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Optimizing Portfolio Liquidation Under Risk-Based Margin Requirements

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Abstract

This paper incorporates risk-based margin requirements into portfolio liquidation procedures in a novel fashion. The approach is analytic and, as a result, more efficient than conventional numerical liquidation methods. The margin requirement calculation is a self-contained inner optimization problem and is traditionally solved by choosing the worst scenario amongst a discrete set of scenarios. We address the inner problem by first generalizing the risk-based haircuts calculation into a continuous region and then using a trust region optimization algorithm to derive the closed-form solution. The solution is typically obtained in less than two iterations and our procedure significantly improves the efficiency of the main portfolio liquation problem. We implement the algorithm on example portfolios and show advantages over traditional approaches.

JEL Codes: G00, C61, O16
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1 Background

Many investment portfolios must conform to specific regulatory constraints regarding the risk-exposure resulting from the portfolio's allocations. For example, the Securities and Exchange Commission (SEC) net capital rule places risk-based requirements on the assets held by broker-dealers. These requirements are meant to ensure broker-dealers have the ability to meet their obligations. Similarly, the SEC recently approved the use of risk-based margin requirements on customer portfolios in order to limit the leverage in a customer's account based on the underlying assets' risk profile.

Although the portfolio management approach advocated in this paper is widely applicable, we choose to focus on margin accounts and the risk-based margin requirements placed on these accounts. This is perhaps one of the most complex applications of the methodology and can be altered by individual portfolio managers given specific requirements.

The CFA Institute – a nonprofit organization of investment professionals – suggests that

> [m]anagers who implement complex and sophisticated investment strategies should [...] understand the various risks and conduct statistical analysis (i.e., stress testing) to determine how the strategy will perform under different conditions. By undertaking adequate due diligence, Managers can better judge the suitability of investments for their clients.[4]

This general suggestion for analysis of portfolio's exposure to risk provides motivation for portfolio managers to determine a risk-based constraint on client portfolios. Once a constraint has been specified, the optimization algorithm outlined in this paper can be used to efficiently enforce such constraints.

1.1 Margin Account and Meeting Margin Calls

A margin account is typically opened with a broker and involves a loan from the broker to the client using securities in the portfolio as collateral.

[4] CFA Institute Program Curriculum, Level III, Volume 1 (Ethical and Professional Standards), 2012.

The margin an investor is required to post in order to establish a position in a security is called the *initial margin.* After the position is established, a generally more lenient requirement – called the *maintenance margin* – is placed on the collateralization of the account.

When the market value of the securities in the portfolio fall below the regulatory requirement for the level of collateralization required to hold the loan, a *margin call* is issued. The margin call essentially notifies the investor that their loan is no longer appropriately collateralized and requires the investor to either deposit additional funds (or securities) into the account or initiates the partial liquidation of the positions in the portfolio. Liquidation can either proceed through the direction of the client or at the discretion of the brokerage firm.

Market value fluctuations effect the ability of the securities to adequately collateralize the brokerage loan. Overly generous loan collateralization guidelines run the risk that investors will obtain excessive levels of leverage in their portfolios.

The portfolio liquidation literature considers the secondary effects of liquidating a portfolio through large transactions. [1] model the effect of execution time lags and liquidation discounts when unwinding a position in a risky asset with the objective of maximizing the terminal value of the cash position in the portfolio. [2] determine, within a certain liquidity model, the optimal liquidation strategy for increasing absolute risk aversion investors and decreasing absolute risk aversion investors. [3] minimize the expected liquidity costs through the optimal placement of market orders in the liquidation. [4] maximize a constant absolute risk aversion investor's expected utility when the investor must liquidate a basket of assets within a finite time horizon. For a recent review of the optimal liquidation literature, see [5].

This paper adapts some of the powerful results of the trust-region literature to derive the optimal strategy to liquidate a portfolio in order to meet a margin call.[5] The main portfolio liquidation problem is a classical portfolio rebalancing problem embedding a margin requirement constraint which itself is an optimization problem.

We call the problem to determine the portfolio margin requirement the

[5]By optimal, we mean the liquidation procedure that alters the positions of the original portfolio minimally.

inner problem. It is equivalent to determining the maximum loss a portfolio would experience within a set of possible scenarios. The traditional way to solve the inner problem is a brute force search over a finite set of scenarios, while in our paper, we expand the set of scenarios to be a continuous set. This continuous extension facilitates the use of option Greeks in the calculation of margin sensitivities to portfolio allocation changes. As a result, the calculation of portfolio margin has clear connections to trust-region problems wherein one tries to find the extreme value of a function within a bounded region. In order to obtain a fast inner problem solutions whose closed-form solution can be passed into the main liquidation problem, this paper uses analysis contained within [6], a modified dog-leg method. A standard method solves the trust-region problem numerically, but our closed-form solution can be utilized directly in the main liquidation problem. For example, the margin requirement criterion's gradient and Hessian information are derived so that the liquidation problem is solved in a more efficient manner. For an introduction to the trust region methods in a more general context with pedagogical discussions, see [7].

We explore the optimal strategy to liquidate a portfolio in order to meet a margin call. To simplify analysis of portfolio liquidation, we assume that the securities in the underlying portfolio are sufficiently liquid so that we can ignore any possible secondary effects on the market price of securities from placing market orders. For retail investors with relatively small positions in their margin accounts, these effects should be small. We consider the more direct problem of determining the set of securities to liquidate within a portfolio to meet a margin call issued by a brokerage firm that minimally changes the portfolio.

1.2 History of Margin Requirements

Margin requirements were developed in the Securities Act of 1933, the Banking Act of 1933 and the Securities Exchange Act of 1934 [8]. The Securities Exchange Act of 1934 gave the Federal Reserve Board the power to "set initial, maintenance, and short sale margin requirements on all securities traded on a national exchange for purposes of regulating the securities credit extended by exchange members" [9]. Regulation T (Reg T) codified the rules pertaining to the fraction of an exchange-traded security's current market value an agent was allowed to lend. Reg T is a *rules-based* calculation methodology. For example, to establish a long position in a stock, one currently needs to deposit a minimum of 50% net equity (1 - loan value/security current market value) according to Reg T requirements.[6] Reg T specifies initial margin requirements and maintenance margin requirements are specified by self-regulatory organizations (SROs). The downside of rules-based margin calculations is that offsetting transactions are not considered as a whole. As a result, some risky portfolios have the same margin requirement as conservative portfolios.

Margin calculations that take into account offsetting transactions within a portfolio are *risk-based*. In this approach, the portfolio is stressed in several scenarios wherein the parameters of the underlying assets (asset price and/or volatility) are varied by prespecified amounts. In general, larger haircuts are applied to higher risk, less diversified, assets.[7] For example, a regulatory body may have determined that asset price changes of up to 15% and asset volatilities changing by up to 10% are reasonable scenarios to consider for the single day potential profit-and-loss of a portfolio. Depending on a portfolio's exposure to the underlying asset, the portfolio could experience a large loss based upon these parametric fluctuations. As such, the required margin is determined to be the maximum loss the portfolio could reasonably expect to experience on the next trading day. One could also view this portfolio margin requirement as the largest value-at-risk for the portfolio under the scenarios considered.[8]

The risk-based haircut methodology may be used to calculate capital charges based upon theoretical option pricing models.[9] The Theoretical Intermarket Margining System (TIMS), developed by The Options Clearing Corporation (OCC) and approved by the SEC in 2006 following a pilot program, uses a similar risk-based calculation methodology for determining the portfolio margin given the positions in a customer's portfolio.[10] TIMS has since been replaced

[6] Regulation T: Code of Federal Regulations, Title 12, Chapter II, Subchapter A, Part 220.

[7] A *haircut* is simply another term used to describe the margin requirement or the maximum expected loss within a reasonable set of scenarios for a given position.

[8] Reg T margin accounts are required to have $2,000 net equity in their account whereas portfolio margin accounts are required to have $100,000 net equity.

[9] Securities Exchange Act of 1934, Section 15c3-1.

[10] http://www.theocc.com/risk-management/cpm/

by the System for Theoretical Analysis and Numerical Simulations (STANs) developed by the OCC utilizing a large-scale Monte Carlo-based risk management methodology.[11] For a complete introduction to portfolio margining, see [10]. For a comparative analysis of risk-based and rules/strategy-based margin methodologies as well as a more complete description of the historical evolution of margin requirements, see [11]. For the possible effects of the recent Dodd-Frank Act on portfolio margin utilization, see [12].

