Theoretical Mathematics & Applications, vol. 7, no. 4, 2017, 1-16
ISSN: 1792- 9687 (print), 1792-9709 (online)
Scienpress Ltd, 2017

Some aspects of partially ordered multisets

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Abstract

The paper outlines some structural properties of a partially ordered multiset (pomset). In the sequel, the *width* and *height* of a pomset are characterized into minimum number of mset chains and mset antichains, respectively. A set of necessary and sufficient conditions is given for $|C_i \cap A_j| = 1$, provided the intersection is not empty.

Mathematics Subject Classification: 06A07; 03E04; 06F25; 91B16

Keywords: Partially ordered multisets; multiset chains; multiset antichains

1 Introduction

An mset is an unordered collection of objects in which repetition of objects is

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Article Info: *Received:* June 7, 2017. *Revised:* July 5, 2017.
Online appearance : December 1, 2017.

significant. For an mset M the *root set* (or support) of M, denoted by M^*, is given by the set $\{x \in S|M(x) > 0\}$, where S is a base set. An mset is called finite if the root set is finite and also, multiplicities are finite. In this paper, we shall confine our attention to finite msets. The *cardinality* of an mset is the sum of the multiplicities of all its distinct elements. *Objects* in an mset M represent the elements of the root set of M. An mset can be represented in various forms. For instance, the mset $M = [1,1,1,1,2,4,4,5,5]$ can be denoted by $[1,2,4,5]_{4,1,2,2}$ or $[1^4, 2^1, 4^2, 5^2]$ or $\{4/1, 1/2, 2/4, 2/5\}$. In this paper, we choose to denote an mset M by $[m_1x_1, m_2x_2, \dots, m_nx_n]$, where m_i is the multiplicity of x_i in M, hence m_ix_i will denote a point in M. We will denote the class of all finite mset defined on a set S by $M(S)$. Let $M, N \in M(S)$, then M is a *submset* of N, denoted by $M \subseteq N$, if $M(x) \leq N(x)$ for all $x \in S$, and $M \subset N$ if and only if $M(x) < N(x)$ for at least one x. A submset of a given mset that contains all multiplicities of common elements is called a *whole submset*. A *full submset* contains all objects of the parent mset. The *union* of two msets M and N is the mset given by $(M \cup N)(x) = max\{m, n\}$ such that $mx \in M$ and $nx \in N$ for all $x \in S$. The *intersection* of M and N is the mset given by $(M \cap N)(x) = min\{m, n\}$ such that $mx \in M$ and $nx \in N$ for all $x \in S$ (see [2], [17] and [18] for details on msets). Some works have appeared dealing with infinite multiplicities as well as involving negative multiplicities [3, 22]. In this work, we consider only nonnegative integral multiplicities of objects in an mset.

It is well-known that partially ordered multisets constitute one of the most basic models of concurrency [8, 15, 16]. The problem of extending various mathematical notions and results related to partially ordered sets (posets) (see [20] and [21] for an exposition on posets) to pomsets has attracted serious attention during the last couple of decades [6, 9, 11, 10]. In this paper, we introduce an ordering $\preccurlyeq\leq$ on an mset M and study some properties of the structure $\mathcal{M} = (M, \preccurlyeq\leq)$, in particular, characterization of the width and height of a pomset. In section 2, we define the ordering $\preccurlyeq\leq$ and investigate some properties of the multiset structure $\mathcal{M}$. We discuss mset chains and mset antichains in section 3 and prove some related results. In section 4, we present bounds of pomsets. An extension of Dilworth's decomposition theorem and its dual to pomsets are presented in section 5.

2 Partially Ordered Multisets (Pomsets)

Let $M = [m_1x_1, m_2x_2, \dots, m_nx_n]$ be an mset such that the points are ordered. We write $m_ix_i \bowtie m_jx_j$ whenever the two points m_ix_i and m_jx_j in M are *comparable* under the defined order and $m_ix_i || m_jx_j$ whenever m_ix_i and m_jx_j are *incomparable*.

