Journal of Statistical and Econometric Methods, vol.1, no.3, 2012, 79-96
ISSN: 1792-6602 (print), 1792-6939 (online)
Scienpress Ltd, 2012

Mixed-fractional Models to Credit Risk Pricing

Xichao Sun[1] **and Litan Yan**[2]

Abstract

This paper proposes a mixed fractional Brownian motion version of a well-known credit risk pricing structural model: the Merton model. Assume that the value of the firm obeys to a geometric mixed fractional Brownian motion, default probability, pricing of bonds, values of stocks and credit spreads are derived. Figures are given to illustrate the effectiveness of the result and show that the mixed-fractional models to credit risk pricing is a reasonable one.

Mathematics Subject Classification : 60H05, 60G15, 60G18
Keywords: Credit Risk, Structural Models, Credit Spreads, stochastic integration

1 Introduction

Briefly speaking, credit risk is an investor's risk of loss arising from a borrower who does not make payments as promised. For modeling credit risk, there are two main approaches: structural models and the reduced form models. Structural models, pioneered by Merton [19], assume that the default

[1] College of Information Science and Technology, Donghua University, 2999 North Renmin Rd., Songjiang, Shanghai 201620, P.R. China, e-mail: sunxichao626@126.com
[2] Department of Mathematics, Donghua University, 2999 North Renmin Rd., Songjiang, Shanghai 201620, P.R. China, e-mail: litanyan@dhu.edu.cn

Article Info: *Received* : October 19, 2012. *Revised* : November 16, 2012
Published online : November 30, 2012

time-point is typically specified as the first moment at which the firm's asset value reaches a specific threshold boundary. The major investigation within these firm value models is to characterize the evolution of the firm's value, as well as the firm's capital structure, see related papers Black-Cox [4], Geske [10] and Leland [15]. More realistic assumptions are allowed in Shimko et al. [23], such as the possibility of default before maturity, coupon payments, stochastic interest rates. Another alternative to structural models is the reduced form approach originated with Jarrow and Turnbull [12], which directly models the default process of risky debt. In combination with the assumptions on the evolution of the risk-free rate and the recovery rate in the event of default, this is used to value risky debt. For the literature on the reduced form models, we refer to Duffie-Singleton [9], Madan-Unal [16], Su and Wang[24] and references therein.

Though Merton model [19] has become the most popular method for credit risk and its generalized version has provided mathematically beautiful and powerful results on credit risk, they are still theoretical adoptions and not necessarily consistent with empirical features of financial return series, such as nonlinearity, long-range dependence, etc, which contradict to the traditional Merton assumption. For example Hsieth[11], Mariani et al.[18], Ramirez et al.[22] and Willinger et al.[27] showed that returns are of long-range (or short-range) dependence, which suggests strong time-correlations between different events at different time scales (e.g., see Mandelbrot [17], and Cajueiro and Tabak [5, 25]). In the search for better credit risk models for describing long-range dependence in financial return series, a fractional Brownian (fBm) model has been proposed as an improvement of the classical Merton model, see Biagini et al.[2], Leccadito and Urga [14] and references therein.

In this paper, we propose the mixed fractional Brownian motion version of a Merton credit risk model. The presence of long memory in credit spreads time series would provide a justification for the theoretical models proposed, that, in turn, would be able to explain realized credit spreads better than traditional credit risk structural models.

The remainder of this paper is organized as follows. Section 2 presents the mixed fractional Brownian motion version of the Merton models. The default probability is discussed in Section 3. In Section 4, we investigate the values of stocks and bonds. Section 5 describes the credit spreads.

2 Merton mixed-fractional model

In the structural approach to credit risk the firm liabilities (equity and bonds et al.) are viewed as derivative contracts on the market value of a firms assets. A stochastic process for the evolution of the firm underlying assets V and the conditions under which a default is triggered as well as the payoff of the risky debt in the event of default are specified. Merton [19] assumes that the firm has only issued zero coupon bonds with maturity T and total face value L, that default may happen only at maturity. Denote by $M(V_T)$ and $N(V_T)$ the prices in T of a defaultable zero coupon bond and the equity respectively. Both M and N are functions of V and more generally all claims on the firm's value are evaluated as derivative securities with the firm's value as underlying. The term structure of interest rate is assumed to be deterministic and the firm pays no dividend over the life of the debt. In case of default bondholders are assumed to have absolute priority, i.e. bond value at time T is $M(V_T) = \min(L, V_T)$ and the equity is simply a call option, $N(V_T) = \max(V_T - L, 0)$.

