X_t and U_t are independent in each time t [3, 4]. For example [5] examined the time series studies of air pollution (effect of PM_{10}) and mortality, if X_t and Y_t are respectively the average personal exposure to PM_{10} and measured ambient PM_{10} concentration on day t, then the slope γ_1 measures the change in personal exposure per unit change in a measured ambient concentration and the intercept γ_0 represents personal exposure to particles that does not drive from external sources, but arises from particle clouds generated by personal activities or unmeasured micro-environments.

Since Autoregressive process has a clear structure in presence measurement errors, therefore it is a popular choice for modeling of time series in many fields. This is especially true in population dynamics where $AR(1), AR(2)$ models are often employed (see e.g., [6, 7]). [4] develops tests to identification of measurement error in long and short memory series. [8] studies the parameters estimation of Autoregressive models using naive, ARMA and corrected naive methods under the classical error model and heteroskedastic measurement errors. [9] studies the asymptotic properties of estimators based on Yule-Walker equations in autoregressive models with measurement error.

The goal of this paper is to estimate the parameters of $AR(p)$ process under the non-classical error model (1), and compare the bias and corresponding standard errors of the estimators with the classical error model.

In model (1), $\{x_t\}$ the true time series agent which is unobserved, instead we observed $\{y_t\}$, and $\{u_t\}$ is measurement error with heteroskedastic variance i.e. $\sigma^2_{u_t}$.

Without loss of generality, suppose that unobserved time series $\{x_t\}$ is mean zero Autoregressive process model of order p, AR(p),

$$X_t = \phi_1 X_{t-1} + ... + \phi_p X_{t-p} + e_t \qquad t = p+1, ..., T \tag{2}$$

where e_t is white noise (Random variables $e_1, e_2, ..., e_n$ are independent and identically distributed with a mean of zero. This implies that the variables all have the same variance σ^2_e and $Cor(e_i, e_j) = 0$ for all $i \neq j$. If, in addition, the variables also follow a normal distribution (i.e., $e_t \sim N(0, \sigma^2_e)$) the series is called Gaussian white noise). This paper is summarized as follows. In the next Section, the homoskedastic and heteroskedastic measurement errors are described. In Section 3, we obtain the AR estimators and theirs variances using different method under the non-classical error model. We compare the extracted estimators with that [8] examined under the classical error model. Section 4 provides the result of simulation studies. It should be noted that method based on the ARMA result among the existing (naive and corrected naive methods) is free from the estimation of $\hat{\sigma}^2_{u_t}$.

2 Hemosckedastic and Heteroskedastic Measurement Errors

Hemosckedastic measurement error refers to case that the variance of measurement error is constant. While historically constant measurement error variances have been used, there are many situations that changing variances be allowed, i.e., with respect to definition of the Heteroskedastic measurement error, we are allowed for the fact that measurement error variances could change across observation. The variances of the Heteroskedastic measurement errors introduce as a function of sampling effort and unobserved variable, i.e.

$\sigma^2_{u_t}(x_t)$ which may depend on x_t through a function $h(x_t)$ and may depend on the sampling effort at time t. For asymptotic purpose it is assumed when the measurement error is Heteroskedastic, σ^2_u is defined as the limit average of $\sigma^2_{u_t}$ as follows [8],

$$lim_{T\to\infty}\frac{\sum_{t=1}^{T}\sigma^2_{u_t}}{T}=\sigma^2_u \tag{3}$$

3 Estimators under the Non-Classical Error Model

In this section, we obtain the parameter estimation of AR process using different methods. To make clear our example, we focus on the AR(p) and provide our simulation on AR (1). We first assume Hemosckedastic measurement errors, we obtain four estimators so called the propose ARMA, naive estimators, and corrected estimators.

3.1 Gold Estimator (GE)

Suppose $\{X_t\}$, is an AR(p) process as model (2) with the unknown parameter $\phi_u = [\phi_1, , \phi_p]'$ and σ^2_e. Parameter $\widehat{\phi}_{Gold}$, without presence the measurement error and using Yule- walker estimation method (YW), derived from the following equations,

$$[\widehat{\gamma}_{(1)}, , \widehat{\gamma}_{(p)}]' - \widehat{\Gamma}\phi_{Gold} = 0, \qquad \frac{-1}{\sigma_e} + \frac{\phi'_u \hat{G}\phi_u}{\sigma_e^3} = 0$$

where$\widehat{\gamma}_{(k)}$, is lag k of sample Autocovariance from $\widehat{\Gamma}$, $p\times p$ sample covariance matrix and $\widehat{G}$ are defined as follows,

$$\widehat{\Gamma}=\begin{bmatrix}\widehat{\gamma}_{(0)} & \cdots & \widehat{\gamma}_{(p-1)}\\ \vdots & \ddots & \vdots\\ \widehat{\gamma}_{(p-1)} & \cdots & \widehat{\gamma}_{(0)}\end{bmatrix}, \widehat{G}=\begin{bmatrix}\widehat{\gamma}_{(0)} & -\widehat{\gamma}_{(1)} & -\widehat{\gamma}_{(2)}\cdots & -\widehat{\gamma}_{(p)}\\ -\widehat{\gamma}_{(1)} & \widehat{\gamma}_{(0)} & \widehat{\gamma}_{(1)}\cdots & \widehat{\gamma}_{(p-1)}\\ \vdots & \vdots & \vdots & \vdots\\ -\widehat{\gamma}_{(p)} & \widehat{\gamma}_{(p-1)} & \widehat{\gamma}_{(p-2)}\cdots & \widehat{\gamma}_{(0)}\end{bmatrix}.$$