Definition 2.1

For any pair of points m_ix_i *and* m_jx_j in $M \in M(S)$, $m_ix_i \preccurlyeq\leq m_jx_j$ if and only if $x_i \preccurlyeq x_j$, and the points m_ix_i and m_jx_j coincide i.e., $m_ix_i == m_jx_j$ if and only if $x_i = x_j$ (this follows from the principle of uniqueness of the multiplicity of an object in an mset). Also, $m_ix_i \neq\neq m_jx_j$ if and only if $x_i \neq x_j$. Moreover, $m_ix_i \bowtie m_jx_j$ if and only if $m_ix_i \preccurlyeq\leq m_jx_j \vee m_jx_j \preccurlyeq\leq m_ix_i$ otherwise $m_ix_i || m_jx_j$. The strict order associated with $\preccurlyeq\leq$ is the ordering $\prec<$, where $m_ix_i \prec< m_jx_j$ implies that $m_ix_i \preccurlyeq\leq m_jx_j$ and $m_ix_i \neq\neq m_jx_j$.

Definition 2.2

The ordering $\preccurlyeq\leq$ on M is said to be *reflexive* if and only if $m_ix_i \preccurlyeq\leq m_ix_i$ for all $m_ix_i \in M$, *symmetric* if and only if $m_ix_i \preccurlyeq\leq m_jx_j$ implies $m_jx_j \preccurlyeq\leq m_ix_i$, *antisymmetric* if and only if $m_ix_i \preccurlyeq\leq m_jx_j \wedge m_jx_j \preccurlyeq\leq m_ix_i$ implies that $m_ix_i == m_jx_j$, and *transitive* if and only if $m_ix_i \preccurlyeq\leq m_jx_j \wedge m_jx_j \preccurlyeq\leq m_kx_k$ implies $m_ix_i \preccurlyeq\leq m_kx_k$.

Definition 2.3

A relation R is called a *quasi-mset order* (or a *pre-mset order*) if it is reflexive and transitive, and a *strict mset order* if it is irreflexive and transitive. The relation R is called a *partial mset order* (or simply *mset order*) if it is reflexive, antisymmetric and transitive. R is a *linear* (or *total*) mset order if it is a partial mset order and for all pairs of point m_ix_i, m_jx_j in M, we have $m_ix_iRm_jx_j \vee m_jx_jRm_ix_i$.

Definition 2.4

A pomset $\mathcal{M}$ is a pair $(M, \preccurlyeq\leq)$, where $M \in M(S)$, and $\preccurlyeq\leq$ is a partial mset order defined on M.

Theorem 2.1

Let $(S, \preccurlyeq)$ be a poset and $M \in M(S)$. Then $\mathcal{M} = (M, \preccurlyeq\leq)$ is a pomset.

Proof

For any m_ix_i in M, since $x_i \preccurlyeq x_i$ we have $m_ix_i \preccurlyeq\leq m_ix_i$, implying that $(M, \preccurlyeq\leq)$ is reflexive.

Let $m_ix_i \preccurlyeq\leq m_jx_j$ and $m_jx_j \preccurlyeq\leq m_ix_i$ in $\mathcal{M}$. Then, $x_i \preccurlyeq x_j$ and $x_j \preccurlyeq x_i$, and hence $x_i = x_j$.

In particular, $m_ix_i == m_jx_j$, hence $\preccurlyeq\leq$ is antisymmetric.

Let m_ix_i, m_jx_j, m_kx_k be points in M such that

$m_ix_i \preccurlyeq\leq m_jx_j$ and $m_jx_j \preccurlyeq\leq m_kx_k$.

We have $x_i \preccurlyeq x_j \preccurlyeq x_k$. Thus transitivity holds.

Therefore, $(M, \preccurlyeq\leq)$ is a pomset. □

Definition 2.5

For two mset orders $\preccurlyeq_1\leq_1$ and $\preccurlyeq_2\leq_2$ on an mset M, the mset order $\preccurlyeq\leq$ is said to be an intersection of $\preccurlyeq_1\leq_1$ and $\preccurlyeq_2\leq_2$ if and only if $m_ix_i \preccurlyeq\leq m_jx_j \Longrightarrow m_ix_i \preccurlyeq_1\leq_1 m_jx_j \wedge m_ix_i \preccurlyeq_2\leq_2 m_jx_j$, for all $m_ix_i, m_jx_j \in M$.

Theorem 2.2

If $\mathcal{M} = (M, \preccurlyeq_1 \leq_1)$ and $\mathcal{N} = (M, \preccurlyeq_2 \leq_2)$ are pomsets corresponding to $(S, \preccurlyeq_1)$ and $(S, \preccurlyeq_2)$, then $\mathcal{M} \cap \mathcal{N} = (M, \preccurlyeq \leq)$ is also a pomset, where $\preccurlyeq \leq = \preccurlyeq_1 \leq_1 \cap \preccurlyeq_2 \leq_2$.