Whereas the original model assumes a Geometric Brownian motion for the firm value, in this paper we consider the following dynamics for V:

$$dV_t = \mu V_t dt + dX_t, \tag{1}$$

where X_t denotes a mixed fractional Brownian motion (mfBm) and the stochastic integration is divergence-type. The so called mfBm X_t defined by Cheridito [8] is linear combination of a Brownian motion W_t and an independent fractional Brownian motion B_t^H with Hurst parameter $0 < H < 1$ defined on the same probability space (Ω, F, P), i.e.,

$$X_t = \sigma B_t^H + \varepsilon W_t,$$

where σ and ε are two real constants such that $(\sigma, \varepsilon) \neq (0, 0)$. Mixed fractional Brownian motions form a special class of long memory processes when Hurst parameter $H > \frac{1}{2}$. Cheridito [8] has proved that, the mixed fractional Brownian motion is equivalent to a multiple of Brownian motion if $H = \frac{1}{2}$ and equivalent to Brownian motion if $H \in (\frac{3}{4}, 1)$, and hence it is arbitrage-free. More works for mixed fractional Brownian motions and their applications in finance can be found in Charles [6], Cheridito[7], Kuznetsov[13], Mishura [20, 21], Su and Wang [24], Wang et al. [26], Yu and Yan [28], Zahle [29] and references therein.

In what follows we denote by $\Phi(\cdot)$ the cumulative probability distribution function of a standard normal random variable:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}u^2\right) du$$

and by $\varphi(\cdot) = \Phi'(\cdot)$ the density function.

3 Default probability of the Firm

The value of the put option reflects the risk of default. The higher the default risk, the higher is the value of the option, i.e. the more the firm holder has to pay to the lender to 'convince' him. An important credit risk measure is the probability that a default will occur given the information at time t under the real-world measure P.

According to Alós *et al* [1] (see also Yu and Yan [28]), we have the following:

Lemma 3.1. *The solution to Equation* (1) *is given by*

$$V_t = V_0 \exp\left(\mu t + \sigma B_t^H + \varepsilon W_t - \frac{1}{2}\sigma^2 t^{2H} - \frac{1}{2}\varepsilon^2 t\right). \tag{2}$$

Remark 1. The log-returns $R_{t,t+s} = \log \frac{V_{t+s}}{V_t}$ of mixed fractional Blacks-Scholes is given by

$$R_{t,t+s} = \mu s + \sigma(B_{t+s}^H - B_t^H) + \varepsilon(W_{t+s} - W_s) - \frac{1}{2}\sigma^2[(t+s)^{2H} - t^{2H}] - \frac{1}{2}\varepsilon^2 s.$$

Obviously, it is non stationary. However, we know the standard Black-Scholes model is Markovian and the log-returns are stationary independent Gaussian random variables.

Denote by $M(t, V_t, T)$ and $N(t, V_t, T)$ the values in t of a defaultable zero coupon bond and the equity respectively.

Theorem 3.1. *The default probability of the firm is*

$$p = P(V_T < L) = \Phi\left(\frac{\ln\frac{L}{V_0} - \mu T + \frac{1}{2}\sigma^2 T^{2H} + \frac{1}{2}\varepsilon^2 T}{\sqrt{\sigma^2 T^{2H} + \varepsilon^2 T}}\right),$$

where $l_0 = \frac{L}{V_0}$ *is the firm leverage.*

Clearly for $\sigma = 0$ we get default probability of the classical Merton model

$$p = \Phi\left(\frac{\ln\frac{L}{V_0} - \mu T + \frac{1}{2}\varepsilon^2 T}{\varepsilon\sqrt{T}}\right).$$

Proof of Theorem 3.1. Because the mixed fractional Brownian motion X_t is a centered Gaussian process, according to Lemma 3.1, we obtain

$$\begin{aligned} p = P(V_T < L) &= P\left(V_0 \exp\left(\mu T + \sigma B_t^H + \varepsilon W_t - \frac{1}{2}\sigma^2 t^{2H} - \frac{1}{2}\varepsilon^2 t < L\right)\right) \\ &= P\left(\sigma B_t^H + \varepsilon W_t < \ln\frac{L}{V_0} - \mu T + \frac{1}{2}\sigma^2 t^{2H} + \frac{1}{2}\varepsilon^2 t\right) \end{aligned}$$

Since $\frac{\sigma B_t^H + \varepsilon W_t}{\sqrt{\sigma^2 t^{2H} + \varepsilon^2 t}}$ is a standard normal random variable, it follows

$$p = \Phi\left(\frac{\ln\frac{L}{V_0} - \mu T + \frac{1}{2}\sigma^2 T^{2H} + \frac{1}{2}\varepsilon^2 T}{\sqrt{\sigma^2 T^{2H} + \varepsilon^2 T}}\right)$$

The proof is completed. □

In Figure 1, 2, for $T \in [0, 50]$, we plot the values of default probability as a function of time to maturity T for three different values of the parameters

$$(\sigma, \varepsilon) \in \{(0.15, 0.15), (0.15, 0.35), (0.25, 0.35)\}$$

and three values of the parameter $H \in \{0.55, 0.70, 0.90\}$.

It's clear that the default probability is an increasing function of the maturity time T. For values of the memory parameter H bigger than $\frac{1}{2}$, default probability is increasing.