3.2 Naive Estimator (NE)

Using some regularity conditions, the convergence of sample Autocovariances can be determined as follows [10]

$$\widehat{\gamma}_{y,|t-s|} \xrightarrow{p} \gamma_1^2\gamma_{|t-s|} \qquad \forall t \neq s$$

and

$$\widehat{\gamma}_{y,(0)} \xrightarrow{p} \gamma_1^2\gamma_{(0)} + \sigma_u^2 \qquad \forall t = s$$

where $\xrightarrow{p}$ stands for convergence in probability. As $T \to \infty$, the parameters estimating equation for ϕ is following;

$$\begin{bmatrix} \gamma_1^2\widehat{\gamma}_{(1)} \\ \vdots \\ \gamma_1^2\widehat{\gamma}_{(p)} \end{bmatrix} - \begin{bmatrix} \gamma_1^2\widehat{\gamma}_{(0)} + \sigma_u^2 & \cdots & \gamma_1^2\widehat{\gamma}_{(p-1)} \\ \vdots & \ddots & \vdots \\ \gamma_1^2\widehat{\gamma}_{(p-1)} & \cdots & \gamma_1^2\widehat{\gamma}_{(0)} + \sigma_u^2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \vdots \\ \phi_p \end{bmatrix} = 0.$$

Hence, the naive estimators $(\widehat{\phi}_{naive}, \widehat{\sigma}^2_{e,naive})$ under the error model (1) and using YW methods will converge in probability to the following equations;

$$\widehat{\phi}_{naive} \xrightarrow{p} (\gamma_1^2\Gamma + \sigma_u^2 I)^{-1}\gamma_1^2\gamma^T = (\gamma_1^2\Gamma + \sigma_u^2 I)^{-1}\gamma_1^2\Gamma\phi_{Gold},$$

and

$$\sigma^2_{e,naive} \xrightarrow{p} \gamma_1^2\gamma_{(0)} + \sigma_u^2 - \gamma_1^2\gamma^T(\gamma_1^2\Gamma + \sigma_u^2 I)^{-1}\gamma_1^2\gamma.$$

Also the asymptotic bias of $\sigma^2_{e,naive}$ will converge in probability to;

$$\widehat{\sigma}^2_{e,naive} - \widehat{\sigma}^2_{e,Gold} \xrightarrow{p} ((\gamma_1^2 - 1)\gamma_{(0)}) + \sigma_u^2 - \gamma_1^2\gamma^T(\gamma_1^2\Gamma + \sigma_u^2 I)^{-1}\gamma_1^2\gamma + \gamma^T(\Gamma)^{-1}\gamma.$$

For example when $p = 1$, i.e. $AR(1)$, it is implied that,

$$\widehat{\phi}_{naive} \xrightarrow{p} \frac{\gamma_1^2\gamma_{(1)}}{\gamma_1^2\gamma_{(0)} + \sigma_u^2} = \lambda\phi_{GOld},$$

where $\lambda = \frac{\gamma_1^2\gamma_{(0)}}{\gamma_1^2\gamma_{(0)}+\sigma_u^2}$.

Using the results of [8], the naive estimator of ϕ under the classical error model $(\gamma_0 = 0, \gamma_1 = 1)$,

$$\widehat{\phi}_{naive} \xrightarrow{p} \frac{\gamma_{(1)}}{\gamma_{(0)} + \sigma_u^2} = \lambda\phi_{GOld},$$

where $\lambda = \frac{\gamma_{(0)}}{\gamma_{(0)}+\sigma_u^2}$.

3.3 ARMA Estimator

ARMA method is based on the fact that if X_t is an $AR(p)$ process with parameters ϕ and u_t is i.i.d with mean zero and constant variance σ_u^2, then ([11, 12]),

$$\phi(B)(y_t - \gamma_0) = \gamma_1 e_t + \phi(B)u_t = \theta(B)b_t,$$

follows $ARMA(p,p)$, where parameters in the ARMA model are the same as in the original Autoregressive model. Also b_t is white noise with mean zero and variance σ_b^2, $\phi(z) = 1 - \phi_1 z - ... - \phi_p z^p$, $\theta(z) = 1 - \theta_1 z - - \theta_p z^p$ and B is the backward shift operator with $B^j W_t = W_{(t-j)}$. A simple approach to estimate ϕ_{ARMA}, is to fit an $ARMA(p,p)$ model to the observed $\{Y_t - \gamma_0\}$ series. The asymptotic properties of $\widehat{\phi}_{ARMA}$, can be computed. For example, when p=1 ([10], chapter 8),

$$\begin{pmatrix} \widehat{\phi}_{ARMA} \\ \widehat{\theta}_{ARMA} \end{pmatrix} \cong N \left(\begin{pmatrix} \phi \\ \theta \end{pmatrix}, T^{-1} \begin{pmatrix} (1-\phi^2)^{-1} & (1+\phi\theta)^{-1} \\ (1+\phi\theta)^{-1} & (1-\theta^2)^{-1} \end{pmatrix}^{-1} \right).$$