Proof

For any point $m_i x_i$ in M, clearly $m_i x_i \preccurlyeq_1 \leq_1 m_i x_i$ and $m_i x_i \preccurlyeq_2 \leq_2 m_i x_i$ since $\preccurlyeq_1 \leq_1$ and $\preccurlyeq_2 \leq_2$ are partial mset orders.

Thus, $m_i x_i \preccurlyeq \leq m_i x_i$ (reflexive property).

Let $m_i x_i$ and $m_j x_j$ be points in M such that

$m_i x_i \preccurlyeq \leq m_j x_j$ and $m_j x_j \preccurlyeq \leq m_i x_i$. (1)

From (1) we have,

$m_i x_i \preccurlyeq_1 \leq_1 m_j x_j$ and $m_j x_j \preccurlyeq_1 \leq_1 m_i x_i$. (2)

Since $\preccurlyeq_1 \leq_1$ is antisymmetric, we have $m_i x_i == m_j x_j$ (3)

Similarly,

$m_i x_i \preccurlyeq_2 \leq_2 m_j x_j$ and $m_j x_j \preccurlyeq_2 \leq_2 m_i x_i$ imply $m_i x_i == m_j x_j$. (4)

From (2) - (4) we can conclude that,

$m_i x_i \preccurlyeq \leq m_j x_j$ and $m_j x_j \preccurlyeq \leq m_i x_i$ imply $m_i x_i == m_j x_j$.

Therefore, $\preccurlyeq \leq$ is antisymmetric.

For transitivity let $m_i x_i, m_j x_j$ and $m_k x_k$ be points in M such that,

$m_i x_i \preccurlyeq \leq m_j x_j$ and $m_j x_j \preccurlyeq \leq m_k x_k$.

We need to show that $m_i x_i \preccurlyeq \leq m_k x_k$.

Now, $m_i x_i \preccurlyeq \leq m_j x_j$ and $m_j x_j \preccurlyeq \leq m_k x_k$ imply

$m_i x_i \preccurlyeq_1 \leq_1 m_j x_j$ and $m_j x_j \preccurlyeq_1 \leq_1 m_k x_k$.

Since $\preccurlyeq_1 \leq_1$ is transitive, we have $m_i x_i \preccurlyeq_1 \leq_1 m_k x_k$. (5)

Similarly,

$m_i x_i \preccurlyeq_2 \leq_2 m_j x_j$ and $m_j x_j \preccurlyeq_2 \leq_2 m_k x_k$ imply $m_i x_i \preccurlyeq_2 \leq_2 m_k x_k$. (6)

From (5) and (6), we obtain $m_i x_i \preccurlyeq \leq m_k x_k$, hence $\preccurlyeq \leq$ is transitive.

Therefore, $\mathcal{M} \cap \mathcal{N} = (M, \preccurlyeq \leq)$ is a pomset. □

Theorem 2.3

Let $(S, \preccurlyeq)$ be a poset. An mset $M \in M(S)$ is partially ordered if and only if its root set is a subposet of $(S, \preccurlyeq)$.

Proof

Suppose $M \in M(S)$ is partially ordered. Thus, for $m_i x_i \in M$, $m_i x_i \preccurlyeq\leq m_i x_i$ holds. The definition of $\preccurlyeq\leq$ implies that

$x_i \preccurlyeq x_i$ for all $x_i \in M^*$, with $i \in [1, n]$. (7)

Also, for all $m_i x_i, m_j x_j \in M$, we have

$m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_i x_i \Longrightarrow m_i x_i == m_j x_j$.

Again by the ordering $\preccurlyeq\leq$, it must be the case that

$x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_i \Longrightarrow x_i = x_j$ for all $x_i, x_j \in M^*$. (8)

Now, let $m_i x_i, m_j x_j, m_k x_k$ be any three points in M. Since M is partially ordered we have

$m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_k x_k \Longrightarrow m_i x_i \preccurlyeq\leq m_k x_k$, and

$x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_k \Longrightarrow x_i \preccurlyeq x_k$ for all $x_i \in M^*$. (9)

From (7) through (9), it follows that $(M^*, \preccurlyeq\leq)$ is a subposet of $(S, \preccurlyeq)$.