To better understand the preference of our model, we compute the default probability using our model and make comparisons with the results of the

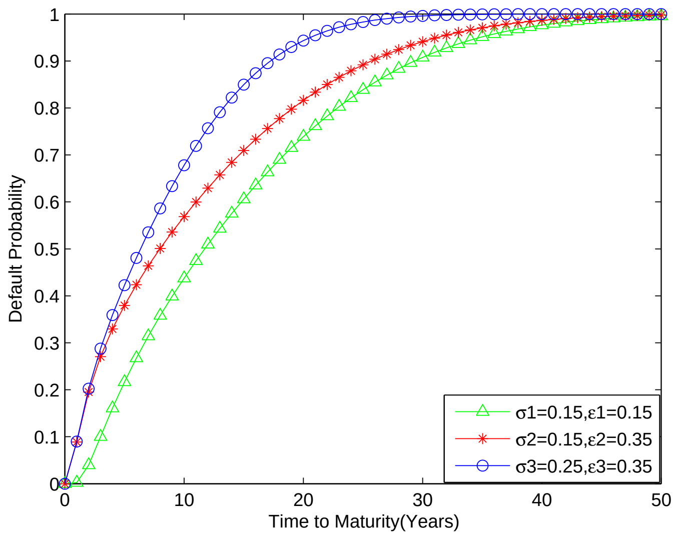

Figure 1: Default probability resulting in the mixed-fractional Merton model against maturity time T when $l_0 = 0.6$, $\mu = 0.06$, $H = 0.70$, and $0 < T < 50$

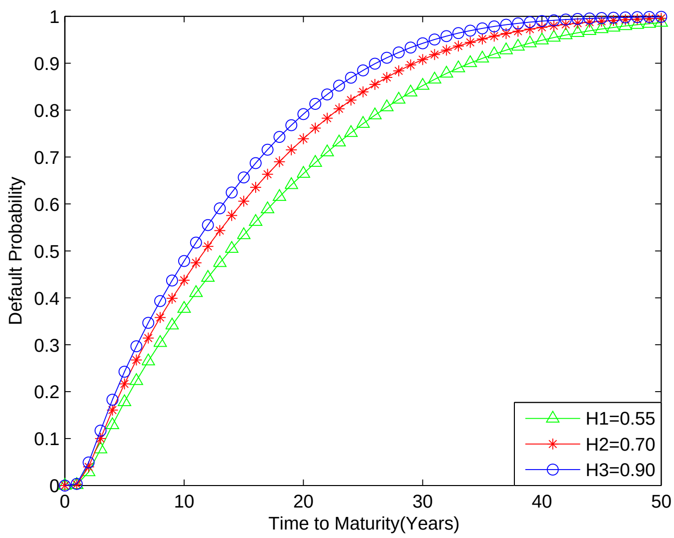

Figure 2: Default probability resulting in the mixed-fractional Merton model against maturity time T when $l_0 = 0.6$, $\mu = 0.06$, $\sigma = 0.15$; $\varepsilon = 0.15$ and $0 < T < 50$.

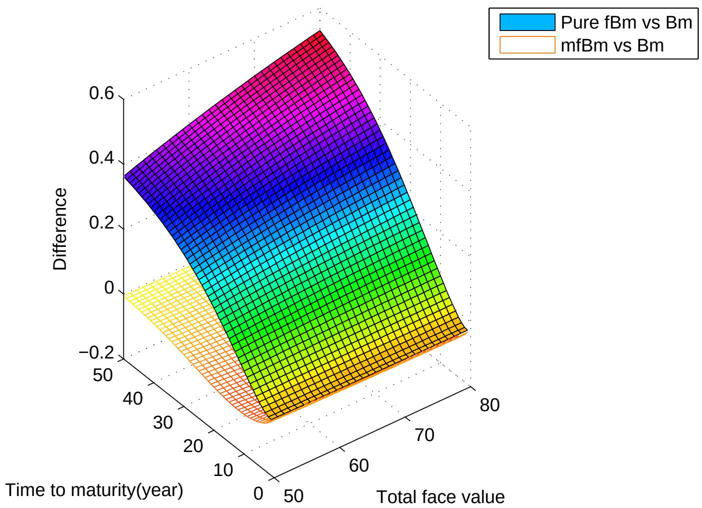

Figure 3: Relative difference of the default probability among the Bm model, pure fBm model and mfBm model

pure fractional Brownian motion model (as shown in the Appendix) based on Leccadito and Urga [14].

Now, we compare the three default probabilities: the theoretical prices derived from the Brownian motion(Bm), pure fractional Brownian motion and mfBm models. We choose these valuation of the parameters: $V_0 = 100$, $\mu = 0.06$, $\sigma = 0.15$, $\varepsilon = 0.35$, $H = 0.55$, maturity $T \in [0, 50]$ and total face value $L \in [50, 80]$. The figure 3 shows the theoretical default probability differences by the Bm model and the pure fractional Brownian motion model and our mfBm, respectively. In the figure, the vertical axis denotes the differences between the default probability, and the horizontal axis denotes time to maturity and the total face value. The figure shows that the default probability by our mixed model is better fitted to the Bm default probability than that by the pure fractional Brownian motion model. Hence, when compared to the figure, our mfBm model seems reasonable.

