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Essential spectrum of the Cariñena operator

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Abstract

This paper addresses the proof that the Cariñena operator is self-adjoint and has only discrete spectrum consisting of isolated eigenvalues with finite multiplicities.

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1 Introduction

Cariñena et al. analyzed in [2] the non-polynomial one-dimensional quantum potential

$$V_c = x^2 + 2g_a \frac{x^2 - a^2}{(x^2 + a^2)^2}, \quad g_a > 0 \tag{1}$$

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where a is a positive real parameter. This potential represents an oscillator which is intermediate between the harmonic oscillator and the isotonic oscillator obtained from V_c when $a \to \infty$ and $a \to 0$ respectively, if g_a remains constant. They proved that the particular case $a^2 = \frac{1}{2}$ is Schrödinger solvable and obtained eigenvalues and eigenfunctions which have properties closely related to those characterizing the harmonic oscillator. They thus enlarged the restricted family of Schrödinger solvable potentials. In [3], Fellows et al. showed that these results can be obtained much more simply by noticing that this potential is a supersymmetric partner potential of the harmonic oscillator.

Through out what follows we call Cariñena operator the operator defined in the Hilbert space $\mathcal{H} = L^2(\mathbb{R})$ of square integrable complex functions defined on $\mathbb{R}$ by the formal relation

$$H_c = -\frac{d^2}{dx^2} + V_c. \tag{2}$$

We prove that the Cariñena operator is self-adjoint with empty essential spectrum using Kato-Rellich and Weyl theorems in perturbation theory.

The paper is organized as follows. In Section 2 we recall two main theorems from the perturbation theory which we use to state our main results in Section 3.

2 Preliminary notes

The following definition of relatively boundedness can be found in [4], page 190.

Definition 2.1. *Let A and T be densely defined linear operators on a Hilbert space. The operator A is said to be relatively bounded with respect to T or T-bounded if $D(A) \supset D(T)$ and there exists $\alpha > 0, \beta > 0$ such that*

$$\|Af\| \leq \alpha\|f\| + \beta\|Tf\|, \quad \forall f \in D(T). \tag{3}$$

The T-bound of A is defined as the greatest lower bound of the possible values of β.

The following theorem which is a fundamental perturbation result due to Kato and Rellich (see [4], page 287) has been found to be very convenient for establishing the self-adjointness of various operators that appear in applications.

Theorem 2.2. *Let T be self-adjoint. If A is symmetric and T-bounded with T-bound smaller than 1, then $T+A$ is also self-adjoint. In particular $T+A$ is self-adjoint if A is bounded and symmetric with $D(A) \supset D(T)$.*

Let T be a closed operator and $\xi \in \mathbb{C}$. If $T-\xi$ is invertible with $(T-\xi)^{-1}$ bounded then ξ is said to belong to the resolvent set of T. The complementary set $\sigma(T)$ of the resolvent set in the complex plane is called the spectrum of T. Let us denote by $\sigma_d(T)$ the discrete spectrum of the operator T, i.e. the set of isolated eigenvalues with finite multiplicities. By definition the essential spectrum of T is the set $\sigma_e(T) = \sigma(T) \setminus \sigma_d(T)$. If $\sigma_e(T) = \emptyset$, we say that T is an operator with pure point spectrum.

To go further let us recall the definition of relatively compactness (see for instance in [1], page 173).

Definition 2.3. *Let A and T be densely defined linear operators on a Hilbert space. T is said to be relatively compact with respect to A or $A-$compact if $D(T) \supset D(A)$ and $T(A-i)^{-1}$ is compact.*

The following stability theorem due to Weyl can be found in [4], Theorem 5.35 or [1], page 174.

Theorem 2.4. *The essential spectrum of a self-adjoint operator A is stable with respect to a symmetric A-compact perturbation T i.e*

$$\sigma_e(A+T) = \sigma_e(A).$$

3 Main results

Let us denote by H_0 the one-dimensional normal harmonic oscillator

$$H_0 := -\frac{d^2}{dx^2} + x^2. \tag{4}$$

The Hermite functions

$$h_n = (2^n n!)^{-\frac{1}{2}}(-1)^n \pi^{-\frac{1}{2}} \exp(\frac{1}{2}x^2)\frac{d^n}{dx^n}\exp(-x^2) \tag{5}$$

satisfy the relation

$$H_0 h_n = (2n+1)h_n \tag{6}$$

i.e the Hermite functions h_n are the harmonic oscillator wave functions with eigenvalues $2n+1$. The set $\{h_n\}_{n=0}^{\infty}$ is an orthonormal basis of $L^2(\mathbb{R})$. For all $f \in L^2(\mathbb{R})$, we have $f = \sum_n \lambda_n h_n$ where $\lambda_n \in \mathbb{R}$ for all n. The following relations