Example: The risk-based portfolio margin calculation methodology significantly reduces the required margin in many scenarios. Under the Reg T margin rules, the initial margin requirement for equities is 50% of market value, and for options 100% of option premium. The following example portfolio illustrates the margin reduction.[12] The put option expires in 90 days and we assume that Citigroup has a constant, continuously-compounded dividend yield of 1%, that the risk-free rate is a constant, continuously-compounded, 3% and the implied volatility is 15%. Table 1 summarizes the results.

Table 1: Comparison of portfolio and Reg T margin requirements

	Reg T Margin	Portfolio Margin	Margin Reduction
Long 1,000 shares Citigroup stock @ $30.00	$15,000.00	$4,500.00	$10,500.00
Long 10 shares Citigroup put $30.00 @ $0.82	$820.00	$807.46	$12.54
Portfolio	$15,820.00	$969.89	$14,850.11 (94%)

[11]The system developed by the Chicago Mercantile Exchange in 1988, known as the "Standard Portfolio ANalysis of Risk" (SPAN), is another such a risk-based margin calculation methodology. See: http://www.cmegroup.com/clearing/risk-management/span-overview.html.

[12]The scenario that gave rise to the largest loss to the portfolio was one in which the stock price decreased by 15% (largest fluctuation considered) and the implied volatility decreased by 15%.

2 Outline of the Problem

2.1 Preliminary Definitions

Consider a portfolio with initial value V consisting of a set of J equities $\{S_j | \, j = 1, 2, \cdots, J\}$ and European options on those equities. We denote the price of the securities in the portfolio by $P_{i,j}$ where $P_{1,j}$ denotes the price of equity j and $P_{i,j}$ represents the price of European option i on underlying asset j. More explicitly, we define

$$P_{i,j} = \begin{cases} S_j & i = 1 \\ O(S_j, \sigma_i, K_i, r, q, T_i, CP_i) & i > 2 \end{cases} \tag{1}$$

where $\{\sigma_i, K_i, T_i, CP_i\}$ represent the implied volatility, strike price, time to expiration and call-put indicator for option i respectively. For simplicity and analytic tractibility, we use the Black-Scholes model for the valuation of the options within the portfolio [13, 14]. In particular, we have

$$O(S, \sigma, K, r, q, T, CP) = \begin{cases} C(S, \sigma, K, r, q, T) & CP = \text{Call} \\ P(S, \sigma, K, r, q, T) & CP = \text{Put} \end{cases} \tag{2}$$

and explicit formulas for these functions can be found in Appendix A.

The initial value of the portfolio may then be written as

$$V = \sum_{i,j} P_{i,j} n^0_{i,j}$$

where $n^0_{i,j}$ denote the quantities within the portfolio for each of the underlying securities: $n^0_{i,j} < 0$ reflects a short position and $n^0_{i,j} > 0$ reflects a long position.

Consider the situation wherein the investor with the above portfolio is issued a margin call. In this case, the net liquidation value (NLV) of the portfolio defined by

$$\text{NLV}(n^0_{i,j}) = \sum_{i,j} P_{i,j} n^0_{i,j} - L$$

where L is margin loan, has fallen below some specified margin requirement of either the brokerage house or a regulatory body. As long as the net liquidation value is greater than zero, the requirement can be satisfied by liquidating the entire portfolio and decreasing all positions to zero. It is also generally possible to satisfy the margin requirement by liquidating only a portion of the portfolio.

2.2 Liquidation Optimization Problem

An optimal strategy to liquidate a portfolio in order to meet a margin call first needs a definition of the quantity being optimized during the liquidation. One approach to the optimization problem is simply to minimize transaction costs incurred during the liquidation. Assume the initial portfolio positions are given by $n^0_{i,j}$ and that the new positions in the portfolio are given by $n_{i,j} = n^0_{i,j} - \text{sign}(n^0_{i,j})\Delta n_{i,j}$, where $\Delta n_{i,j} \geq 0$. The transaction costs associated with liquidating $\Delta n_{i,j}$ from the initial position $n^0_{i,j}$ can be written generally as $f_{i,j}(\Delta n_{i,j})$. In order to minimize transaction costs, we need to determine

$$\min_{\Delta n_{i,j}} \sum_{i,j} f_{i,j}(\Delta n_{i,j})$$

such that the margin call is satisfied: $\text{NLV}(n_{i,j}) \geq \text{Margin}(n_{i,j})$. In order to make progress in this problem, we first make the simplifying assumption that transaction costs are uniform across securities within the portfolio: $f_{i,j}(\Delta n_{i,j}) = f(\Delta n_{i,j})$. We then assume that transaction costs are roughly linearly dependent on changes in positions: $f(\Delta n_{i,j}) \propto \Delta n_{i,j}$.

The optimization problem we address in this paper is therefore given by

$$\min_{\Delta n_{i,j}} \sum_{i,j} \Delta n_{i,j} \tag{3}$$

subject to the constraints:

$$\text{NLV}(n_{i,j}) \geq \text{Margin}(n_{i,j}), \tag{4}$$

$$0 \leq \Delta n_{i,j} \leq |n^0_{i,j}|, \tag{5}$$

$$n_{i,j} = n^0_{i,j} - \text{sign}(n^0_{i,j})\Delta n_{i,j}. \tag{6}$$

We leave the study of the efficacy of objective functions different from (3) to future research.

The altered positions $n_{i,j}$ will be different from the starting positions, $n^0_{i,j}$, of the portfolio, when the initial margin requirement is breached.[13] The changes in position are $\Delta n_{i,j} = (n_{i,j} - n^0_{i,j}) \cdot \text{sign}(n^0_{i,j})$. The constraint in (5) ensures that changes in the positions are not larger than the initial positions.[14] The

[13]Therefore, the starting point $\Delta n_{i,j} = 0$ is not a feasible solution.

[14]In principle, one could imagine a situation wherein a margin requirement is satisfied by changing a short position to a long position. Since this alteration would presumably change the strategy implemented by the account holder, we do not consider such portfolio alterations.

constraint in (6) serves to prevent positions in the portfolio from increasing in magnitude. If $n^0_{i,j} < 0$ – representing a short position in security i with underlying asset j – then the constraint ensures $n_{i,j} \geq n^0_{i,j}$. Similarly, if $n^0_{i,j} > 0$ – representing a long position in security i with underlying asset j – then the constraint ensures $n_{i,j} \leq n^0_{i,j}$.

The net liquidation value of a portfolio does not change as the positions are changed, so long as transactions only internally alter the portfolio.[15] As a result, the net liquidation value of the portfolio is independent of positions

$$\mathrm{NLV}(n_{i,j}) = \mathrm{NLV}(n^0_{i,j}).$$

As noted previously, the optimization problem always has at least one feasible solution: the portfolio is completely liquidated ($n_{i,j} = 0$ for all securities i and underlying assets j). In this case, $\Delta n_{i,j} = \mathrm{sign}(n^0_{i,j})n^0_{i,j}$.

The optimization procedure outlined in this paper is more general than the specific margin liqudiation context. For example, rather than implementing constraints based upon the net liquidation value of a customer's margin account, one could implement a similar constraint based upon the SEC net capital rule and apply the procedures outlined in the paper to the portfolio of assets held by a broker-dealer.

2.3 Margin Calculation Optimization Inner Problem

The primary constraint – Equation (4) – in the liquidation optimization problem requires that the net equity in the account is larger than the required margin. When portfolio margin requirements are implemented the broker needs to determine the scenario in which the portfolio would experience the maximum loss. As a result, the calculation of the margin requirement is in itself an optimization problem. This section explores this requirement and extends the conventional methodology to facilitate an analytically tractible approximation. The analytically tractible approximations for the margin requirement developed later in the paper can then be implemented to determine the optimal liquidation in the face of a margin call.

[15] This means that no other securities or cash is deposited. The change in position of one security results in a change in the cash or the position in another security.

2.3.1 Conventional Discrete Margin Calculation

The calculation of portfolio margin requirements is based upon the "risk-based haircuts" methodology wherein the profit and loss of the portfolio is determined within a variety of scenarios. The scenarios typically involve the underlying assets and/or underlying asset volatilities increasing or decreasing in value within a reasonable range of their original values.[16] For example, a portfolio consisting of a single short position in a stock would experience the maximum loss (within the considered scenario range) if the stock increased in value. The maximum loss a portfolio would incur among a set of scenarios is defined to be the portfolio margin.