4 Bond Pricing and Valuation of Equity

In case of default bondholders are assumed to have absolute priority, the values of the zero coupon bond and equity at maturity T are given as follows: From the Table 1, we see that the value of the zero coupon bond at maturity

Table 1:

	V_T	$M(V_T)$	$N(V_T)$
.non-default	$V_T \geq L.$	$L.$	$V_T - L.$
.default	$V_T < L.$	$V_T.$	$0.$

time T is given by

$$M(V_T) = \min\{V_T, L\} = L - \max((L - V_T), 0).$$

and the value of the equity at maturity time T is given by

$$N(V_T) = \max\{(V_T - L), 0\}.$$

Obviously, $N(V_T)$ is just like the price of a European call option with strike price L and maturity time T.

Theorem 4.1. *Assuming absolute priority for the bondholders, a geometric mixed-fractional Brownian motion* (2) *with* $H \in [\frac{1}{2}, 1)$ *for the firm asset, and that the firm has only issued zero coupon bonds with maturity* T *and total face value* L*, when the risk-free rate is equal to* $r_s, s \in [0, T]$*, the value of the equity at time* t *is*

$$N(t, V_t, T) = V_t \Phi(d_1) - L e^{-\int_t^T r_s ds} \Phi(d_2), \tag{3}$$

where

$$d_1 = \frac{\ln \frac{V_t}{L} + \int_t^T r_s ds + \frac{\sigma^2}{2}(T^{2H} - t^{2H}) + \frac{\varepsilon^2}{2}(T - t)}{\sqrt{\sigma^2(T^{2H} - t^{2H}) + \varepsilon^2(T - t)}}, \tag{4}$$

and

$$d_2 = \frac{\ln \frac{V_t}{L} + \int_t^T r_s ds - \frac{\sigma^2}{2}(T^{2H} - t^{2H}) - \frac{\varepsilon^2}{2}(T - t)}{\sqrt{\sigma^2(T^{2H} - t^{2H}) + \varepsilon^2(T - t)}}. \tag{5}$$

Proof. Elementary calculations yield

$$N(t, V_t, T) = e^{-\int_t^T r_s ds} \mathbb{E}[N(V_T)] = V_t \Phi(d_1) - L e^{-\int_t^T r_s ds} \Phi(d_2)$$

where d_1 and d_2 is given by (4) and (5). □

The value of the zero coupon bonds at maturity time T is a portfolio: buy a risk-less bond with value L and maturity time T and sell a European put option with with strike price L and maturity time T. Then we have the following:

Theorem 4.2. *Under the assumptions of Theorem 4.1, the price of the bond at time t is given by*

$$M(t, V_t, T) = Le^{-\int_t^T r_s ds}\Phi(d_2) + V_t\Phi(-d_1). \tag{6}$$

where d_1 and d_2 is given by (4), (5).

Proof. Simple calculations show

$$\begin{aligned} M(t, V_t, T) &= e^{-\int_t^T r_s ds}\mathbb{E}[M(V_T)] = e^{-\int_t^T r_s ds}\mathbb{E}\left[L - \max\{(L - V_T), 0\}\right] \\ &= e^{-\int_t^T r_s ds}L - P(t, V_T, L, \sigma, r, T), \end{aligned}$$

where

$$P(t, V_T, L, \sigma, r, T) = Le^{-\int_t^T r_s ds}\Phi(-d_2) - V_t\Phi(-d_1).$$

The proof is completed. □

Remark 2. (1)Clearly for $\sigma = 0$, we get the classical Merton model and (3) reduces to the Black-Scholes formula for a call option.

(2) The relationship between the price of the bond and the value of the equity at time t is given by

$$M(t, V_t, T) + N(t, V_t, T) = V_t.$$

In Figure 4, 5, 6, 7, we plot the values of stocks and bonds as a function of time to maturity T for three different values of the parameters

$$(\sigma, \varepsilon) \in \{(0.15, 0.15), (0.15, 0.35), (0.25, 0.35)\}$$

and three values of the parameter $H \in \{0.55, 0.70, 0.90\}$.

It is clear that the value of a stock is an increasing function of the maturity time T and the value of a bond is decreasing with respect to T. The value of

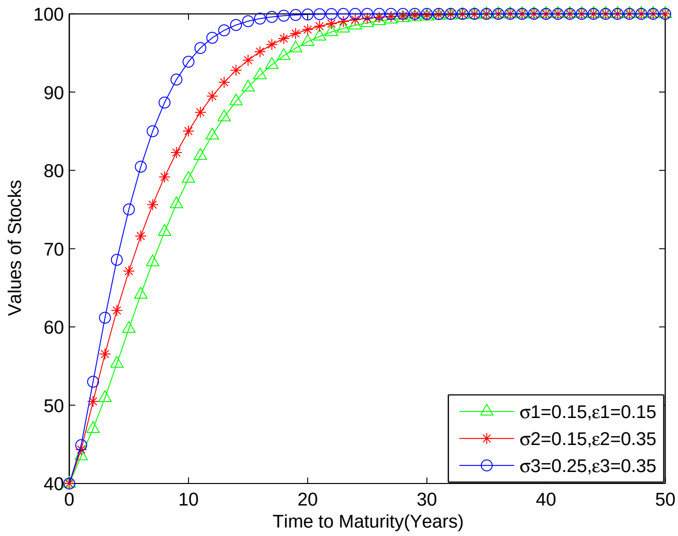

Figure 4: Values of stocks at time zero resulting in the mixed-fractional Merton model against maturity time T when $l_0 = 0.6$, $r = 0.06$, $H = 0.70$, and $0 < T < 50$