It can be conclude $\widehat{\phi}_{ARMA} \cong AN\left(\phi, \frac{(1+\phi\theta)^2(1-\phi^2)}{T(\phi+\theta)^2}\right)$, where AN is denoted approximately normal. There are two solutions for θ_1, but only one leads to a stationary process. Defining,

$$k = \frac{\gamma_1^2\sigma_e^2 + \sigma_u^2\left(1+\phi_1^2\right)}{\phi_1\sigma_u^2},$$

then,

$$\theta_1 = \frac{-k + \sqrt{k^2-4}}{2}, \quad \text{if} \quad 0 < \phi_1 < 1$$

and

$$\theta_1 = \frac{-k - \sqrt{k^2-4}}{2}, \quad \text{when} \quad -1 < \phi_1 < 0.$$

Therefore for classical error model, one can obtain the similar result as non-classical one just by substituting σ_e^2 instead of $\gamma_1^2\sigma_e^2$, because $\gamma_1 e_t$ is white noise with mean zero and variance $\gamma_1^2\sigma_e^2$.

3.4 Corrected Naive Estimator (CNE)

The problem with the naive estimator arises from the bias in the estimator of $\gamma_{y,(0)}$. For the eliminate bias, we obtain the following corrected estimator;

$$\widehat{\phi}_{CNE} = \left(\widehat{\Gamma}_Y - \widehat{\sigma}_u^2 I_p\right)^{-1} \left(\widehat{\gamma}_{Y,(1)}, ..., \widehat{\gamma}_{Y,(p)}\right)^T$$

where $\widehat{\sigma}_u^2 = \frac{\sum_{t=1}^T \widehat{\sigma}_{u_t}^2}{T}$. Like moment estimator in linear regression, $\widehat{\phi}_{CNE}$ can have problems for small samples. One reason is the distribution of the corrected estimator of $\gamma_{y,(0)}, \widehat{\gamma}_{y,(0)} - \widehat{\sigma}_u^2$ can have positive probability around zero, so the estimate of $\gamma_{y,(0)}$ may be negative (Bounaccorsi, 2010). To gard this problem,we propose a modified version of this estimator; if $\widehat{\Gamma}_Y - \widehat{\sigma}_u^2 I_p$ is non positive definite, then $\widehat{\phi}_{CNE} := \widehat{\phi}_{naive}$. In line with the proposition of [8], the estimator $\widehat{\phi}_{CNE}$ is consistent if,

$$lim_{T\to\infty}\widehat{\sigma}_u^2 = \frac{\sum_{t=1}^T \widehat{\sigma}_{u_t}^2}{T} \xrightarrow{p} \sigma_u^2 \tag{4}$$

Theorem 1 Consider model AR(1), assuming independence U_t and $\sigma_{U_t}^2$ of X_t, and using the following equations,

$$\sigma_u^2 = lim_{T\to\infty}\frac{\sum_{t=1}^T \sigma_{u_t}^2}{T}, \lambda = \frac{\gamma_1^2\gamma_{(0)}}{\gamma_1^2\gamma_{(0)} + \sigma_u^2}, \acute{\lambda} = \frac{\gamma_{(0)}}{\gamma_{(0)} + \sigma_u^2},$$

and

$$\sigma_{\sigma_u^2}^2 = lim_{T\to\infty} var\left(\widehat{\sigma}_u^2\right), \overline{\sigma_u^4} = lim_{T\to\infty}\frac{\sum_t \sigma_{u_t}^4}{T}, \overline{E(u^4)} = lim_{T\to\infty}\frac{\sum_t E(u_t^4)}{T},$$

then the asymptotic variance of $\widehat{\phi}_{CNE}$ under non-classical error model is as follows,

$$avar(\widehat{\phi}_{CNE}) = \frac{1}{T}\left[\left[\gamma_1^4\left(1-\phi^2\right)\left[\frac{2-\lambda}{\lambda}\right]\right] + \frac{\left(1-\phi^2\right)\overline{\sigma_u^4}}{\gamma_{(0)}^2} + \frac{\phi^2\overline{E(u^4)}}{\gamma_{(0)}^2} + \frac{\phi^2}{\gamma_{(0)}^2}\sigma_{\sigma_u^2}^2\right].$$