When calculating portfolio margin, typically brokerage houses will stress the portfolio in class groups – securities with the same underlying asset – and then aggregate the maximum losses a portfolio would experience into the total portfolio margin requirement. This procedure makes sense intuitively since securities with different underlying assets should move independently of one another (ignoring correlation effects as a first approximation).

If we let $\mathbb{S} = \{1, 2, \ldots, S\}$ be the finite set of scenarios with unique parameter changes $\vec{x}_j = \left(\frac{\Delta S_j}{S_j}, \frac{\Delta \sigma_j}{\sigma_j}\right)^T$. If $\vec{x}_j^*$ is defined as the scenario under which the subportfolio incurs the largest loss, then

$$\vec{x}_j^* = \arg \min_{\vec{x}_k, k \in \mathbb{S}} \Delta V_j(\vec{x}_k | n_{i,j}) \tag{7}$$

where the change in value of a subportfolio is given by,

$$\Delta V_j(\vec{x}_j | n_{i,j}) = \sum_i \left(P_{i,j}(\vec{x}_j) - P_{i,j}(0)\right) n_{i,j}.$$

As a result, the portfolio margin requirement is given by

$$\text{Margin}(n_{i,j}) = -\sum_j \Delta V_j(\vec{x}_j^* | n_{i,j}) \tag{8}$$

where $\Delta V_j(\vec{x} | n_{i,j})$ is the change in the value of the subportfolio corresponding to asset j, given the positions $n_{i,j}$, due to changes in the parameters of the

[16]The range of market value fluctuations is approximately 15% for stock baskets and options. A smaller range of market value fluctuations is required for high-capitalization broad-based indexes such as the S&P 500. Some brokerage houses consider their own, more volatile, set of scenarios.

underlying asset given by the vector $\vec{x}_j$.[17] See Figure 1 for an illustration of a discrete set of scenarios considered during the stress-test of a subportfolio.

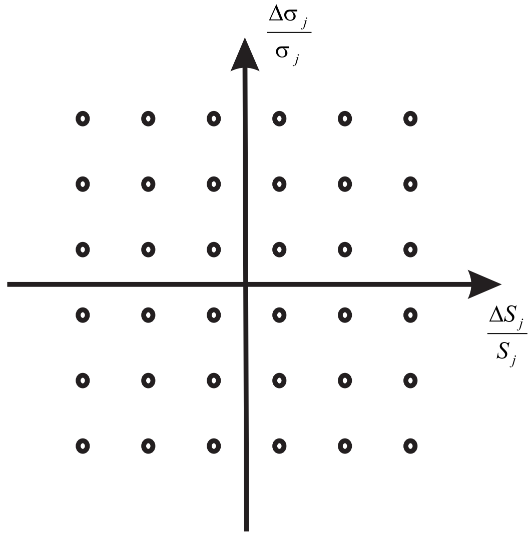

Figure 1: Scenarios considered in the OCC Risk-Based Haircuts methodology.

A finite set of scenarios is convenient for regulatory bodies and brokerage houses to compute the portfolio margin. The optimization inner problem thus defined in (7) is essentially a discrete optimization over a finite set. For each set of new positions $n_{i,j}$, calculating the portfolio margin requirement involves solving x_j^* for the optimization inner problem. This requires one to compute the portfolio changes ΔV_j a total of S times and to then select the maximum loss. Once $x_j^*|n_{i,j}$ is solved, it is plugged back to the main liquidation optimization problem (3)–(4) to determine optimal liquidation sizes $\Delta n_{i,j}$. As a result of the discretization, there is no analytical form for the solution $x_j^*|n_{i,j}$ and therefore the portfolio liquidation problem is computationally difficult to solve.

With infinitesimal changes of positions $n_{i,j}$, the worst scenario typically stays the same, meaning there is little change in the solution $\vec{x}_j^*$ for the inner problem. Using this fact, we know that the margin surface is piecewise linear because the portfolio loss linearly changes with positions.

Example: Consider a simple hypothetical portfolio containing a short straddle. See Figure 2.3.1 for the margin surface. The portfolio is short 1,000 Eu-

[17]By considering subportfolios rather than individual securities, [15] showed how brokers can set margin levels to increase revenue from lending more money.

ropean put options and 1,000 European call options linked to the same stock, both with strike price \$60 and both maturing in 3 months. The current stock price of the underlying asset is $S_1 = \$60$, the risk-free rate for the options is 3%, the implied volatilities for both options is 15% and the asset's dividend yield is 1%. Cash held in the portfolio is \$8,000. According to Black-Scholes European option valuation formulas, each call option costs \$1.92 and each put option costs \$1.63. The portfolio's net liquidation value is therefore \$4,444.78. For the stress test, we consider the following six scenarios

$$\begin{aligned} x_1 \in \mathbb{S} \;\; = \;\; & \big\{(15\%, 0), (-15\%, 15\%), (-15\%, -15\%), \\ & (15\%, 0), (15\%, 15\%), (15\%, -15\%)\big\}. \end{aligned} \tag{9}$$

At the initial position $\{n_{2,1} = -1000, n_{3,1} = -1000\}$ (Point 1 in the figure), the worst scenario corresponds to $x_1^* = (15\%, 15\%)$ – stock price increases 15% and volatility increases 15%.[18] The portfolio margin requirement is \$5,922.83, larger than the \$4,444.78 net equity held in the account and a margin call is issued.

We have plotted the portfolio margin requirement surface in Figure 2.3.1. As long as $n_{i,1}$ changes within a reasonable range, the margin requirement changes linearly with $n_{i,1}$ – on Plane 1, this is a result of the fact that the maximum losses correspond to the same risk scenario. When the positions $n_{i,j}$ change more substantially, the margin surface switches to Plane 2, corresponding to another scenario $\vec{x}_1^* = (-15\%, 15\%)$.

The optimal solution for the main liquidation problem is $(\Delta n_{2,1}, \Delta n_{3,1}) = (175, 234)$ corresponding to the positions $(n_{2,1}, n_{3,1}) = (-825, -766)$ or Point 3 in Figure 2.3.1. In other words, only by liquidating more than one security simultaneously will the portfolio margin requirement be satisfied. The graph shows that by first liquidating call options, the margin will be reduced (from Point 1 to Point 2). Once the portfolio is liquidated to Point 2, buying put options or call options individually only serves to increase the margin requirement. The optimal liquidation involves the simultaneous purchase of call and put options (from Point 2 to Point 3).

[18] $n_{1,1} = n_{1,1}^0 = 0$ since the portfolio does not contain an investment in the underlying stock.

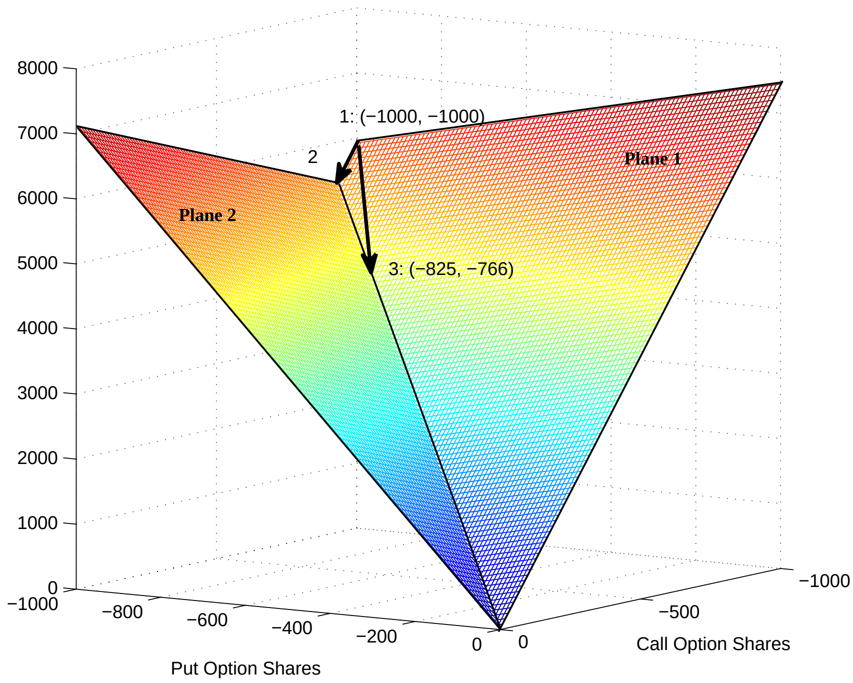

Figure 2: Portfolio with a Short Straddle.