$$H_0 f = \sum_n \lambda_n H_0 h_n = \sum_n \lambda_n (2n+1)h_n \tag{7}$$

lead to the fact that

$$H_0 f \in L^2(\mathbb{R}) \text{ if and only if } \sum_n \lambda_n^2 (2n+1)^2 < \infty \tag{8}$$

as consequence of the Parseval equality. The domain of the harmonic oscillator H_0 can then be described as follows :

$$D(H_0) = \{f \in L^2(\mathbb{R}) : f = \sum_n \lambda_n h_n, \sum_n \lambda_n^2 (2n+1)^2 < \infty\} \tag{9}$$

In the other hand, we consider the maximal multiplication operator V_a determined by the continuous function

$$V_a(x) = 2g_a \frac{x^2 - a^2}{(x^2+a^2)^2} \tag{10}$$

with domain of definition and action given by

$$D(V_a) = L^2(\mathbb{R}),\ V_a f = 2g_a \frac{x^2-a^2}{(x^2+a^2)^2} f. \tag{11}$$

where $V_a f$ is the conventional product of the functions V_a and f. The domain of the operator V_a is the whole Hilbert space $L^2(\mathbb{R})$ because the function $x \mapsto V_a(x)$ is a real-valued bounded function on $\mathbb{R}$. Another consequence of the latter is that the operator V_a is symmetric and bounded. We look at the Cariñena operator as a perturbation of the harmonic oscillator by the potential V_a. Its domain and action are given by

$$D(H_c) = D(H_0) \cap D(V_a) = D(H_0) \tag{12}$$

and

$$H_c f = -\frac{d^2 f}{dx^2} + x^2 f + V_a f. \tag{13}$$

Theorem 3.1. *The operator V_a is H_0-bounded.*

Proof. We have $D(V_a) = L^2(\mathbb{R}) \supset D(H_0)$. In the other hand

$$\|V_a f\| \leq \|V_a\|\|f\| \leq \|V_a\|\|f\| + \frac{1}{2}\|H_0 f\|,\ \forall f \in D(H_0). \tag{14}$$

Hence the operator V_a is relatively bounded with respect to H_0. □

Theorem 3.2. *The Cariñena operator is self-adjoint.*

Proof. The Cariñena operator H_c is the sum of the operators H_0 and V_a. The harmonic oscillator H_0 is self-adjoint and the multiplication operator V_a is bounded and symmetric with $D(V_a) \supset D(H_0)$. Then according to Theorem 2.2 the Cariñena operator H_c is self-adjoint. □

Theorem 3.3. *The following equality holds :*

$$\sigma_e(H_c) = \sigma_e(H_0).$$

We may prove the following lemma.

Lemma 3.4. *The operator $C = V_a(H_0 - i)^{-1}$ is Hilbert-Schmidt.*

Proof. Let us first notice that

$$\forall x \in \mathbb{R},\ |V_a(x)| \leq \frac{2g_a}{a^4}. \tag{15}$$

We also have

$$Ch_n = (2n + 1 - i)^{-1} V_a h_n. \tag{16}$$

Then

$$\sum_n \|Ch_n\|^2 = \sum_n \|(2n + 1 - i)^{-1} V_a h_n\|^2 \tag{17}$$

$$\leq \sum_n |2n + 1 - i|^{-2} \|V_a h_n\|^2 \tag{18}$$

$$\leq \frac{4g_a^2}{a^8} \sum_n |2n + 1 - i|^{-2} < \infty. \tag{19}$$

So the operator C is Hilbert-Schmidt. □

Proof of Theorem 3.3. We have $D(V_a) \supset D(H_0)$. The operator $C = V_a(H_0 - i)^{-1}$ is Hilbert-Schmidt, hence compact. Thus V_a is H_0-compact. Then according to Theorem 2.4, one has

$$\sigma_e(H_c) = \sigma_e(H_0).$$

□

It is well known that the harmonic oscillator H_0 has empty essential spectrum. Therefore we derive the following consequence for the Cariñena operator H_c.

Corollary 3.5. *We have $\sigma_e(H_c) = \emptyset$. In other words, the Cariñena operator H_c has only discrete spectrum consisting of isolated eigenvalues with finite multiplicities.*

Remark that the main property of $V_a(x)$ used to achieve the results is that $V_a(x)$ is bounded on $\mathbb{R}$. Therefore, we can state the following theorem which is more general.

Theorem 3.6. *Let $V \in L^\infty(\mathbb{R})$. Then the operator $H = H_0 + V(x)$ is self-adjoint with pure point spectrum.*

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