By assuming variance σ_u^2 to be constant and $\overline{E(u^4)} = \delta\sigma_u^4$,

$$avar(\widehat{\phi}_{CNE}) = \frac{1}{T}\left[\left[\gamma_1^4\left(1-\phi^2\right)\left[\frac{2-\lambda}{\lambda}\right]\right] + \frac{\phi^2}{\gamma_{(0)}^2}\sigma_{\sigma_u^2}^2 + \frac{(1-\acute{\lambda})^2[\phi^2(\delta-1)+1]}{\acute{\lambda}^2}\right],$$

and under normality measurement error, i.e $\delta = 3$;

$$avar(\widehat{\phi}_{CNE}) = \frac{1}{T}\left[\left[\gamma_1^4\left(1-\phi^2\right)\left[\frac{2-\lambda}{\lambda}\right]\right] + \frac{\phi^2}{\gamma_{(0)}^2}\sigma_{\sigma_u^2}^2 + \frac{(1-\acute{\lambda})^2[2\phi^2+1]}{\acute{\lambda}^2}\right].$$

(For more details of proof see Appendix).

Finally using the proposition (3) of [8], it can be concluded that the asymptotic variance of $\widehat{\phi}_{CEE}$ has approximately normal distribution. Under the classical error model, similar results can be observed by substituting $\gamma_0 = 0, \gamma_1 = 1$.

4 Simulation study

In this section, simulation studies are carried out for $AR(1)$ process. In order to investigate the bias of the estimators and theirs standard errors, a simulation study is done for the case where $\{x_t\}$ is $AR(1)$ process. The simulation is done in three steps:

- At first step, T segment was simulated from $AR(1)$ for $\{x_t\}$
- At second step, by assuming normality of the measurement error u_t, y_t was simulated from model (1) for all the different values of γ_1 between (0.5, 2).
- At third step, the biases and standard errors of estimators are evaluated, with 500 replication for steps (1) and (2), given different values of $\beta = 0.1, 0.3, 0.5, 0.7, 0.9$ and $T = 100, 200, 500, \lambda = 0.5, 0.75, 0.9$ under the non-classical error model.

Figure 1, represents the simulation result under the non-classical measurement error model. The left and right panels are respectively bias and standard error of the estimators. It is clear that biases are decreased when the values of T are increased, but bias behavior of the estimators is uniform by changing values of the γ_1, that is it shows the low influence of the γ_1. Also it can be observed that the bias of the naive estimator is more than the other estimators. Similarly, We observe that the influence of γ_1 and T respectively in the naive and ARMA estimators standard error.

Figure 2 shows the second scenarios of the obtained estimator through the considered values of ϕ. It can be seen that, for the small values of ϕ, the bias of the ARMA estimator is more than the other estimators, because identification from of the ARMA model be harder.

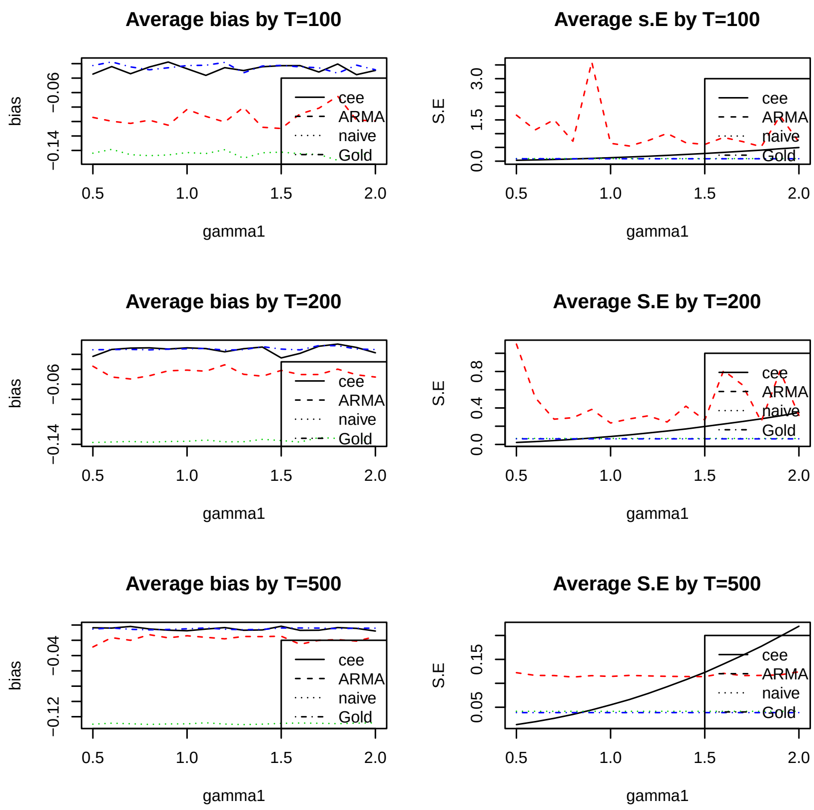

Figure 1: bias of the estimators of the Autoregressive process of order one for $\gamma_1 = (0.5, 2)$ and $T = 100, 200, 500$ and $\phi = 0.5, \lambda = 0.75$.