The converse part is straightforward. Suppose that $(M^* \preccurlyeq)$ is a subposet of $(S, \preccurlyeq)$. Clearly, $x_i \preccurlyeq x_i$ for all $x_i \in M^*$. Let m_i be the multiplicity of x_i in $M \in M(S)$. From the definition of $\preccurlyeq\leq$, we have $m_i x_i \preccurlyeq\leq m_i x_i$ (reflexivity of $\preccurlyeq\leq$). Also, $x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_i \Longrightarrow x_i = x_j$ for all $x_i, x_j \in M^*$, this in turn gives, $m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_i x_i \Longrightarrow m_i x_i == m_j x_j$ (antisymmetry of $\preccurlyeq\leq$). And for all $x_i, x_j, x_k \in M^*$, we will have $x_i \preccurlyeq x_j \wedge x_j \preccurlyeq x_k \Longrightarrow x_i \preccurlyeq x_k$. Again, it follows that $m_i x_i \preccurlyeq\leq m_j x_j \wedge m_j x_j \preccurlyeq\leq m_k x_k \Longrightarrow m_i x_i \preccurlyeq\leq m_k x_k$ (transitivity of $\preccurlyeq\leq$). □

3 Mset Chains and Mset Antichains

Definition 3.1

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset. A point $m_i x_i$ in M is *maximal* in $\mathcal{M}$ if for any other point $m_j x_j \in M$ with $m_i x_i \preccurlyeq\leq m_j x_j$ we have $m_i x_i == m_j x_j$. Similarly, a point $m_i x_i$ in M is *minimal* if for any other point $m_j x_j \in M$ with $m_j x_j \preccurlyeq\leq m_i x_i$ we have $m_i x_i == m_j x_j$. If such points are unique, we call them *maximum* and *minimum* respectively.

Theorem 3.1

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset. If $\mathcal{M}$ is totally ordered then maximal and maximum points coincide.

Proof

Let $m_i x_i$ *and* $m_j x_j$ be points in M such that $m_i x_i$ is a maximal point in $\mathcal{M}$ and $m_j x_j$ is a maximum point in $\mathcal{M}$.

Since $\mathcal{M}$ is totally ordered, we will have either $m_i x_i \preccurlyeq\leq m_j x_j$ or $m_j x_j \preccurlyeq\leq m_i x_i$.

Now, suppose that $m_i x_i \preccurlyeq\leq m_j x_j$, then, by definition of a maximal point

$m_i x_i == m_j x_j$.

Similarly, the other case follows. □

A similar argument holds for minimal and minimum points if $\mathcal{M}$ is totally ordered.

Definition 3.2

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and N, a submset of M. A suborder $\preccurlyeq\leq_{\mathcal{K}}$ is the restriction of $\preccurlyeq\leq$ to pairs of points in the submset N of M such that $n_i x_i \preccurlyeq\leq_{\mathcal{K}} n_j x_j \Leftrightarrow m_i x_i \preccurlyeq\leq m_j x_j$, where $n_i x_i, n_j x_j \in N$ and $n_i \leq m_i$. The pair $(N, \preccurlyeq\leq_{\mathcal{K}})$ is called a subpomset of $\mathcal{M}$.

Definition 3.3

A subpomset C of a pomset $\mathcal{M} = (M, \preccurlyeq\leq)$ is called an *mset chain* if C is linearly (or totally) ordered.

A subpomset A of $\mathcal{M}$ is called an *mset antichain* if no two points in A are comparable.

A pomset $\mathcal{M}$ is *connected* (or is an mset chain) if $m_i x_i \bowtie m_j x_j$ for all distinct pairs of points $m_i x_i, m_j x_j \in M$. $\mathcal{M}$ is an mset antichain if $m_i x_i || m_j x_j$ for all distinct pairs of points $m_i x_i, m_j x_j$ in M.

Definition 3.4

An mset chain C in a pomset $\mathcal{M}$ is *maximal* if it is not strictly contained in any other mset chain of $\mathcal{M}$. An mset chain C in a pomset $\mathcal{M}$ is a *maximum mset chain* if it contains maximum number of points. Maximal and maximum mset antichains are defined analogously.

Remark 3.1

A pomset can contain more than one maximal mset chain. Also, in a pomset, maximal and maximum mset chains may coincide. The following example illustrates this.

