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Periodical Rivers

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Abstract

In this paper, we prove the existence of periodical solutions for a vector field L_θ of the plan, presenting periodically saddle-points. The main interest is devoted to the slow-fast vector field of $\mathbb{R}^3$ studied in the paper [5].

Mathematics Subject Classification : 34D10,34D15,34D20
Keywords: Canards trajectories; singular perturbations; Morse form; Slow-fast vector eld; Non standard analysis; Poincares return maps

1 Introduction

In the literature many authors are interested in the slow-fast vector fields of the plan presenting critical points of Morse's type (or saddle point) [1], [2] and river type solutions [3],[4]. In the present study, we give an example of

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plan field L_θ presenting a priori periodically saddle points. We show in this paper the transition between plan systems and $\mathbb{R}^3$ systems vector fields with primary integrals. This allows us to show the existence of periodic solutions of L_θ from the existence of limit cycles of E_ε.

2 Problematic

Let us consider the following system

$$E_\varepsilon \begin{cases} x' &= -y \\ y' &= x \\ \varepsilon z' &= y^2 - z^2 \end{cases} \qquad (1)$$

as a local model for a system:

$$E_\varepsilon \begin{cases} x' &= Q(x,y) \\ y' &= P(x,y) \\ \varepsilon z' &= h_1(y,z)h_2(y,z) \end{cases}$$

where ε is a small positive parameter, P and Q of class S^1, satisfying the following hypothesis : the primitives of $Pdx - Qdy = 0$ are primary integrals. $h_1$and h_2 standard functions verifying close to the critical points the following conditions:

$Jac(h_1, h_2) \neq 0$, $\overrightarrow{grad}h_1 = \overrightarrow{grad}h_2$, $h'_{1z} = h'_{2z}$

The fold is defined by $h_1(y,z) = h_2(y,z) = 0$. The system (1) is a slow-fast vector field, with a slow-manifold's equation $y^2 - z^2 = 0$; it admits primary integral $F(x,y) = x^2 + y^2$.

Let be the standard cylinder's equation : $x^2 + y^2 = \rho^2$, then the critical points $A_1(\rho,0,0), A_2(-\rho,0,0)$ belonging to the fold (fold's equation : $z = 0, y = 0$) are Morse's points.

According the proposition 4.3 in [5], A_1, A_2 are Morse's point of false canard type (because $P = x$ and $h = 0$ see [5]) and the curve C of equation

$$\begin{cases} z &= |y| \\ x^2 + y^2 &= \rho^2 \end{cases} \text{ is positively stable.}$$

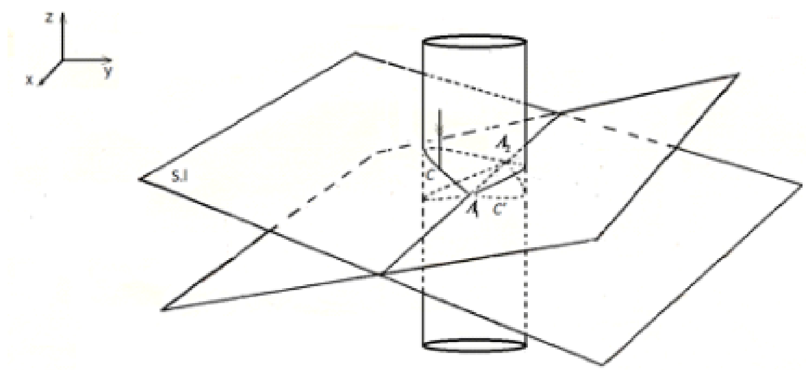

Figure 1: intersection of the cylinder and the slow manifold

Noting by C', the reflection curve of C across the plan $z = 0$, is given by the equation C' :

$$\begin{cases} z &= -|y| \\ x^2 + y^2 &= \rho^2 \end{cases}$$

Proposition 2.1. *for any ρ positive, E_ϵ admits two closed orbits, the first one in the halo of C, the second one in the halo of C' (see Figure 1).*

Proof. It's sufficient to prove for ρ standard since the property is internal. A_1, A_2 are of Morse's type (by the proposition 4.3 [5]) with false-canards.