Figure 3 represents the simulation result of the corresponding estimator's standard error. It clearly can be seen that by changing the considered values of ϕ, the naive and ARMA estimators considerably vary. This might be because of the impact of increasing of the standard error of $\hat{\gamma}_1$.

Figure 4 shows the presentation of the estimators behavior and corresponding standard errors by different values of the λ, where λ is function of γ_1 and σ_u^2, i.e., $\lambda = \frac{\gamma_1^2 \gamma_{(0)}}{\gamma_1^2 \gamma_{(0)} + \sigma_u^2}$. As is shown, when λ values is increased, bias is decreased. Because it might be the les impact of σ_u^2. The standard error of the naive estimator is more than the ARMA estimator when the values of γ_1 and λ is simultaneously increased.

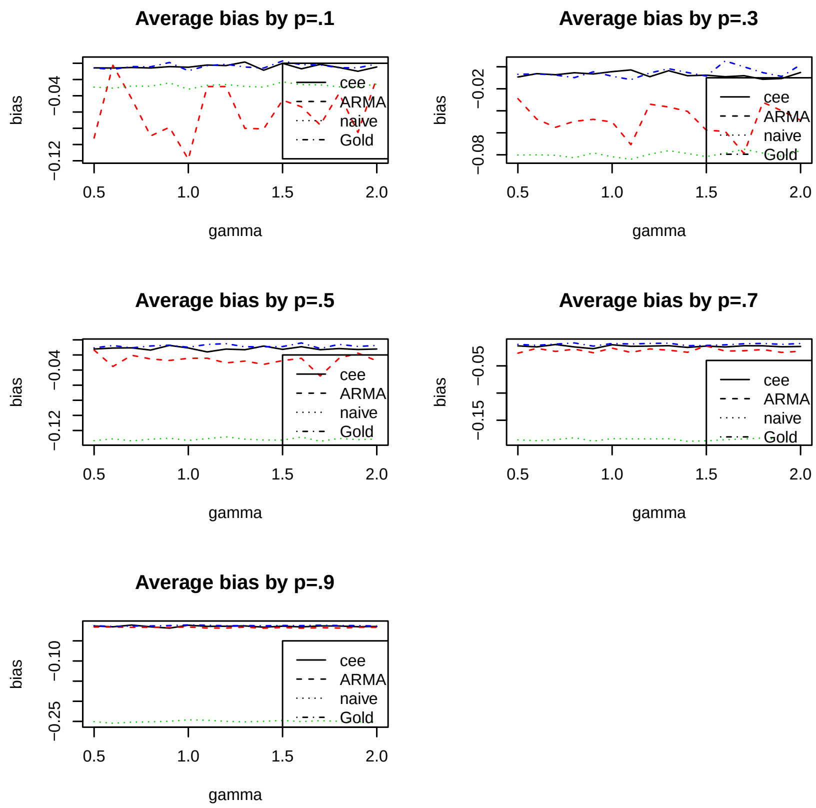

Figure 2: bias of the estimators of the Autoregressive process of order one for $\gamma_1 = (0.5, 2)$ and $\phi = 0.1, 0.3, 0.5, 0.7, 0.9$ and $T = 300, \lambda = 0.75$.

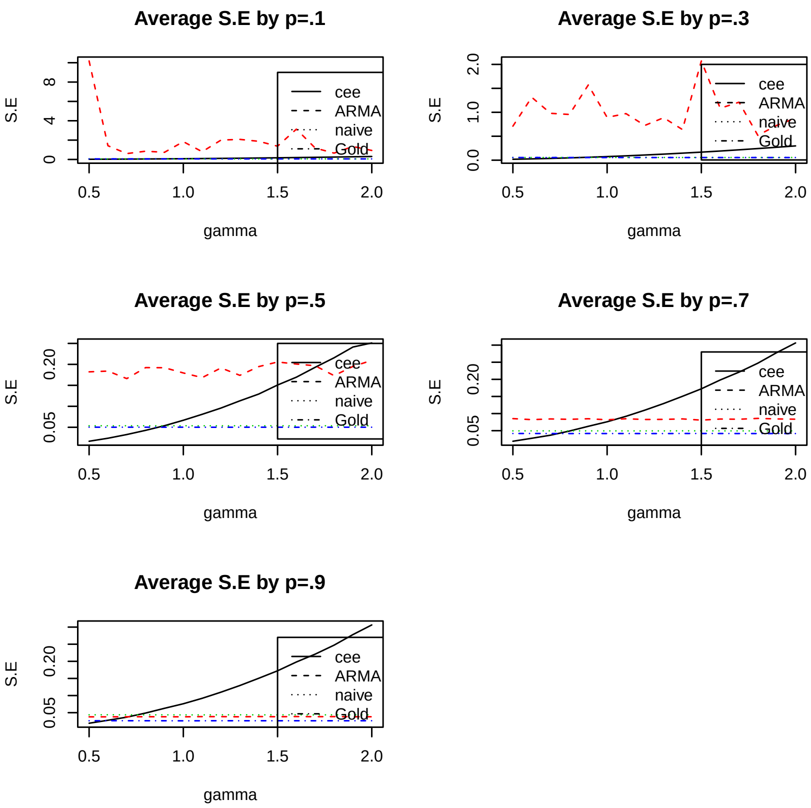

Figure 3: standard error of the estimators of the Autoregressive process of order one for $\gamma_1 = (0.5, 2)$ and $\phi = 0.1, 0.3, 0.5, 0.7, 0.9$ and $T = 300, \lambda = 0.75$.