2.3.2 Continuous Extension of the Margin Calculation

There are many computational barriers when the inner optimization involves a discrete set of scenarios. One of the main contributions of this paper is to extend the set of scenarios to a continuous region. This extension is analytically advantageous and allows practioners to apply results from the trust-region literature. Furthermore, [16] argue that increased margin precision – through the consideration of a larger set of scenarios – promotes greater efficiency of options trading.

The problem of determining the portfolio margin requirement has clear connections to trust-region problems wherein one tries to find the extreme value of a function within a bounded region. The analysis of this section, and indeed this paper, relies heavily on the analysis contained within [6] and the material contained within [7].

The generalized inner problem that we consider involves determining the maximum loss a portfolio would experience if the underlying asset price and underlying asset volatility were altered in magnitude. In Figure 3, we show the distinction between the two approaches with our new generalized approach depicted in the right panel.

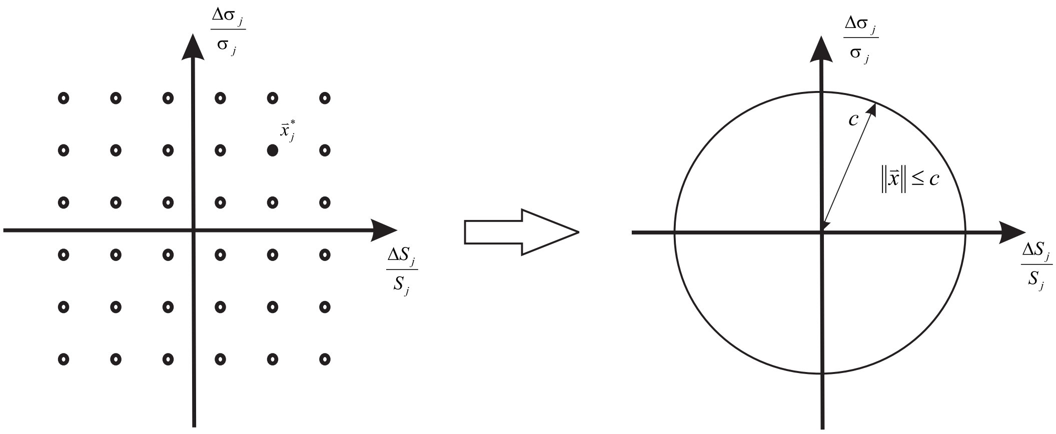

Figure 3: A continuous generalization of the discrete scenarios considered in the OCC Risk-Based Haircuts methodology.

The region we consider in our calculation of portfolio margin includes an uncountably infinite set of scenarios for the underlying asset, volatility combinations. Furthermore, if the radius of the circle c bounding the set of scenarios is equal to $\max(|\frac{\Delta S}{S}| \times |\frac{\Delta\sigma}{\sigma}|)$ then the continuous region contains all of the discrete scenarios of the conventional risk-based haircuts analysis. As a result of this, the analysis we suggest here will give a more stringent portfolio margin requirement than the conventional risk-based haircuts analysis.

Our generalized definition of (8) is given as follows. Let $\vec{x}_j = \left(\frac{\Delta S_j}{S_j}, \frac{\Delta\sigma_j}{\sigma_j}\right)^T$ be constrained such that $||\vec{x}_j|| \leq c$ for some $c > 0$, where $|| \cdot ||$ represents the Euclidean norm.[19] The inner problem changes to a similar well defined problem with $\vec{x}_j^*$ defined by

$$\vec{x}_j^* = \arg \min_{||\vec{x}|| \leq c} \Delta V_j(\vec{x}|n_{i,j}). \tag{10}$$

The generalized portfolio margin requirement (8) with $\vec{x}_j^*$ defined in (10).

An important improvement of the new formulation of portfolio liquidation over the conventional approach is that the solution constrained within the

[19]Although we assume c is the same number for all equities in the portfolio, generalization to non-uniform sensitivities – reflecting more or less risky assets – is straight-forward.

circle is continuous. Any small variation in the positions $n_{i,j}$ for the main optimization problem results in small changes of optimal scenario $\vec{x}_j^*$ determining the portfolio margin. The changes are continuous but may not be smooth since the scenario corresponding to the maximum loss may move beyond the feasible region. If analytical solutions of $\vec{x}_j^*$ given $n_{i,j}$ are available, then the portfolio margin can be expressed as a function of $n_{i,j}$. We may further take the gradient, or determine the Hessian matrix of the margin function to facilitate the main liquidation optimization problem.

2.3.3 Alternative Extension of the Margin Calculation

Rather than generalizing the portfolio margin calculation from the discrete set to the continuous set as in Figure 3, one could choose to consider a box region ($||\vec{x}||^{\infty} \leq c$) as depicted in Figure 4. The primary advantage of using

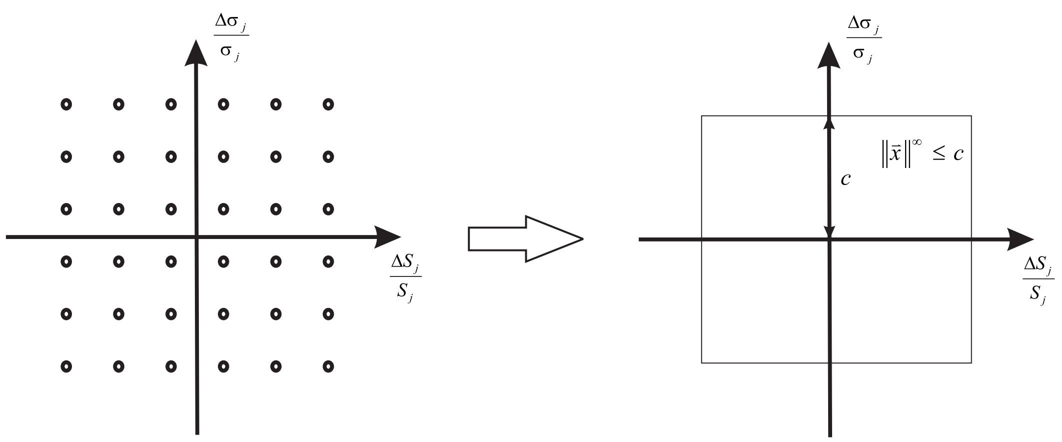

Figure 4: An alternative continuous generalization of the discrete scenarios considered in the OCC Risk-Based Haircuts methodology.

this approach is that the portfolio margin estimated using the box region is a closer proxy to the result of the discrete set. Because there is a larger number of scenarios considered in the circular region and as a result that approach is more conservative (placing a larger margin requirement).

The inner optimization problem given the box region is essentially a quadratic programming problem. After laying out the KKT conditions, the inner problem is equivilent to solving a 6 dimensional linear programming problem. The inner problem becomes numerically simple, but analytically intractible. After we considered the pros and cons of the two approaches, we concluded that the circular region depicted in Figure 3 is the most convenient generalization of the conventional discrete portfolio margin calculation.

3 Margin Calculations

Now that we have discussed our continuous extension of the portfolio margin requirement, we develop the analytically tractible approximations for the margin requirement.[20] These approximations will then be implemented to determine the optimal liquidation of a portfolio of securities in the face of a margin call. The solutions to the main liquidation problem using the approximate margin requirement will have approximately the same objective value to the true solution.

In Section 3.1, we use the first order expansion of the change in the portfolio value to estimate the margin requirement. In Section 3.2, we use the second order expansion. The second order approach provides a better estimation than the first order approach; however, the second order approximation also turns out to be more computationally involved.

3.1 First Order Margin Estimation

Consider the first-order change in value of the portfolio, ΔV, given by the changes in the underlying asset values $S \to S+\Delta S = S(1+R_S)$ and underlying asset implied volatilities $\sigma \to \sigma + \Delta\sigma = \sigma(1+R_\sigma)$ where we consider scenarios with $(\frac{\Delta S}{S})^2 + (\frac{\Delta \sigma}{\sigma})^2 = R_S^2 + R_\sigma^2 \leq c^2$ where c is some positive real number.[21] The change in the portfolio value (ΔV) is estimated by the first order change

[20] Obviously having an exact analytic solution is preferable; however this is almost never possible.