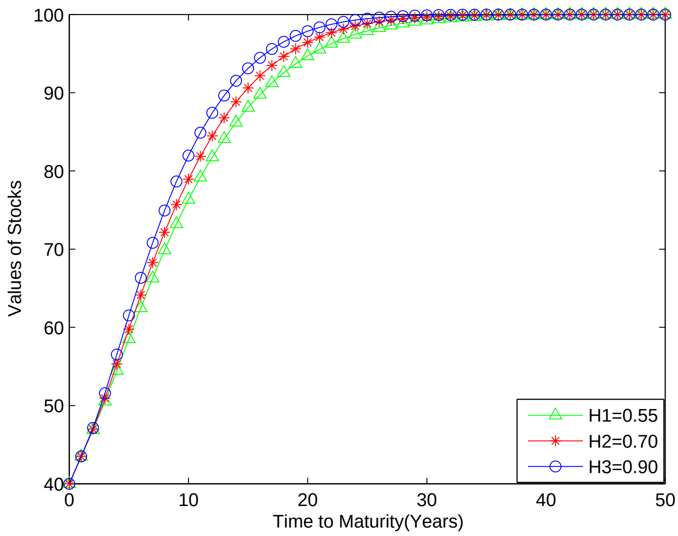

Figure 5: Values of stocks at time zero resulting in the mixed-fractional Merton model against maturity time T when $l_0 = 0.6$, $r = 0.06$, $\sigma = 0.15$; $\varepsilon = 0.15$ and $0 < T < 50$.

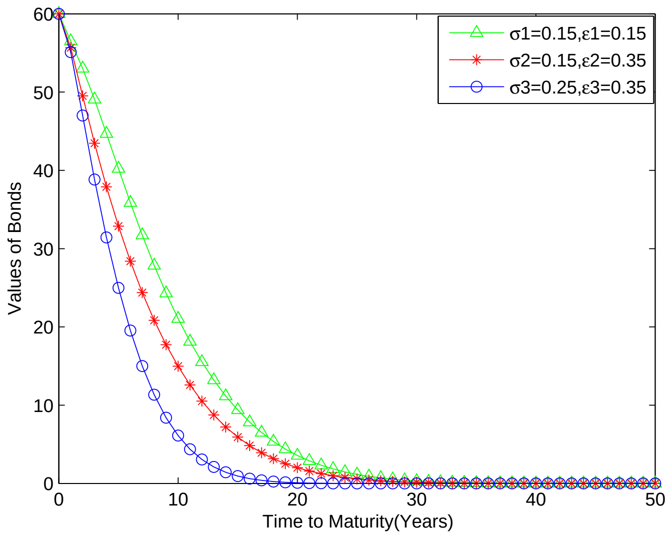

Figure 6: Values of bonds at time zero resulting in the mixed-fractional Merton model against maturity time T when $l_0 = 0.6$, $r = 0.06$, $H = 0.70$, and $0 < T < 50$

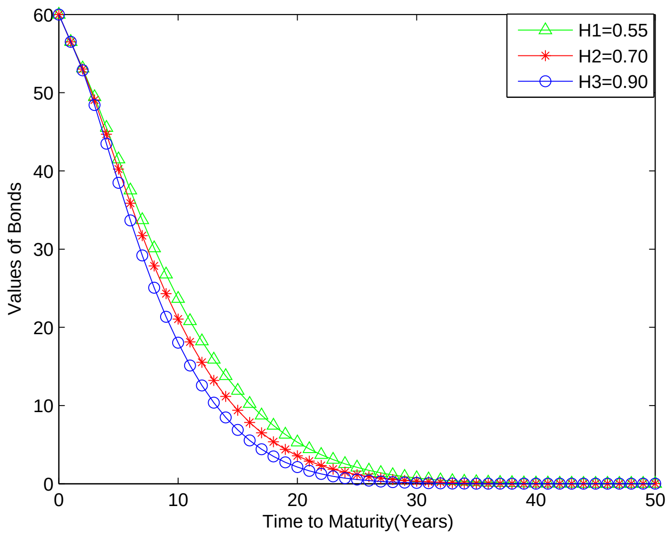

Figure 7: Values of bonds at time zero resulting in the mixed-fractional Merton model against maturity time T when $l_0 = 0.6$, $r = 0.06$, $\sigma = 0.15$; $\varepsilon = 0.15$ and $0 < T < 50$.