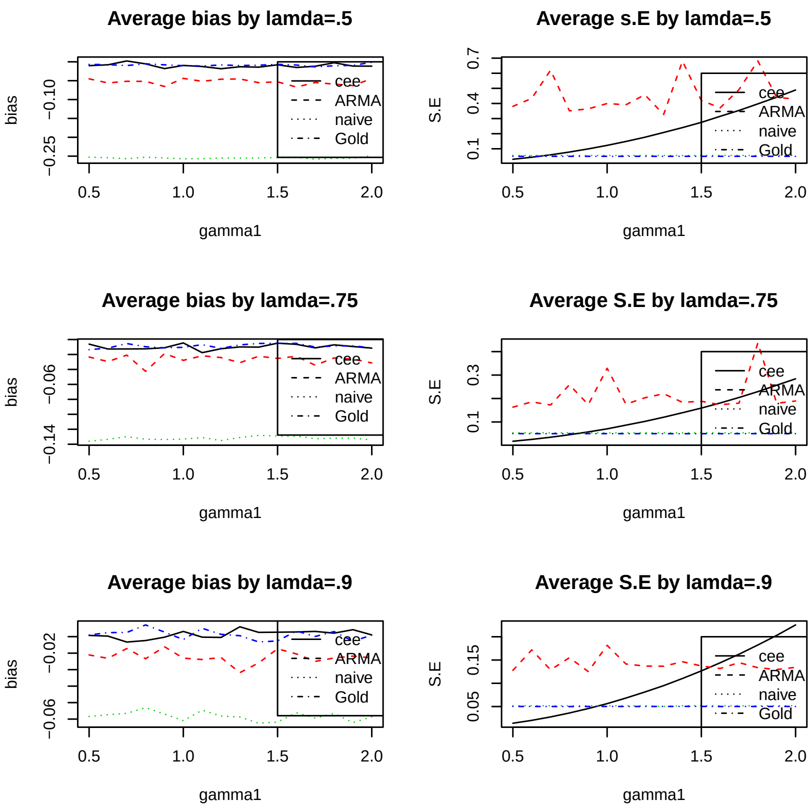

Figure 4: bias of the estimators of the Autoregressive process of order one for $\gamma_1 = (0.5, 2)$ and $\lambda = 0.5, 0.75, 0.9$ and $T = 300, \phi = 0.5$.

5 Conclusion

This study concerns the estimation of parameter for $AR(1)$ process in the presence of non-classical error model. From theoretical point of view and also simulation study, it is concluded that Corrected naive and Gold estimators have similar behaviors from biases and the standard errors in the presence of the non-classical measurement error model. Moreover, when the parameters of the model are like that the model is markedly far from to be a white noise, i.e., ϕ and γ_1 are increased simultaneously, it can be seen that the bias and standard error of the ARMA estimator closed to CNE and GE. Further the naive estimator has not good performance, because with increasing γ_1 parameter, the corresponding standard error is increased. A comparison between the estimators, it can be obtained that Corrected naive estimator is more efficient than the other estimators. These results show the trace of non-classical measurement error model and measurement error, and considering that tends to be more valuable and informative.

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Appendix

In this Section we shad light on some formulas that we have pointed out in the text.
The asymptotic behavior of $\sqrt{T}\begin{pmatrix}\widehat{\gamma}_y-\gamma_y\\ \widehat{\sigma}_u^2-\sigma_u^2\end{pmatrix}$ is obtained, where $\gamma_y=\left[\gamma_{y,(0)},...,\gamma_{y,(p)}\right]'$.
The variance of $\widehat{\phi}_{CEE}$ is determined by using the delta method.
If $Z_t'=[Y_t^2,Y_tY_{t+1},...,Y_tY_{t+p}]$ and $\overline{Z}=\frac{\sum_t Z_t}{T}$, then the asymptotic behavior $\sqrt{T}\begin{pmatrix}\widehat{\gamma}_y-\gamma_y\\ \widehat{\sigma}_u^2-\sigma_u^2\end{pmatrix}$ is equivalent to $\sqrt{T}\overline{Z}$. By taking the limit exists, we can evaluate $lim_{T\to\infty}Cov(\sqrt{T}\overline{Z})=Q$, where

$$Q=\begin{bmatrix} Q_{\gamma} & Q_{\gamma,\sigma_u^2}\\ Q_{\sigma_u^2,\gamma} & Q_{\sigma^2_{\sigma_u^2}}\end{bmatrix}.$$