[21] Portfolio margin calculations based upon the "risk-based haircuts" methodology of the Options Clearing Corporation (OCC) have values of c on the order of 0.05 to 0.15 depending upon the composition of the portfolio.

in the portfolio value given by,

$$\Delta\widetilde{V}(\vec{x}_1,\ldots,\vec{x}_J|n_{i,j}) = \sum_j \Delta\widetilde{V}_j(\vec{x}_j|n_{i,j}) = \sum_j \vec{g}_j^T\vec{x}_j, \tag{11}$$

where $\vec{g}_j^T = (\sum_i D_{i,j}n_{i,j}, \sum_i V_{i,j}n_{i,j})$, $\vec{x}_j = \left(\frac{\Delta S_j}{S_j}, \frac{\Delta\sigma_j}{\sigma_j}\right)^T$ and

$$D_{i,j} = \frac{\partial P_{i,j}}{\partial S_j}S_j \quad \text{and} \quad V_{i,j} = \frac{\partial P_{i,j}}{\partial \sigma_j}\sigma_j.$$

The partial derivatives in the equations above are the conventional greeks delta and vega. For a derivation of the first and second order partial derivatives of option values within the Black-Scholes model, see Appendix A.

To solve this problem of calculating the margin requirement more completely, we have to determine for a given set of initial asset positions

$$\min_{||\vec{x}_j||\le c} \Delta\widetilde{V}_j(\vec{x}_j|n_{i,j}) = \min_{||\vec{x}_j||\le c} \vec{g}_j^T\vec{x}_j. \tag{12}$$

Since the objective function for the inner problem is a linear function, the solution must be parallel with $-\vec{g}$ (direction of steepest descent) subject to the constraint $||\vec{x}_j^*|| = c$ in order to maximize the reduction in portfolio value. Therefore the solution is simply

$$\vec{x}_j^* = -\frac{c\vec{g}_j}{||\vec{g}_j||}. \tag{13}$$

Figure 5 illustrates this analytic solution.

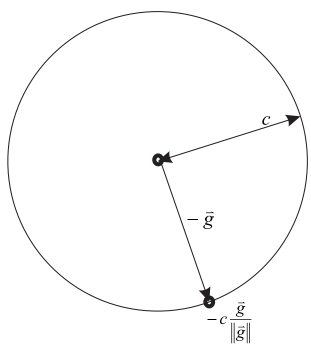

Figure 5: Result of first order minimization of portfolio changes.

Now that we have found the explicit value of $\vec{x}_j^*$, given the holdings $\{n_{i,j}\}$, we have found an approximation of the generalized margin requirement. We can use these results to estimate the margin requirement as follows

$$\text{Margin}(n_{i,j}) \approx -\Delta\widetilde{V}(\vec{x}_1^*, \ldots, \vec{x}_J^* | n_{i,j}) = c \sum_j ||\vec{g}_j||. \tag{14}$$

As a result of this first order margin estimation, we replace the portfolio liquidation constraint in (4) by the following constraint

$$\text{NLV}(n_{i,j}) \geq c \sum_j ||\vec{g}_j||. \tag{15}$$

For regions in which the value of the portfolio does not vary substantially, the first order margin calculation and resulting portfolio liquidation procedure outlined above is efficient and accurate. To be more precise, as long as the following limit holds

$$\left| \frac{\partial^2 V}{\partial X_i \partial X_j} \right| \ll \left| \left(\frac{\partial V}{\partial X_i} \right) \left(\frac{1}{V} \right) \left(\frac{\partial V}{\partial X_j} \right) \right| \tag{16}$$

for all $\{i, j\}$, where $X_i \in \{\sigma_i, S_i\}$, and for all $|\Delta X_i \Delta X_j| < \epsilon^2 X_i X_j$ for some $\epsilon > 0$. For portfolios with values that depend smoothly on the parameter values, there will always be an $\epsilon > 0$ where this is the case. In cases where this ϵ is comparable to the parameter c defining the size of the feasible region, the first-order analysis should be sufficient to determine the portfolio margin requirement.

In practice, the first-order estimation will not always give an accurate estimate of the margin requirement; however, in certain special cases, the first order approach is quite accurate. 1) When there is a large equity component dominating the value of the portfolio. Since the equity itself has a margin surface that is linear with respect to changes in the characteristics for the asset, the first order approximation tends to be more accurate. 2) For broadbased indices, the sensitivity tests are based upon smaller parameter ranges. The smaller range of values for the stress test in this case will significantly improve the accuracy of the first order expansion.

3.2 Second Order Margin Estimation

The change in value of the portfolio, ΔV, to second order in $\vec{x}_j^T = \left(\frac{\Delta S_j}{S_j}, \frac{\Delta \sigma_j}{\sigma_j} \right)$

is estimated by

$$\Delta\widetilde{V}(\vec{x}_1,\ldots,\vec{x}_J|n_{i,j}) = \sum_j \left(\vec{g}_j^T\vec{x}_j + \frac{1}{2}\vec{x}_j^T B_j \vec{x}_j\right), \tag{17}$$

where $\vec{g}_j$ is defined in (11) and the symmetric 2×2 matrix B_j is defined as follows

$$B_j = \sum_i \begin{pmatrix} \frac{\partial^2 P_{i,j}}{\partial S_j^2} n_{i,j} & \frac{\partial^e P_{i,j}}{\partial S_j \partial \sigma_j} n_{i,j} \\ \frac{\partial^2 P_{i,j}}{\partial \sigma_j \partial S_j} n_{i,j} & \frac{\partial^2 P_{i,j}}{\partial \sigma_j^2} n_{i,j} \end{pmatrix}. \tag{18}$$

This matrix is the Hessian matrix for the subset of portfolio value changes due to changes in the characteristics of the underlying asset j. This estimation is more accurate since it includes smaller effects resulting from the changing characteristics of the underlying asset.

The second order margin estimation is now a quadratic function of the changes in the underlying assets. The matrix B_j is not necessarily positive-definite.[22] Several approximate approaches to estimate the extreme value of the function in (17) within the feasible region falter when the matrix B_j is not positive-definite. For example, the single dogleg method of [17] and the modified double dogleg method of [18], although powerful, are ill-equipped to handle the case of an indefinite Hessian.

The following theorem sets up the machinery required to solve the portfolio margin calculation exactly.[23] Although we do not use this exact approach due to its computational complexity, we include the theorem here as a motivation for the approximate approach that we develop and advocate.

Theorem 3.1. *The vector $\vec{x}^*$ is a global solution of the trust-region problem*

$$\min_{\vec{x}\in\mathbb{R}^n} \left(\vec{g}^T\vec{x} + \frac{1}{2}\vec{x}^T B\vec{x}\right) \quad \textit{such that} \quad ||\vec{x}|| \le c \tag{19}$$

if and only if $\vec{x}^*$ is feasible and there is a scalar $\lambda \ge 0$ such that the following conditions are satisfied:

$$\begin{aligned} (B+\lambda I)\vec{x}^* &= -\vec{g}, & (20)\\ \lambda(c - ||\vec{x}^*||) &= 0, & (21)\\ (B+\lambda I) & \quad \textit{is positive-semidefinite.} & (22) \end{aligned}$$

[22]A real valued $n \times n$ matrix A is defined to be positive-definite if $\vec{x}^T A\vec{x} > 0$ for all n-dimensional real vectors $\vec{x} \neq \vec{0}$. Similarly, a real valued $n \times n$ matrix A is defined to be positive-semidefinite if $\vec{x}^T A\vec{x} \ge 0$ for all n-dimensional real vectors $\vec{x} \neq \vec{0}$.

[23]For a proof of the theorem, see, for example, [19].

where I is the $n \times n$ identity matrix.

Additional pedagogical discussion concerning this theorem and related topics can be found in [7].

The theorem is constructive in the sense that the second condition, in (21), requires that either $\lambda = 0$ or that the solution to the problem is on the boundary of the feasible region. Of course if $\lambda = 0$, then the third condition requires B to be positive-semidefinite and the first condition gives $\vec{x}^* = -B^{-1}\vec{g}$ so long as $||\vec{x}^*|| \leq c$. If $||\vec{x}^*|| = c$, then λ is only constrained by the fact that it must be larger in magnitude than the smallest negative eigenvalue of B. If this condition is satisfied, then $(B+\lambda I)$ is positive-definite and $\vec{x}^* = -(B+\lambda I)^{-1}\vec{g}$.

3.3 Computing the Closed-Form Solutions for the Optimization Inner Problem

There are numerous algorithms to solve the margin requirement optimization inner problem, which is now converted to a standard trust-region subproblem. Once the numerical solution is obtained, it can be plugged into the main liquidation problem to value the degree of the violations of the margin constraint. However, this is a slow process when the size the portfolio is large ($n \geq 100$) because the liquidation problem has to be solved using a derivative free method since margin constraint (4) does not have derivative information.