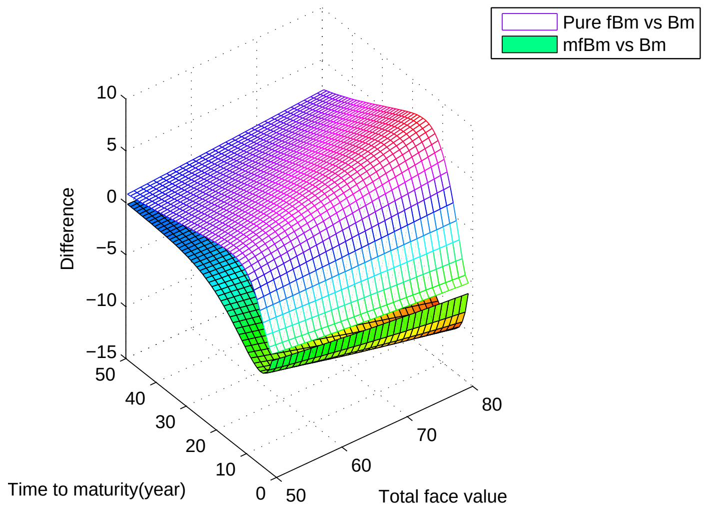

Figure 8: Relative difference of the values of stocks among the Bm model, pure fBm model and mfBm model

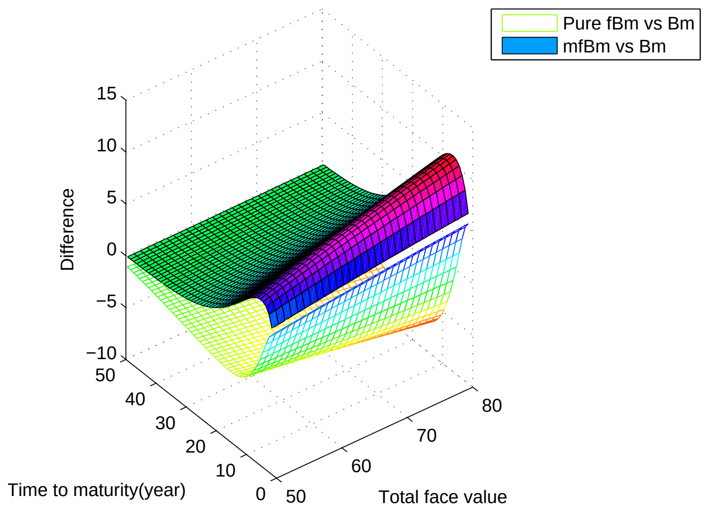

Figure 9: Relative difference of the values of bonds among the Bm model, pure fBm model and mfBm model

a stock is increasing with respect to H and the value of a bond is decreasing with respect to H.

To better understand the preference of our model, we compute the the values of stocks and bonds using our model and make comparisons with the results of the pure fractional Brownian motion model (as shown in the Appendix) based on Leccadito and Urga [14].

Now, we compare three values of stocks and bonds: the theoretical prices derived from the Bm, pure fractional Brownian motion and mfBm models. We choose parameters: $V_0 = 100$, $r = 0.06$, $\sigma = 0.15$, $\varepsilon = 0.35$, $H = 0.55$, t=0.5, maturity $T \in [0.5, 50]$ and total face value $L \in [50, 80]$. The figures 8,9 show the theoretical stocks or bonds differences by the Bm model and the pure fractional Brownian motion model and our mfBm, respectively. In the figures 8 and 9, the vertical axis denotes the differences between the prices of stocks and bonds respectively, and the horizontal axis denotes time to maturity and the total face value. The figures show that the prices by our mixed model is better fitted to the Bm model than that by the pure fractional Brownian motion model. Hence, when compared to the figures 8 and 9, our mfBm model seems reasonable.

5 Credit Spread

A credit spread, or net credit spread, involves a purchase of one option and a sale of another option in the same class and expiration but different strike prices. Investors receive a net credit for entering the position, and want the spreads to narrow or expire for profit. In contrast, an investor would have to pay to enter a debit spread. Credit spread are written on the spread between the rate of return of a zero coupon corporate bond and the risk-free bond. The spread is the extra return offered by the corporate bond to compensate for the risk of default or downgrading. The buyer of an option pays a premium to transfer the risk of a loss in value due to a downgrading of the issuer of a reference instrument. From the definition of credit spread, we can easily check the following result.

Theorem 5.1. *Under the assumptions of Theorem 4.2, the credit spread of* *the bond at time zero is*

$$s(0,T) = -\frac{1}{T}\ln\left[\Phi(d_2) + \frac{V_0}{L}e^{\int_0^T r_s ds}\Phi(-d_1)\right],$$

where

$$d_1 = \frac{\ln\frac{V_0}{L} + \int_0^T r_s ds + \frac{\sigma^2}{2}T^{2H} + \frac{\varepsilon^2}{2}T}{\sqrt{\frac{\sigma^2}{2}T^{2H} + \frac{\varepsilon^2}{2}T}},$$

and

$$d_2 = \frac{\ln\frac{V_0}{L} + \int_0^T r_s ds - \frac{\sigma^2}{2}T^{2H} - \frac{\varepsilon^2}{2}T}{\sqrt{\frac{\sigma^2}{2}T^{2H} + \frac{\varepsilon^2}{2}T}}.$$

In Figure 10, 11, we plot the values of credit spreads as a function of time to maturity T for three different values of the parameters

$$(\sigma,\varepsilon) \in \{(0.15,0.15),(0.15,0.35),(0.25,0.35)\}$$

and three values of the parameter $H \in \{0.55, 0.70, 0.90\}$.