For computation of the matrix Q, we should obtain elements of the matrix Q, i.e., $lim_{T\to\infty}Cov(\widehat{\gamma}_{y,(p)},\widehat{\gamma}_{y,(r)})$ and $lim_{T\to\infty}Cov(\widehat{\gamma}_{y_{(r)}},\widehat{\sigma}^2_u)$. To do so, we obtain

$$\begin{aligned}
&lim_{T\to\infty}Cov(\widehat{\gamma}_{y,(p)},\widehat{\gamma}_{y,(r)})\\
&\quad= lim_{T\to\infty}\frac{1}{T}E(\sum_t(Y_t-E(Y_t))(Y_{t+p}-E(Y_{t+p}))\sum_s(Y_s-E(Y_s))(Y_{s+r}-E(Y_{s+r})))\\
&\quad- lim_{T\to\infty}TE(\widehat{\gamma}_{y,(p)})E(\widehat{\gamma}_{y,(r)})\\
&\quad= lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_s E\left((\gamma_1X_t+U_t)(\gamma_1X_{t+p}+U_{t+p})(\gamma_1X_s+U_s)(\gamma_1X_{s+r}+U_{s+r})\right))\\
&\quad- \gamma_{y,(p)}\gamma_{y,(r)}.
\end{aligned}$$

Now it follows from the latter that,

$$E(\gamma_1X_t+U_t)(\gamma_1X_{t+p}+U_{t+p})(\gamma_1X_s+U_s)(\gamma_1X_{s+r}+U_{s+r}).$$

Assuming independence of X_t and U_t, we have,

$$\begin{aligned}
&= I(r=0)[\gamma_1^2E(X_tX_{t+p})E(U_s^2)] + I(p=0)[\gamma_1^2E(X_sX_{s+r})E(U_t^2)]\\
&\quad + I(p=r=0)[E(U_t^2U_s^2)] + I(p=r\neq 0)[E(U_t^2U_{t+p}^2)]\\
&\quad + I(s=t+p-r)[\gamma_1^2E(X_tX_s)E(U_{t+p}U_{s+r})]\\
&\quad + I(s=t+p)[\gamma_1^2E(X_tX_{s+r})E(U_{t+p}U_s)]\\
&\quad + I(s=t-r)[\gamma_1^2E(X_{t+p}X_s)E(U_tU_{s+r})]\\
&\quad + I(s=t)[\gamma_1^2E(X_{t+p}X_{s+r})E(U_tU_s)] + [\gamma_1^4E(X_tX_{t+p}X_sX_{s+r})].
\end{aligned}$$

Replacing in the previous equation, it can be concluded,

$$\begin{aligned}
&lim_{T\to\infty}Cov(\widehat{\gamma}_{y,(p)},\widehat{\gamma}_{y,(r)})\\
&\quad= lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_s I(r=0)[\gamma_1^2E(X_tX_{t+p})E(U_s^2)])\\
&\quad+ lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_s I(p=0)[\gamma_1^2E(X_sX_{s+r})E(U_t^2)])\\
&\quad+ lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_s I(p=r=0)[E(U_t^2U_s^2)])\\
&\quad+ lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_s I(p=r\neq 0)[E(U_t^2U_{t+p}^2)])\\
&\quad+ lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_s I(s=t+p-r)[\gamma_1^2E(X_tX_s)E(U_{t+p}U_{s+r})])
\end{aligned}$$

$$+ lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_s I(s=t+p)[\gamma_1^2 E(X_tX_{s+r})E(U_{t+p}U_s)])$$
$$+ lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_s I(s=t-r)[\gamma_1^2 E(X_{t+p}X_s)E(U_tU_{s+r})])$$
$$+ lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_s I(s=t)[\gamma_1^2 E(X_{t+p}X_{s+r})E(U_tU_s)])$$
$$+ lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_s[\gamma_1^4 E(X_tX_{t+p}X_sX_{s+r})]) - \gamma_{y,(p)}\gamma_{y,(r)}.$$

If $Cov(\widehat{\gamma}_{y,(p)},\widehat{\gamma}_{y,(r)}) = q_{\gamma rp}$, then using simple calculation for different form of $r,p=0,1$, we will have;

$$q_{\gamma,00} = \gamma_1^4 q_{00} + 4\gamma_1^2\gamma_{(0)}\sigma_u^2 + \overline{E(u^4)} - \overline{\sigma_u^4}$$
$$q_{\gamma,01} = \gamma_1^4 q_{01} + 4\gamma_1^2\gamma_{(1)}\sigma_u^2$$
$$q_{\gamma,11} = \gamma_1^4 q_{11} + 2\left[\gamma_1^2\gamma_{(0)}\sigma_u^2 + \gamma_1^2\gamma_{(2)}\sigma_u^2\right] + lim_{T\to\infty}\frac{\sum_{t=1}^T \sigma_{u_t}^2\sigma_{u_{t+1}}^2}{T}$$

and also it implies $Q_{\widehat{\gamma},\widehat{\sigma}_u^2} = 0$, because

$$lim_{T\to\infty}Cov(\widehat{\gamma}_{y,(p)},\widehat{\sigma}_u^2) = lim_{T\to\infty}\frac{1}{T}(\sum_t\sum_r E(Y_tY_{t+p})\widehat{\sigma}_{u_t}^2) - \gamma_{y,(p)}\sigma_u^2 = 0.$$