Example 3.1

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ and let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the root set for the mset $M = [2x_1, 3x_2, 4x_3, 6x_4, 8x_5, 16x_6]$ where X is partially ordered as follows: $x_1 \preccurlyeq x_3 \preccurlyeq x_5 \preccurlyeq x_6$, $x_1 \preccurlyeq x_4$, and $x_2 \preccurlyeq x_4$.

The following are mset chains in $\mathcal{M}$:

$C_1 = [2x_1, 4x_3, 8x_5, 16x_6]$ $C_2 = [2x_1, 6x_4]$ $C_3 = [3x_2, 6x_4]$ $C_4 = [4x_3, 8x_5]$

Clearly, C_1, C_2 and C_3 are maximal mset chains. Where C_1 is the maximum.

Definition 3.5

A pomset $\mathcal{M} = (M, \preccurlyeq\leq)$ is said to be well-ordered if for any submset N of M, there exists a point $n_i x_i$ in N, such that $n_i x_i$ is the minimum point with respect to the defined order.

Lemma 3.2

Every well-ordered pomset is an mset chain.

Proof

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and $m_i x_i, m_j x_j$ be any arbitrary pair of distinct points in M. Since $\mathcal{M}$ is well-ordered, the submset $[n_i x_i, n_j x_j]$ has a minimum point.

Thus, either $n_i x_i \prec\!< n_j x_j$ or $n_j x_j \prec\!< n_i x_i$.

Since this condition holds for every pair of distinct points in M, it follows that $\mathcal{M}$ is totally ordered.

4 Bounds of pomsets

Definition 4.1

Let $\mathcal{K} = (N, \preccurlyeq\leq_{\mathcal{K}})$ be a subpomset of a pomset $\mathcal{M} = (M, \preccurlyeq\leq)$. A point $m_i x_i \in M$ is an upper bound for $\mathcal{K}$ if $m_i x_i \succcurlyeq\geq n_j x_j$ for all points $n_j x_j$ in N. Dually, $m_i x_i \in M$ is a lower bound of $\mathcal{K}$ if $m_i x_i \preccurlyeq\leq n_j x_j$ for all points $n_j x_j$ in N.

Lemma 4.1

If an mset chain C is maximal in a pomset $\mathcal{M}$, then C necessarily contains its upper bound.

Proof

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and let $C = (N, \preccurlyeq\leq_C)$ be a maximal mset chain in $\mathcal{M}$. Since C is linearly ordered, for some i we will have a point $n_i x_i \in N$ such that $n_i x_i >\!\succ n_j x_j$ for all other points $n_j x_j \in N$. This implies that $n_i x_i$ is a maximum point. Suppose a point $m_k x_k \notin N$ is an upper bound for C. Now C is maximal implies that for any point $m_k x_k \notin N$, we would have either $m_k x_k || n_i x_i$ or $m_k x_k \preccurlyeq\leq n_i x_i$ since $n_i x_i$ is the maximum point.

If $m_k x_k || n_i x_i$, then $m_k x_k$ cannot be an upper bound for C.

Now, suppose that $m_k x_k \preccurlyeq\leq n_i x_i$, by the definition of upper bound we have a contradiction, hence the result. □

Theorem 4.2

Let $\mathcal{M}$ be a pomset and let $\mathcal{C}$ be a collection of all maximal mset chains in $\mathcal{M}$. If K is an mset containing all upper bounds of the elements of $\mathcal{C}$. Then any two distinct points in K are incomparable.

Proof

Let $C_1, \dots, C_n$ be the maximal mset chains in $\mathcal{M}$. Suppose that $m_1 x_1, m_2 x_2, \dots m_n x_n$ are upper bounds for the mset chains $C_1, C_2, \dots, C_n$, then $K = [m_1 x_1, \dots, m_n x_n]$.

Let $m_i x_i \text{ } and \text{ } m_j x_j$ be distinct points in K, then there exists maximal mset chains C_i and C_j in $\mathcal{C}$ such that $m_i x_i$ is an upper bound for C_i and $m_j x_j$ is an upper bound for C_j say.

Now, $C_i \cup [m_j x_j]$ is not an mset chain since C_i is maximal in $\mathcal{M}$. Similarly, $C_j \cup [m_i x_i]$ is not an mset chain. Assume that $m_i x_i \bowtie m_j x_j$, then either $m_i x_i \prec\!< m_j x_j$ or $m_j x_j \prec\!< m_i x_i$ holds.