In order to compute the derivative of the margin constraint, we begin by obtaining an approximate solution to the inner optimization problem. The solution is then used to analyze the constraint surface. This procedure significantly improves the efficiency with which the liquidation problem is solved. In Section 4, we compare the computational resources used in the two approaches.

First, consider the case when the matrix B is positive-definite. The approximate procedure that we follow in this paper can be seen as a generalization of the approach found in [17]. In Powell's approach (known as the Dogleg Method), the approximate solution to the quadratic problem in (19) is found by taking a linear combination of the vector corresponding to the direction of the Cauchy point and the vector corresponding to the direction of the Newton point subject to the condition that the solution is feasible.[24] This approach is

[24]The Cauchy point is gives the minimum value of the portfolio change in the direction

only limited by the fact that it requires the matrix B to be positive-definite.

In Figure 6, we give a graphical depiction of the minimization procedure when the matrix B is positive-definite. The three concentric circles in Figure 6 represent the possible cases for the size of the feasible region. If the Cauchy point is not within the feasible region, then the approximate solution to the minimization problem is given by $\vec{x}^* = -c\vec{g}/||\vec{g}||$. If the feasible region contains the Cauchy point but not the Newton point, then the solution is given by $\vec{x}^*(\tau) = -(||\vec{g}||^2/(\vec{g}^T B\vec{g}))\vec{g} - \tau B^{-1}\vec{g}$ where τ ensures $||\vec{x}^*(\tau)|| = c$. Finally, if the Newton point is within the feasible region, then $\vec{x}^* = -B^{-1}\vec{g}$.

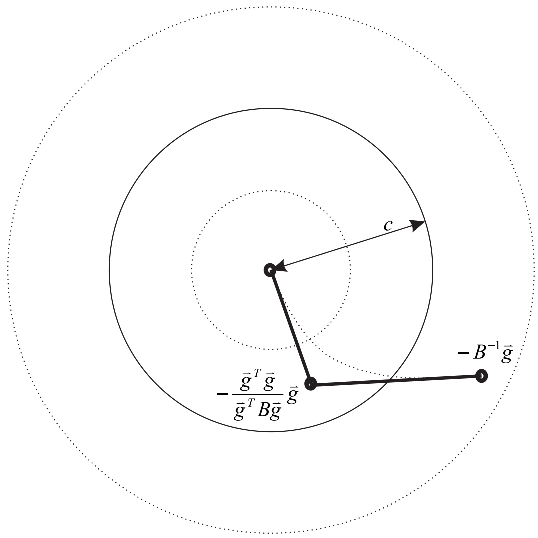

Figure 6: Dogleg methodology.

If the matrix B is not positive-definite, we cannot apply the dogleg method and instead we apply the indefinite dogleg algorithm. Since the matrix B is a real symmetric 2×2 matrix, we can apply the spectral theorem of finite dimensional vector spaces to decompose the matrix B as

$$B = Q^T \Lambda Q \quad \text{where} \quad \Lambda = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

where $Q = (\vec{q}_1, \vec{q}_2)$ is an orthogonal matrix. For sufficiently large λ, the matrix

of steepest descent: $-\left(||\vec{g}||^2/(\vec{g}^T B\vec{g})\right)\vec{g}$. The Newton point gives the minimum value of the unconstrained quadratic model (assuming B is positive-definite): $-B^{-1}\vec{g}$.

$(B + \lambda I)$ is positive-definite. If $\vec{x}(\lambda) = -Q(\Lambda + \lambda I)^{-1} Q^T \vec{g}$, then

$$\vec{x}(\lambda) = -\left(\frac{\vec{q}_1^T \vec{g}}{\lambda_1 + \lambda} \vec{q}_1 + \frac{\vec{q}_2^T \vec{g}}{\lambda_2 + \lambda} \vec{q}_2 \right).$$

The norm of the $\vec{x}(\lambda)$ is given by

$$||\vec{x}(\lambda)|| = \left(\frac{\vec{q}_1^T \vec{g}}{\lambda_1 + \lambda} \right)^2 + \left(\frac{\vec{q}_2^T \vec{g}}{\lambda_2 + \lambda} \right)^2$$

due to the orthogonality of the vectors $\vec{q}_1$ and $\vec{q}_2$.

If the matrix B is not positive-definite, then we must alter the above procedure slightly. Without loss of generality, we assume $\lambda_1 \leq \lambda_2$. We then consider the matrix $(B + \lambda I)$ for some $\lambda \in (-\lambda_1, -2\lambda_1]$ such that the matrix $(B + \lambda I)$ is positive-definite.

In Figure 7, we present a graphical depiction of the indefinite dogleg methodology. Once again, we have three concentric circles representing the possible

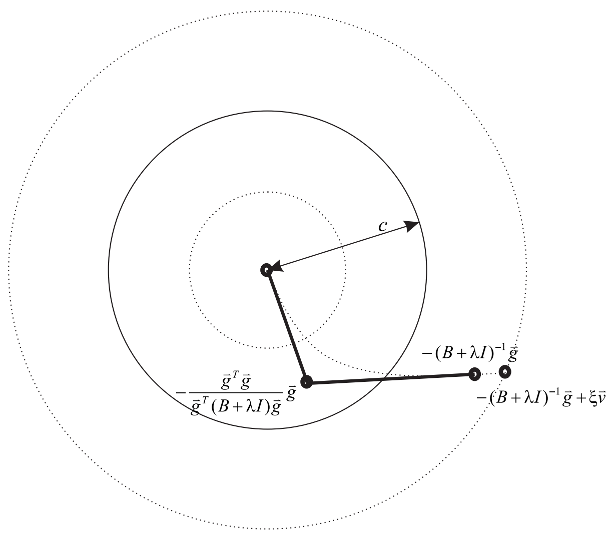

Figure 7: Indefinite dogleg methodology

cases for the size of the feasible region. If the Cauchy point of the modified problem is not within the feasible region, then $\vec{x}^* = -c\vec{g}/||\vec{g}||$. If the Cauchy point is within the region, but the Newton point is not, then the solution is given by $\vec{x}^*(\tau) = -(||\vec{g}||^2/(\vec{g}^T(B + \lambda I)\vec{g})\vec{g} - \tau(B + \lambda I)^{-1}\vec{g}$ where τ ensures $||\vec{x}^*(\tau)|| = c$.[25] Finally, if the Newton point of the modified problem is within

[25] For an iterative approach of determining the value of τ using Newton's root finding

the trust region then $\vec{x}^*(\xi) = -(B + \lambda I)^{-1}\vec{g} + \xi\vec{v}_1$ such that $||\vec{x}^*(\xi)|| = c$.[26] The following summarizes the second order algorithm.

Algorithm:

1. If B is positive-definite:

   (a) If the unconstrained minimizer satisfies $|| - B^{-1}\vec{g}|| \leq c$, then $\vec{x}^* = -B^{-1}\vec{g}$, otherwise perform a dogleg search.

   (b) If the Cauchy point $|| - (||\vec{g}||^2/(\vec{g}^T B\vec{g})\vec{g}|| > c$ is outside the circle then search along the steepest decent direction $-\vec{g}$ and determine $\vec{x}^* = -c\vec{g}/||\vec{g}||$ on the boundary.

   (c) Otherwise look for an interpolation point of the Cauchy point and the unconstrained minimizer $\vec{x}(\tau) = -(||\vec{g}||^2/(\vec{g}^T B\vec{g})\vec{g} - \tau B^{-1}\vec{g}$ such that $||\vec{x}^*(\tau)|| = c$.

2. If B is not positive-definite, find a value λ in $(-\lambda_1, -2\lambda_1]$, such that $B + \lambda I$ is a positive-definite matrix:

   (a) If the point $-(||\vec{g}||^2/(\vec{g}^T(B+\lambda I)\vec{g})\vec{g}$ is outside the circle, then search along the steepest decent direction $-\vec{g}$ and determine $\vec{x}^* = -c\vec{g}/||\vec{g}||$ on the boundary of the region.

   (b) If the unconstrained minimizer $-(B + \lambda I)^{-1}\vec{g}$ is located inside the circle, $\vec{x}^*(\xi) = -(B + \lambda I)^{-1}\vec{g} + \xi\vec{v}_1$ such that $||\vec{x}(\xi^*)|| = c$

   (c) Otherwise determine the interpolation point $\vec{x}(\tau) = -(||\vec{g}||^2/(\vec{g}^T(B+\lambda I)\vec{g})\vec{g} - \tau(B + \lambda I)^{-1}\vec{g}$ such that $||\vec{x}(\tau^*)|| = c$.