Now, is obtained the corrected naive estimator under AR(1) model, $\widehat{\phi}_{CNE} = \frac{\widehat{\gamma}_{y,(0)}}{\widehat{\gamma}_{y,(0)}-\widehat{\sigma}_u^2}$, using three variables of the Taylor expansion evaluated at $(\widehat{\gamma}_{y,(0)},\widehat{\gamma}_{y,(1)},\widehat{\sigma}_u^2)$, we have,

$$\widehat{\phi}_{CNE} = \phi_{Gold} + \frac{1}{\gamma(0)}(\widehat{\gamma}_{y,(1)} - \phi\widehat{\gamma}_{y,(0)}) + \frac{\phi}{\gamma_{(0)}}\widehat{\sigma}_u^2.$$

Then,

$$var(\widehat{\phi}_{CNE}) = \frac{1}{T\gamma_{(0)}^2}\left[q_{\gamma 11} + \phi^2 q_{\gamma 00} - 2\phi q_{\gamma 01}\right] + \frac{\phi^2}{T\gamma_{(0)}^2}\sigma_{\sigma_u^2}^2.$$

Substituting $var(\phi_{Gold}) = \frac{1}{T\gamma_{(0)}^2}[q_{11}+\phi^2 q_{00}-2\phi q_{01}]$ (where q_{rp} is auto-covariance between $\widehat{\gamma}_{x,(p)}$ and $\widehat{\gamma}_{x,(r)}$) in the above equation, it can be compute that,

$$\begin{aligned} var(\widehat{\phi}_{CNE}) &= \gamma_1^4\frac{1-\phi^2}{T} + \frac{1}{T\gamma_{(0)}^2}\left[2\gamma_1^2\sigma_u^2\left(\gamma_{(0)}+\gamma_{(2)}\right) + lim_{T\to\infty}\frac{\sum_t \sigma_{u_t}^2\sigma_{u_{t+1}}^2}{T}\right] \\ &\quad + \frac{1}{T\gamma_{(0)}^2}\left[\phi^2(4\gamma_1^2\gamma_{(0)}\sigma_u^2 + \overline{E(u^4)} - \overline{\sigma_u^4})\right] - \frac{1}{T\gamma_{(0)}^2}\left[8\phi\gamma_1^2\gamma_{(1)}\sigma_u^2\right] \\ &\quad + \frac{\phi^2}{T\gamma_{(0)}^2}\sigma_{\sigma_u^2}^2. \end{aligned}$$

Proof is proved with simplification.

References

[1] A.J. Scott, T.M.F. Smit and R.G. Jones, The application of time series methods to the analysis of repeated surveys, *International Statistical Review*, **45**, (1977), 13-28.

[2] A.J. Scott and T.M.F. Smit, Analysis of repeated survays using time series methods, *Journal American Statistical Association*, **69**, (1974), 674-678.

[3] J.P. Bounaccorsi, *Measurement error models, methods and applications*, New York, Chapman Hall/CRC, Taylor and Francis Group, 2010.

[4] K. Tanka, A unified approach to the measurement error problem in time series Models, *Econometric Theory*, **18**, (2002), 278-296.

[5] F. Dominici and S. Zeger, A measurement error model for time-series studies of air pollution and mortality, *Biostatistics*, **1**, (2000), 157-175.

[6] B. Dennis and M.L. Taper, Density dependence in time series observations of natural populations: estimation and testing, *Ecology*, **64**, (1994), 205-224.

[7] D.W. Williams and A.M. Liebhold, Influence of weather on the synchrony of gypsy moth (Lepidoptera: Lymantriidae) outbreaks in New England, *Envirnomental entomlogy*, **24**, (1995), 987-995.

[8] J. Staudenmayer and J.P. Bounaccorsi, Measurement error in linear autoregressive models, *Journal American Statistical Association*, **471**, (2005), 841-852.

[9] K.C. Chanda, Asymptotic properties of estimators for autoregressive models with errors in variables, *The Annals of Statistics*, **24**, (1996), 423-430.

[10] P.J. Brockwell and R.A. Davis , *Time series: Theory and methods*, Second Edition, New York, Springer, 1991.

[11] G.E.P. Box, G.M. Jenkins and G. Reinsei *Time Series Analysis: Forecasting and Control*, 3rd ed., New York: Prentice-Hall, 1994.

[12] M. Pagano, Estimation of Models of Autoregressive Signal Plus White Noise, *The Annals of Statistics*, **2**, (1974), 99-108.