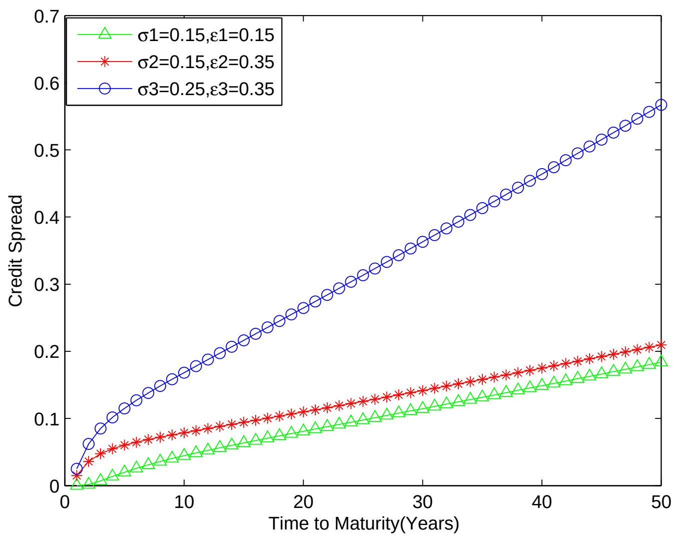

Figure 10: Credit spread resulting in the mixed-fractional Merton model against maturity time T when $l_0 = 0.6$, $r = 0.06$, $H = 0.70$, and $0 < T < 50$

Clearly the credit spreads are increasing for values of σ and ε. The credit spreads are also decreasing with respect to H at first and then increasing.

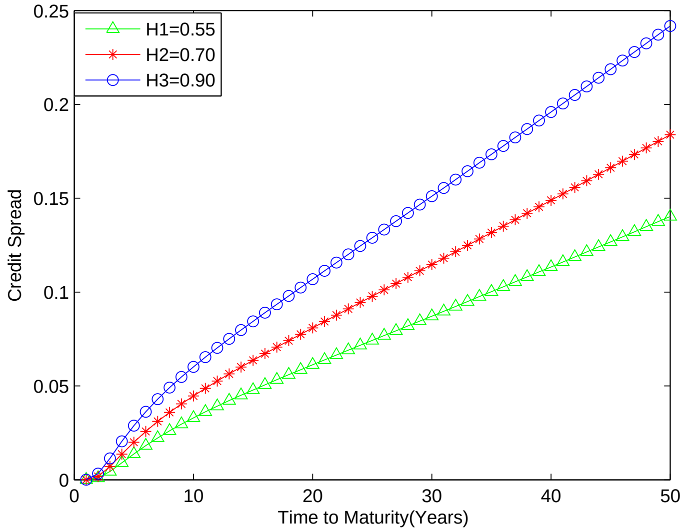

Figure 11: Credit spread resulting in the mixed-fractional Merton model against maturity time T when $l_0 = 0.6$, $r = 0.06$, $\sigma = 0.15$; $\varepsilon = 0.15$ and $0 < T < 50$.

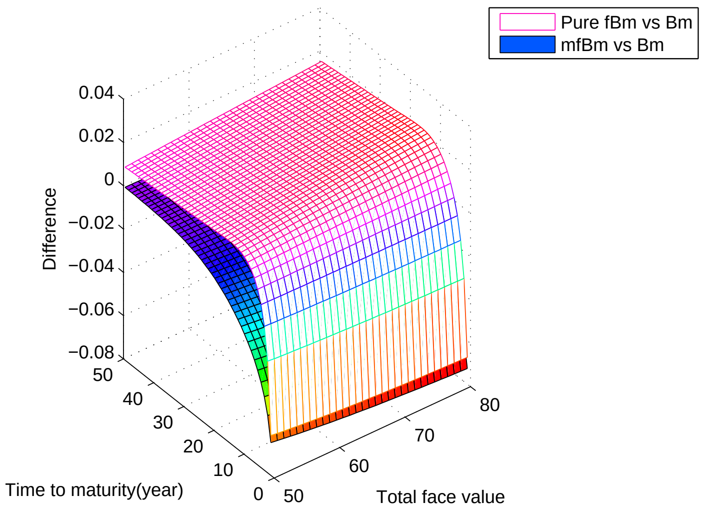

Figure 12: Relative difference of the credit spread among the Bm model, pure fBm model and mfBm model

To better understand the preference of our model, we compute the the values of credit spreads using our model and make comparisons with the results of the pure fractional Brownian motion model (as shown in the Appendix) based on Leccadito and Urga [14].

Now, we compare three values of credit spreads: the theoretical prices derived from the Bm, pure fractional Brownian motion and mfBm models. We choose these valuation of the parameters: $V_0 = 100$, $r = 0.06$, $\sigma = 0.15$, $\varepsilon = 0.35$, $H = 0.55$, maturity $T \in [0, 50]$ and total face value $L \in [50, 80]$. The figure 12 shows the theoretical credit spreads differences by the Bm model and the pure fractional Brownian motion model and our mfBm, respectively. In the figure, the vertical axis denotes the differences between the credit spreads, and the horizontal axis denotes time to maturity and the total face value. The figure shows that the credit spreads by our mixed model is better fitted to the Bm model than that by the pure fractional Brownian motion model. Hence, our mfBm model seems reasonable.