This algorithm exhausts all possible cases for the second order portfolio margin estimation. To implement the algorithm, we determine numerically the case that is appropriate and then proceed to the corresponding closed-form following the algorithm outlined above. This algorithm can efficiently compute an analytically tractable solution. Since B is a 2×2 symmetric

algorithm, see Appendix B. This algorithm converges quite quickly in most situations. This method has achieved efficiency in large scale, high dimensional problems, but functional dependence on the positions is lost when this approach is taken.

[26]Due to the complexity of this last case in which B is not positive-definite and $||(B + \lambda I)^{-1}\vec{g}|| \leq c$, [20] refer to this as the "hard case".

matrix, its eigenvalues λ_1 and λ_2 and corresponding eigenvectors $\vec{v}_1$ and $\vec{v}_2$ can be computed analytically. Furthermore, the key solution points used in the algorithm, such as the Cauchy point and the Newton point can be computed analytically because the matrix B is easily inverted.[27] The parameters τ and ξ in the algorithm above are also easily solved in the two-dimensional space.

In each case, we have shown how to determine the value of $\vec{x}^*$ that minimizes the value of the subportfolio within the feasible region. We can use these results to estimate the margin requirement as follows

$$\text{Margin}(n_{i,j}) \approx -\Delta V(\vec{x}_1^*, \ldots, \vec{x}_J^* | n_{i,j}). \tag{23}$$

As a result of this second order margin estimation, we replace the portfolio liquidation constraint in (4) with the following constraint

$$\text{NLV}(n_{i,j}) \geq -\Delta V(\vec{x}_1^*, \ldots, \vec{x}_J^* | n_{i,j}). \tag{24}$$

The availability of approximate analytic solutions $\vec{x}*$ has many benefits. For example, by replacing the form of $\vec{x}^*$ in the margin function $-\Delta V$, we can determine the portfolio margin as a function of the positions. This allows us to compute the value of margin as well as gradient of margin with respect to $n_{i,j}$.

To solve the main liquidation problem, we can use any one of the many standard non-linear optimization algorithms. For example, we have used the `fmincon` tool in MATLAB. The tool implements either a trust-region-reflective algorithm or an active-set algorithm to solve the non-linear programming problem with non-linear constraints.

4 Example Portfolios

Liquidating a portfolio one security at a time has the disadvantage of not adequately addressing offsetting positions and, in some cases, can end up leaving the portfolio more exposed to market value fluctuations than before the liquidation occurred.

[27] For the eigenvalue decomposition and the inverse of a general 2×2 symmetric matrix, see Appendix C.

4.1 Portfolio 1

The first example is a portfolio containing a long butterfly spread on a stock with current price of \$60. The three call options – all expiring in three months – in the portfolio have different strike prices and positions:

- Long 500 European call options with strike \$55: ($n^0_{2,1} = 500$)
- Short 1,000 European call options with strike \$60: ($n^0_{3,1} = -1000$)
- Long 500 European call options with strike \$65: ($n^0_{4,1} = 500$)

We assume the three options have the same implied volatility 15%, that the risk-free rate is a constant, continuously-compounded 3% and that the dividend yield is a constant, continuously-compounded 1%.

The first method (Figure 8(a)) uses the discrete scenarios based calculations.[28] The six scenarios are given as before:

$$\vec{x}^T \in \mathbb{S} = \{(15\%, 0), (-15\%, 15\%), (-15\%, -15\%), (15\%, 0), (15\%, 15\%), (15\%, -15\%)\}.$$

where $\vec{x}^T = (\Delta S/S, \Delta\sigma/\sigma)$. The margin surface is a combination of linear planes. The second method (Figure 8(b)) uses the first order estimation. As we can see, because the portfolio does not have a position in the underlying asset and the range of stock variation is moderate (15%), the first order estimation does not provide a good estimation. The third method (Figure 8(c)) is the second order estimation and is much more precise than the first order estimation. One feature about the second order estimation is that it smooths out the margin surface. The smooth interpolation between two nearly planar surfaces is a general feature of the second order estimation.

Assuming the portfolio is financed with a margin loan of \$500, the initial margin estimated by the second order expansion is \$1,742 versus the net liquidation value \$505.

The solution to this problem is to liquidate $(\Delta n_{2,1}, \Delta n_{3,1}, \Delta n_{4,1}) = (254, 437, 0)$ resulting in the final positions $(n_{2,1}, n_{3,1}, n_{4,1}) = (246, -563, 500)$.[29]

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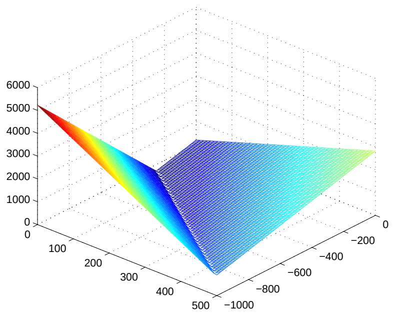
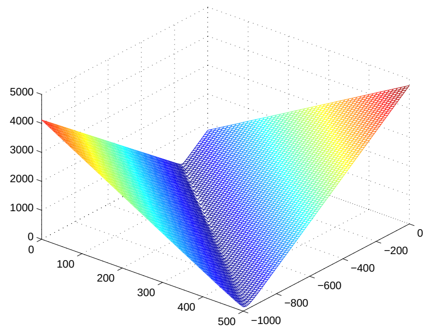
[28] Since it is not possible to plot the margin surface's dependence on three independent variables, we supress the dependence on one independent variable in the figure.

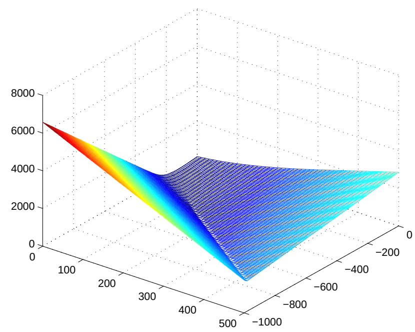
[29] The portfolio contains no investment in the underlying stock and therefore $n_{1,1} = n^0_{1,1} = 0$.



(a) Discrete stress test scenarios



(b) Circle based stress test scenario range and first order estimation



(c) Circle based stress test scenario range and second order estimation

Figure 8: Graphical depiction of the margin surfaces estimated in three different approaches for a portfolio that is long a butterfly spread. Since there are 3 variables involved, we only consider two of them: the x and y axis are based on the long call option (with strike \$55) position and the short call option (with strike \$60) position.

4.2 Portfolio 2

The second portfolio consists of three options strategies. All option positions are on the same stock with current price of \$60 and with a constant, continuously-compounded dividend yield of 1%. The options have different expiration dates and

- Long collar expiring in two months:
  - Long 500 shares of the underlying stock: ($n^0_{1,1} = 500$)
  - Short 500 European call options with strike \$63: ($n^0_{2,1} = -500$)
  - Long 500 European put options with strike \$59: ($n^0_{3,1} = 500$)
- Long butterfly spread expiring in three months:
  - Long 500 European call options with strike \$55: ($n^0_{4,1} = 500$)
  - Short 1,000 European call options with strike \$60: ($n^0_{5,1} = -1000$)
  - Long 500 European call options with strike \$65: ($n^0_{6,1} = 500$)
- Long bear call expiring in one month:
  - Long 300 European call options with strike \$67: ($n^0_{7,1} = 300$)
  - Short 300 European call options with strike \$57: ($n^0_{8,1} = -300$)

We use this complex example to primarily test the efficiency of the optimization problem with/without the gradient information of the contraint (4). Assuming the initial cash amount is -\$30,000, the initial estimated margin is \$2,627 and the net liquidation value is \$248. Since the net equity in the account (\$248) is less than the margin requirement (\$2,627), a margin call is issued.

Using an active-set algorithm in conjunction with the analytic margin approximations presented in this paper, we obtain the liquidation solution of $\{\Delta n_{1,1} = 59, \Delta n_{2,1} = 500, \Delta n_{5,1} = 21\}$ and all other positions liquidated entirely. Without using the gradient information of the margin constraint, it takes a total 344 function valuations to determine the solution. With the gradient information from the margin surface, the number of function valuations reduces by approximately 67% to 114.

If an investor's portfolio does not have too many holdings, the main optimization problem is still computationally manageable. In that case, we do not have to use the gradient information we computed from the analytical solutions in Section 3.2. When the portfolio is very large, using gradient information could save significant computational resources – possibly in excess of 67% as we show in this example.