Suppose $m_i x_i \prec\!< m_j x_j$. Now, $m_i x_i$ is an upper bound for C_i implies that $m_i x_i \succcurlyeq\geq m_k x_k$ for all other points $m_k x_k \in C_i$. By transitivity, it follows that, $m_j x_j \succ\!> m_k x_k$ for all $m_k x_k \in C_i$, which is a contradiction since C_i is maximal in $\mathcal{M}$.

A similar argument holds for the case $m_j x_j \prec\!< m_i x_i$ in C_j.

Hence it must be the case that $m_i x_i || m_j x_j$.

Now $m_i x_i, m_j x_j$ are arbitrary points in K, therefore, no two points in K are comparable. □

5 Height and width of pomset

Definition 5.1

The *height* of a pomset $\mathcal{M}$ denoted by $\hbar$ is the number of points in a maximum mset chain in $\mathcal{M}$. The *width* of a pomset $\mathcal{M}$ denoted by ϖ is the number of points in a maximum mset antichain in $\mathcal{M}$.

Remark 5.1

The number of mset chains in a chain partitioning of $\mathcal{M}$ can be described in relation to the width of $\mathcal{M}$. Likewise, the number of mset antichains in an antichain partitioning of a pomset $\mathcal{M}$ can be described with respect to the height of $\mathcal{M}$. Dilworth's theorem [7], and its dual [14] describe these relationships in the classical setting.

Using the idea of set-based partitioning [10], the next result necessarily guarantees that if the intersection of any mset chain and mset antichain in a pomset is not empty, then its cardinality is at most 1.

Theorem 5.1

Let $\mathcal{M} = (M, \preccurlyeq \leq)$ be a pomset and let C_i, A_j be mset chains and mset antichains in $\mathcal{M}$, respectively with $i, j \in \{1,2, \dots, n\}$. Then $\left|\mathsf{C}_i \cap A_j\right| \leq 1$ for any i, j, if and only if the partitions of the mset antichains are such that each occurrence of the generating object of a point $m_i x_i$ belongs to a different partition i.e. $x_i, x_j \in A_j \Longrightarrow x_i \neq x_j$.

Proof

Assume that $|C_i \cap A_j| \leq 1$. Now, $C_i \cap A_j$ is either empty or has only one point for any i, j. Let the points $l_1 x_1, \dots, l_n x_n$ be in A_j, with $l_i \leq m_i$. The case where $\left|C_i \cap A_j\right| < 1$ is trivial. Suppose $C_i \cap A_j \neq \emptyset$ and let $l_i x_i$ in A_j be a point in $C_i \cap A_j$. Now $|C_i \cap A_j| \leq 1$ implies that $l_i \not> 1$. Hence it must be the case that

$l_i = 1$. We can apply this process inductively on all points $l_1x_1, \dots, l_nx_n \in A_j$ since each point $l_ix_i \in A_j$ must belong to a different mset chain C_i. Hence all points in A_j will be of the form l_ix_i with $l_i = 1$. Therefore, $x_i, x_j \in A_j \Longrightarrow x_i \neq x_j$.
Next, assume the converse. Clearly, for each point $l_ix_i \in A_j$, $l_i \ngtr 1$, otherwise we will have a contradiction. If $C_i \cap A_j = \emptyset$, the result follows. Now assume that $C_i \cap A_j$ is not empty and suppose that $\left|C_i \cap A_j\right| > 1$. Then there will be points say $x_1, \dots, x_n$ of A_j, with $n \leq |A_j|$ in $C_i \cap A_j$. This implies that $x_1, \dots, x_n$ are comparable since they are also points in C_i which is a contradiction. Hence $C_i \cap A_j$ is empty or $\left|C_i \cap A_j\right| = 1$. Therefore, $|C_i \cap A_j| \leq 1$. □

Theorem 5.2
Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset defined over a partially ordered base set. Then $\mathcal{M}$ can be partitioned into exactly ϖ mset chains where ϖ is the width of the pomset $\mathcal{M}$.