ACKNOWLEDGEMENTS. The Project-sponsored by NSFC (11171062) and Innovation Program of Shanghai Municipal Education Commission(12ZZ063).

References

[1] E. Alós, O. Mazet and D. Nualart, Stochastic calculus with respect to Gaussian processes, *Annals of Probabality*, **29**(2), (2001), 766-801.

[2] F. Biagini, H. Fink, and C. Klüppelberg, A fractional credit model with long range dependent default rate, (2009),www-m4.ma.tum.de/Papers/.

[3] F. Black and M. Scholes, The pricing of options and corporate libilities, *Journal of Political Economy*, **81**(3), (1973), 637-659.

[4] F. Black and J.C. Cox, Valuing corporate securities: Some effects of bond indenture provisions, *Journal of Financial and Quantitative Analysis*, **31**(2), (1976), 351-367.

[5] D.O. Cajueiro, B.M. Tabak, Long-range dependence and market structure, *Chaos Solitons Fractals*, **31**(4,) (2007), 995-1000.

[6] E. Charles, The fractional mixed fractional Brownian motion, *Statistics Probability* & *Letters*, **65**(2), (2003), 111-120.

[7] J. Cheridito, *Regularizing fractional Brownian motion with a view toward stock pricing modeling*, Ph.D. Dissertation, ETH Zurich, 2001.

[8] P. Cheridito, Mixed fractional Brownian motion, *Bernoulli*, **7**(6), (2001), 913-934.

[9] D. Duffie and K.J. Singleton, An econometric model of the term structure of interest-rate swap yields, *Journal of Finance*, **52**(4), (1997), 1287-1321.

[10] R. Geske, The valuation of corporate securities as compound options, *Journal of Finance*, **12**(4), (1977), 541-552.

[11] D. A. Hsieth, Chaos and non-linear dynamics: Applications to financial market, *Journal of Finance*, **46**(5), (1991), 1839-1877.

[12] R. Jarrow and S. Turnbull, Pricing derivatives on financial securities subject to credit risk, *Journal of Finance*, **50**(1), (1995), 53-86.

[13] Yu. Kuznetsov, The absence of arbitrage in a model with fractal Brownian motion, *Russian Math. Surveys*, **54**(4), (1999), 847-848.

[14] A. Leccadito and G. Urga, Fractional Models to Credit Risk Pricing, preprint, (2006).

[15] H. E. Leland, Corporate debt value, bond covenants and optimal capital structure, *Journal of Finance*, **49**(4), (1994), 1213-1252.

[16] D. Madan and H. Unal, Pricing the risks of default, *Review of Derivatives Research*, **2**(2), (1998), 121-160.

[17] B.B. Mandelbrot, *The Fractal Geometry of Nature*, W.H. Freeman, San Francisco, 1982.

[18] M.C. Mariani, I. Florescu, M.P. Beccar Varela, E. Ncheuguim, Long correlations and Levy model applied to the study of memory effects in high frequency data, *Physica A*, **388**(8), (2009), 1659-1664.

[19] R.C. Merton, On pricing of corporate debt: the risk structure of interest rate, *Journal of Finance*, **29**(2), (1974), 449-470.

[20] Y. Mishura, *Stochastic calculus fractional Brownian Motions and related processes*, Springer, 2008.

[21] Y. Mishura and E. Valkeila, The absence of arbitrage in a mixed Brownian-fractional Brownian model, *Proceedings of the Steklov Institute of Mathematics*, **237**, (2002), 224-233.

[22] J.A. Ramirez, J. Alvarrez, E. Rodriguez, G.F. Anaya, Time-varying Hurst exponent for US stock markets, *Physica A*, **387**(24), (2008), 6159-6169.

[23] D. Shimko, N. Tejima and D. Van Deventer, The pricing of risky debt when interest rates are stochastics, *Journal of Fixed Income*, **3**(2), (1993), 437-444.

[24] X. Su, and W. Wang, Pricing options with credit risk in a reduced form model, *Journal of the Korean Statistical Society*, **41**(4), (1993), 58-65.

[25] B. M. Tabak, D.O. Cajueiro, Long-range dependence and multifractality in the term structure of LOBOR interest rates, *Physica A*, **373**(1), (2007), 603-614.

[26] X. Wang, E. Zhu, M.T, H. Yan, Scaling and long-range dependence in option pricing II: Pricing European option with transaction costs under the mixed Brownian-fractional Brownian model, *Physica A*, **389**(3), (2010), 445-451.

[27] W. Willinger, M.S. Taqqu, V. Teverovsky, Stock market prices and long-range dependence, Finance and Stochastics, *Finance and Stochastics*, **3**(1), (1999), 1-13.

[28] Z. Yu and L. Yan, European call option pricing under a mixed fractional Brownian motion environment, *Journal of university of science and technology of Suzhou*, **25**(4), (2008), 4-10.

[29] M. Zahle, Long range dependence, no arbitrage and the Black-Scholes formula, *Stochastics and Dynamics*, **265**(2), (2002), 265-280.