5 Discussion

By extending the calculation of risk-based constraints – the inner problem – from a discrete optimization to a continuous optimization, this paper has increased the tractability of implementing risk-based constraints at the portfolio level. Using derivative information of the constraint surface, our approach to portfolio allocation decisions is more efficient than conventional numerical approaches.

We show how one could efficiently enforce specific regulatory requirements on a portfolio under management by determining the securities within the portfolio to liquidate in the event that a regulatory requirement is not satisfied. For the specific example of the portfolio management of margin accounts, this paper developed the optimal liquidation of a portfolio containing equities and European options on those equities to meet a margin call.

We generalized the conventional definition of portfolio margin requirements by extending the discrete set of scenarios to a continuous, uncountably infinite, set of scenarios. Since this generalization considers a larger set of scenarios, our approach necessarily provides a more stringent margin requirement than the conventional discrete scenario analysis. Using this generalization, we proposed an algorithm to find approximate analytic expressions for the margin requirement as a function of the positions in the portfolio. We then implemented a non-linear programming procedure to satisfy the margin call and minimally alter the underlying portfolio. The objective function used in this optimization - Equation (3) - can be altered to any other objective function appropriate for the portfolio management problem being considered.

An important implication of this work specific to the management of margin accounts is that the margin requirement is rarely satisfied by liquidating a single security at a time. Often a manager must liquidate several securities simultaneously to maximally decrease the required margin while minimally altering the positions in the underlying portfolio.

Although we focused on the optimal liquidation given a risk-based margin requirement example, the optimization procedure discussed in this paper is general. For example, a hedge fund manager could use the algorithms developed here to implement a strategy that fixes the β of a portfolio within a specified range. Additional applications of this approach are the subject of ongoing research.

Appendices

A Option Sensitivities in the Black-Scholes Model

The risk-neutral valuation of European options within the Black-Scholes model is conventional, but we include the formulas here for convenience and completeness of presentation. Consider European options with strike price K and expiring T years from now on an underlying asset with spot price S and volatility σ. Assume the underlying has a constant, continuously-compounded dividend yield q and take the risk-free rate r to be constant and continuously-compounded. The Black-Scholes valuation of a European call option with these characteristics is

$$C(S,\sigma,K,r,q,T) = Se^{-qT}N(d_+) - Ke^{-rT}N(d_-)$$

and the valuation of a European put option with these characteristics is

$$P(S,\sigma,K,r,q,T) = Ke^{-rT}N(-d_-) - Se^{-qT}N(-d_+)$$

where

$$d_\pm = \frac{\ln(S/K) + (r - q \pm \sigma^2/2)T}{\sigma\sqrt{T}}$$

and N is the standard normal cumulative distribution function.

We need to know how these option valuation formulas depend on the fractional changes in the asset price $S \to S(1+R_S)$ and volatilities $\sigma \to \sigma(1+R_\sigma)$. Using the chain rule, we have the identities

$$\frac{\partial}{\partial R_S} = \left(\frac{\partial S}{\partial R_S}\right)\frac{\partial}{\partial S} = S\frac{\partial}{\partial S} \quad \text{and} \quad \frac{\partial^2}{\partial R_S^2} = \left(\frac{\partial S}{\partial R_S}\right)^2\frac{\partial^2}{\partial S^2} = S^2\frac{\partial}{\partial S^2}$$

$$\frac{\partial}{\partial R_\sigma} = \left(\frac{\partial \sigma}{\partial R_\sigma}\right)\frac{\partial}{\partial \sigma} = \sigma\frac{\partial}{\partial \sigma} \quad \text{and} \quad \frac{\partial^2}{\partial R_\sigma^2} = \left(\frac{\partial \sigma}{\partial R_\sigma}\right)^2\frac{\partial^2}{\partial \sigma^2} = \sigma^2\frac{\partial^2}{\partial \sigma^2}$$

The first derivative of the option valuation formulas are given by

$$\begin{aligned}
\frac{\partial C}{\partial R_S} &= Se^{-qT}N(d_+), \quad \frac{\partial P}{\partial R_S} = Se^{-qT}(N(d_+) - 1), \\
\frac{\partial C}{\partial R_\sigma} &= Se^{-qT}\sigma\sqrt{T}N'(d_+) = \frac{\partial P}{\partial R_\sigma}.
\end{aligned}$$

The second derivative of the option valuation formulas are given by

$$\begin{aligned}
\frac{\partial^2 C}{\partial R_S^2} &= \frac{Se^{-qT}N'(d_+)}{\sigma\sqrt{T}} = \frac{\partial^2 P}{\partial R_S^2}, \\
\frac{\partial^2 C}{\partial R_\sigma^2} &= Se^{-qT}\sigma\sqrt{T}d_+d_-N'(d_+) = \frac{\partial^2 P}{\partial R_\sigma^2}, \\
\frac{\partial^2 C}{\partial R_\sigma \partial R_S} &= -Se^{-qT}d_-N'(d_+) = \frac{\partial^2 P}{\partial R_\sigma \partial R_S}.
\end{aligned}$$

We use these formulas extensively in evaluating the sensitivity of a portfolio consisting of European options to fractional changes in underlying asset prices and volatilities.

B Root-Finding Algorithm

If B is positive-definite ($\lambda_1 \geq 0$), then either the Newton point – corresponding to the global minimum of the quadratic function – is within the feasible region ($||B^{-1}\vec{g}|| \leq c$), in which case $\vec{x}^* = \vec{x}(0)$, or the Newton point is outside the feasible region $||B^{-1}\vec{g}|| \geq c$, in which case there exists a unique $\lambda = \lambda^*$ such that $||\vec{x}(\lambda^*)|| = c$. In this latter case, $\vec{x}^* = \vec{x}(\lambda^*)$.

If B is not positive-definite then $\lambda_1 \leq 0$. Consider the case in which $\vec{q}_1^T\vec{g} \neq 0$. In this case, one needs to find a value for $\lambda > \lambda_1$ such that $||\vec{x}(\lambda)|| = c$. Following [7], one can implement a Newton's root-finding method to iteratively determine the zero of the function,

$$\phi(\lambda) = \frac{1}{c} - \frac{1}{||\vec{x}(\lambda)||}.$$

If B is not positive-definite then $\lambda_1 \leq 0$. Consider the case in which $\vec{q}_1^T\vec{g} = 0$. The solution to the trust-region problem in this case is given by

$$\vec{x}(\tau) = -\left(\frac{\vec{q}_2^T\vec{g}}{\lambda_2 - \lambda_1}\vec{q}_2\right) - \tau\vec{v}_1$$

where v_1 is the eigenvector corresponding to λ_1, and the real number τ is fixed by the constraint on the norm of the vector $\vec{x}$.[30] Due to the orthogonal

[30] In other words, $\vec{v}_1$ solves the eigenvector equation $(B - \lambda_1 I)\vec{v}_1 = 0$. We are assuming, without loss of generality, that $||\vec{v}_1|| = 1$.

decomposition of B, we have the following equation for τ^*

$$||\vec{x}(\tau^*)||^2 = \left(\frac{\vec{q}_2^T \vec{g}}{\lambda_2 - \lambda_1}\right)^2 + (\tau^*)^2 = c^2.$$

The approximate solution to the trust region problem is therefore $\vec{x}^* = \vec{x}(\tau^*)$ and can be found by determining the zero of the function

$$\phi(\tau) = \frac{1}{c} - \frac{1}{||\vec{x}(\tau)||}.$$

using Newton's root-finding method.[31]

C Linear Algebra

Consider a real symmetric 2×2 matrix given by

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

The eigenvalues and corresponding eigenvectors of this matrix are given by

$$\lambda_\pm = \left(\frac{a+c}{2}\right) \pm \sqrt{\left(\frac{a-c}{2}\right)^2 + b^2} \quad \text{and} \quad \vec{v}_\pm = \begin{pmatrix} b \\ \lambda_\pm - a \end{pmatrix}.$$

If $\lambda_+\lambda_- \neq 0$, then the matrix A is invertible with inverse A^{-1} defined by

$$A^{-1} = \frac{1}{\lambda_+\lambda_-}\begin{pmatrix} c & -b \\ -b & a \end{pmatrix}.$$

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[31] If $\vec{g}$ is orthogonal to both $\vec{q}_1$ and $\vec{q}_2$, then $\vec{g} = 0$. For a non-trivial portfolio, our analysis is almost always complete.

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