In the neighborhood of A_1, V'_a which is an equivalent system to E_ϵ is expressed by :

$$(V'_a)\begin{cases} w' &= \sqrt{\rho^2 - \frac{(\epsilon v + w)^2}{4}} - wv \\ \epsilon v' &= \sqrt{\rho^2 - \frac{(\epsilon v + w)^2}{4}} + wv \end{cases} \quad (II)$$

the reduced system is:

$$(V'_a)\begin{cases} w' &= \sqrt{\rho^2 - \frac{w^2}{4}} - wv \\ 0 &= \sqrt{\rho^2 - \frac{w^2}{4}} + wv \end{cases} \quad (II_0)$$

The system (II_0) is an integrable one, the equation of the slow-curve Cl is: $k(w,v) = \sqrt{\rho^2 - \frac{w^2}{4}} + wv$, it does not admit critical points since $k(w,v) = \partial_v k(w,v) = 0$ does not have solutions.

Hence the fields (II) and (II_0) are equivalent in the halo of C.l and the trajectories of (II_0) are shadows of trajectories of (II).
The equation of C_1' in the plan (w,v) is: $v = \sqrt{\frac{\rho^2}{w^2} - \frac{1}{4}}$ with$-2\rho \prec w \prec 0$. □

2.1 Application of the cross-section

Let consider a Poincaré's section S with the cylinder $F(x,y) = \rho^2$ (see Figure 3), we choose S as the plan $x + y = 0$, denoting by Δ the intersection of S with the cylinder, then Δ corresponds in the plan (w,v) to a straight line of equation $v + w = -\rho\sqrt{2}$(see Figure 2).

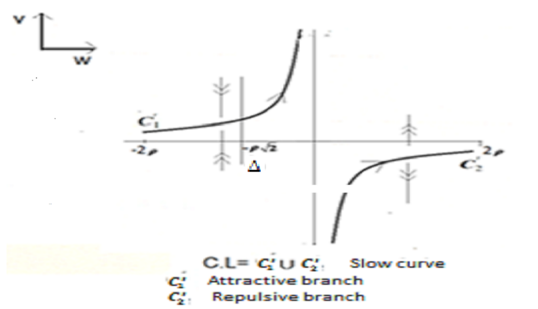

Figure 2:

Let (I) be an interval containing $z_0'(z_0' = -y_0)$, such z_0' is the unique point belonging to the slow manifold $S.l$ (see figure 3, I is limited by two brackets).

(I) is transverse to C and to the field E_ε, from a point of (I), a path of E_ε cross after a period (2π), because the cylinder is an invariant manifold and the functions $x(t)$ and $y(t)$ are 2π periodic.

denoting by $I^+ = \{(x,y,z) \in \mathrm{I} \ / \ y^2 - z^2 \prec 0\}$ and $I^- = \{(x,y,z) \in I/y^2 - z^2 \succeq 0\}$.

The return map T leaves invariant the first two components of a point $P \in I$.

Let be $P_0(x_0, y_0, z_0) \in I$ such that $z(0) = z_0$, $T(z_0) = z(2\pi)$ and successively, we define $T^k(z_0) = z(2k\pi)$, and thus the recursive sequence $z_{k+1} = T(z_k)$.

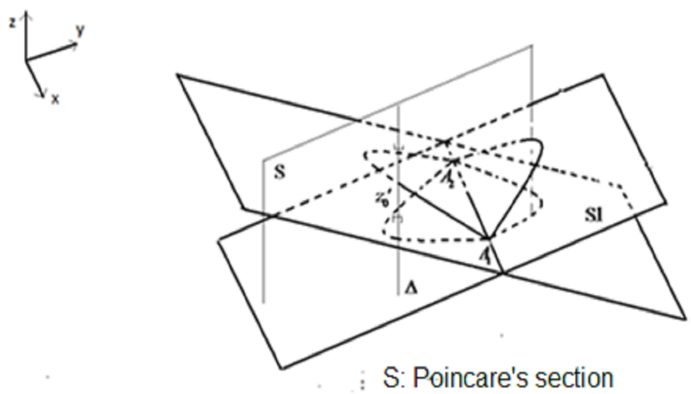

Figure 3: Poincare's section

First case: If $P_0 \epsilon$ I^+ with the condition $y_0^2 - z_0^2 \ll 0$ ($i.e$ $y_0^2 - z_0^2$ non infinitely small and negative), since the flows is fast, the trajectory Γ_0 crossing P_0 is such that z is decreasing.

The sequence z_k^0 is decreasing bounded below by $z_0^{'}$ (see Figure 2) since Γ_0 rushes to $C_1^{'}$ in infinitely small time in the vicinity of $w = -2\rho$ and remain alongside the latter before overlap I^+.

Second case: Let be $P_1(x_0, y_0, z_1) \epsilon$ I^- such that $z(0) = z_1$, the trajectory Γ_1 crossing P_1 is such that z is increasing. So the sequence z_k^1 is increasing bounded above by the previous sequence z_k^0 since we have: $\forall j \geq 0, \forall k \geq 0$ $z_j^1 \leq z_k^0$.