Proof
The case where $\mathcal{M}$ contains only one point m_ix_i is trivial. Suppose the assertion is true for all pomsets $\mathcal{N}_i, i = 1,2, \dots, k$ with $|\mathcal{N}_i| < |\mathcal{M}|$ for each i and let $\mathcal{M} = \mathcal{N}_k \cup [m_ix_i]$, this implies that $|\mathcal{M}| = |\mathcal{N}_k| + |m_ix_i|$. If A is an mset antichain in $\mathcal{M}$ containing only one point m_ix_i, then the assertion is true. Now assume that A contains more than one point and let $\mathcal{C}$ be a maximal mset chain in $\mathcal{M}$, then $\varpi - |A| \leq width(\mathcal{M} \setminus \mathcal{C}) \leq \varpi$. Let F be the subpomset $\mathcal{M} \setminus \mathcal{C}$, if F has width $\varpi - |A|$, by the induction hypothesis F can be partitioned into $\varpi - |A|$ mset chains, together with $\mathcal{C}$ gives a partition into at most ϖ mset chains. Furthermore, if the pomset $\mathcal{M}$ is partitioned into n mset chains then, $n = \varpi$. Observe that since ϖ is the cardinality of a maximum mset antichain, every point in that mset antichain must belong to a different mset chain. Taking $n < \varpi$ will imply that there exist $m_ix_i, m_jx_j \in \mathrm{C}_i$ for some i, j with $m_ix_i || m_jx_j$, which is a contradiction. □

Dually, we present an extension of Mirsky's theorem to pomsets as follows:

Theorem 5.3

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset. Then $\mathcal{M}$ can be partitioned into exactly $\hbar$ mset antichains where $\hbar$ is the height of the pomset $\mathcal{M}$.

Proof

We prove the theorem by induction. If $\mathcal{M}$ is an mset antichain, we have a trivial case. Next, assume that the theorem holds for pomsets of height t where $t < \hbar$. Define $\mathcal{H}$ to be the mset of all maximal points of $\mathcal{M}$. Clearly $\mathcal{H}$ is an mset antichain in $\mathcal{M}$ and every maximal mset chain in $\mathcal{M}$ contains exactly one point $m_i x_i$ from $\mathcal{H}$ which is also the maximum point in that mset chain. Let $\mathcal{B}$ be the pomset $\mathcal{M} \backslash \mathcal{H}$, height of $\mathcal{B}$, denoted height $(\mathcal{B})$, will be $\hbar - (height\ of\ \mathcal{H})$. By the induction hypothesis, height $(\mathcal{B}) < \hbar$ implies that $\mathcal{B}$ is partitioned into $\hbar - (height\ of\ \mathcal{H})$ mset antichains. Therefore the pomset $\mathcal{B}$ together with $\mathcal{H}$ is partitioned into at most $\hbar$ mset antichains. □

Example 5.1

Let $\mathcal{M} = (M, \preccurlyeq\leq)$ be a pomset and

$M = [2x_1, 6x_2, 2x_3, 5x_4, 3x_5, x_6]$

Suppose that the ordering $\preccurlyeq\leq$ on M is defined as follows:

$2x_1 \preccurlyeq\leq 6x_2$, $2x_3 \preccurlyeq\leq 5x_4$, $2x_1 \preccurlyeq\leq 3x_5$.

The pomset $\mathcal{M}$ has $\varpi = 4\ and\ \hbar = 2$.

Observe that, in an mset chain partitioning of $\mathcal{M}$, there are 4 mset chains:

$C_1 = [2x_1, 6x_2], C_2 = [2x_3, 5x_4], C_3 = [3x_5], C_4 = [x_6]$.

In view of Theorem 5.1, a set-based antichain partitioning of the pomset gives the following:

$A_1 = \{x_2, x_4, x_5, x_6\}$, $A_2 = \{x_2, x_4, x_5\}$, $A_3 = \{x_2, x_4, x_5\}$, $A_4 = \{x_2, x_4\}$, $A_5 = \{x_2, x_4\}$, $A_6 = \{x_2\}, A_7 = \{x_1, x_3\}, A_8 = \{x_1, x_3\}$

6 Concluding Remarks

It is known that several characterizations exist for the set of maximal antichains of a poset. An interesting problem will be to characterize the maximal mset antichains of a pomset. In view of wide practical applications of msets, a number of mset orderings have been studied in the literature (see [1, 6, 10, 13]). The orderings defined in the aforementioned literatures are exploited in comparing msets in $M(S)$. With further investigations, the ordering $\preccurlyeq\leq$ can be extended to compare msets in $M(S)$.

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