The relationship holds for $j = k = 0$, if there exists k, j , such that $k \leq j$, then Γ_0 intersects Γ_1 on the cylinder, which is absurd (because of the uniqueness of the trajectory of the Cauchy's problem).

The Poincare's return map is defined as S-continuous on I, since by two points infinitely close in I the paths have the same shadow.

The sequences z_j^1 and z_k^0 converge, and therefore T admits a fixed point l as the limit of z_k^0.

$C_a(x(t), y(t), l(t))$ is a closed orbit (called attractive cycle).

Substitution of the time parameter (ie $t = -t$), leads to a change of the orientation's field, and by the same process we deduce the existence of the cycle C_r for the field $-E_\varepsilon$ which is repellent for E_ε.

C_r is located in the halo of $C^{'}$.

Proposition 2.2. *(**Existence of limit cycles for** E) The unperturbed system: E* $\begin{cases} X' = -Y \\ Y' = X \\ Z' = Y^2 - Z^2 \end{cases}$ *admits two closed orbits.(see Figure 4)*

Proof. It has been shown that E_ε admits two closed orbits (attractive cycle) and (repelling cycle) in the proposition 1 for ε infinitely small.

denoting by $\mathrm{K}_\varepsilon = \{\varepsilon \succ 0/\ E_\varepsilon$admits 2 cycles $\}$ is an internal set not empty since it contains infinitely smalls numbers.

$\mathrm{K}_\varepsilon \supset \{\ \varepsilon \succ 0/\ \ \varepsilon\ i.small\}$ = Half-halo positive of 0, wich is strictly external.

By "Permanence lemma" K_ε contains at least one standard $\alpha \succ 0$.

Therefore E_α admits two closed orbits, but E_α is simply a scaling standard, so it does not distort the topological form of paths.

denote by $x = \alpha X, y = \alpha Y, z = \alpha Z$, we find E. □

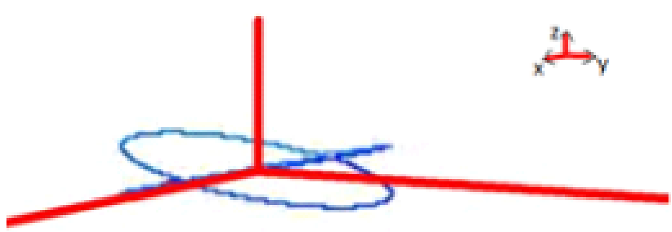

Figure 4: Figures of the two cycles in (x,y,z) for E

Consequences: Existence of periodical solutions for the plan vector field L_θ

Let be the plan vector field : $L_\theta \begin{cases} \theta' = 1 \\ Z' = \rho^2 \sin^2(\theta) - Z^2 \end{cases}$, it admits two periodic solutions.(see Figure 5)

Proof. L_θ is 2π periodic in θ, as seen in $\mathbb{R}^3$ on the plan $\rho = const$ is a transformation [modulo 2π] of the vector field (E) in cylindrical coordinates. □

The cycles C_a and C_r correspond to periodical solutions; the first one attractive and the other repulsive of system L_θ, we qualify them periodical rivers.

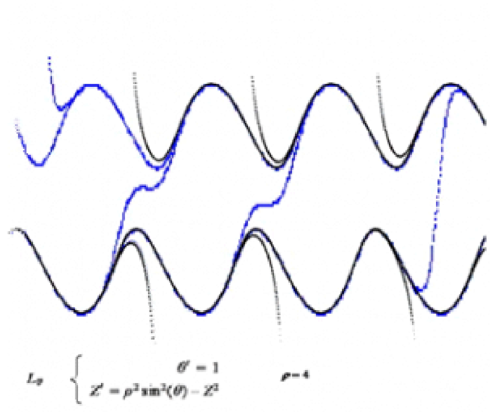

Figure 5: Periodic rivers for L_θ

3 Conclusion

In this paper, the transition between differential systems of $\mathbb{R}^3$ with primary integrals and the one parameter field of the plan, can be usefull, such as the previous example (result 2) In the treated example we note that the existence of periodic solutions of plan systems L_θ depends on the nature of the critical points P.N.P.S (see definition on [5],[6]).

The results can be generalized to perturbations of L_θ of type:

$$L_\theta \left\{ \begin{array}{c} \theta' = 1 \\ Z' = \rho^2 \sin^2(\theta) - Z^2 + \varepsilon h(\theta, \rho, z) \end{array} \right.$$

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