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Analytic Solutions for A New Kind of Auto-Coupled KdV Equation with Variable Coefficients

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Abstract

In this paper, we apply the extended variable-coefficient mapping method to discuss a new kind of auto-coupled KdV equation with variable coefficients. By solving nonlinear differential algebraic equations which are derived from nonlinear evolution equations, many Jacobi elliptic function solutions, hyperbolic function solutions and trigonometric periodic solutions for the auto-coupled KdV equation with variable coefficients are derived. By selecting the appropriate parameter values, some exact solutions of the other forms are also obtained.

Mathematics Subject Classification: 35M10, 35Q51, 35Q53
Keywords:the extended variable-coefficient mapping method; auto-coupled KdV equation with variable coefficients; elliptic function solutions

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1 Introduction

Nonlinear evolution equations play an important role in description of natural phenomena. As the soliton phenomena were first observed by Scott Russell in 1831[1], researchers began to study the explicit solutions of nonlinear evolution equations, many powerful methods to seek explicit exact solitary wave solutions of nonlinear evolution equations have been established and developed. Since constructive methods transform the problem of solving nonlinear evolution equations into the problem of solving the corresponding systems of algebraic equations, the problem of solving nonlinear evolution equations can be simplified, and it can also reveal many of the essential attribute of equations. Some methods have been widely applied and extended, such as Bäcklund transformation[2,3], sine-cosine method[4], homogeneous balance method[5], tanh-function expansion method[6,7], the extended tanh-function expansion method[8,9], the Jacobi elliptic function expansion method[10,11], the Riccati expansion method[12].

At present, people have paid more attention to the KdV equation with variable coefficients. Using different constructive methods, some researchers have obtained exact solutions of the KdV equation with variable coefficients. For example Liu Shi-Kuo et al. applied extended Jacobi elliptic function expansion method to construct the exact solutions of variable coefficients KdV equation, getting the solitary wave solutions and soliton solutions easier[13]. Based on the idea of the homogeneous balance method, Fan En gui obtained the Bäcklund transformation and similarity reductions of general variable coefficient KdV equation[14]. Woopyo Hong et al. found analytic solutions for general variable coefficient KdV equation and made use of both the truncate Painleve expansion and symbolic computation to obtain an auto-Bäcklund transformation and certain soliton-typed analytic solutions[15]. Li Desheng and Zhang Hongqing obtained exact soliton-like, rational formal and trigonometric function solutions of the general variable coefficient KdV and MKdV equations by using the extended tanh-function method[16].

Based on the reduced method, the variable separation method of extended mapping method has been widely applied to solve exact solutions of nonlinear evolution equations[17–19], Zhang Sheng and Xia Tiecheng proposed a variable coefficient extended mapping method and applied this method to the MKdV equaiton with variable coefficients and (2+1)dimensional Nizhnik-Novikov-Vesolov equations. Many new and more general exact solutions including Jacobi elliptic function solutions ,hyperbolic function solutions and trigonometric function solutions are obtained. However, at present, there are less of papers which applied the extended variable coefficient mapping method to the variable coefficient coupled equations. This present work is motivated to apply the method from[19] to a new kind of auto-coupled KdV equations, and some exact solutions are derived. Particularly, by selecting some different parameter values,we obtain the corresponding other forms solutions, including the Jacobi elliptic function solutions, hyperbolic function solutions and tribonometric function solutions.

The rest of this paper is arranged as follows: In Section 2, we shall describe the variable-coefficient extended mapping method for searching solutions of nonlinear evolution equations with variable coefficients and give the main steps of the method. In Section 3, we shall apply this method to a new kind of auto-coupled KdV equation with variable coefficients and obtain some solutions. In Section 4, some conclusions are given.

2 Description of method

For a given partial differential, say, in two variables x and t

$$F(u, u_t, u_x, u_{xx}, u_{xt}, \cdots) = 0, \tag{1}$$

where $u_x = \frac{\partial u}{\partial x}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{xt} = \frac{\partial^2 u}{\partial x \partial t}$, $\cdots$, the same hereafter.

We seek solutions of Eq.(1) in following form[19]:

$$u = a_0 + \sum_{i=1}^{n} a_i f^i(\xi) + \sum_{i=-1}^{-n} b_i f^i(\xi) + \sum_{i=2}^{n} c_i f^{i-2}(\xi) f'(\xi) + \sum_{i=-1}^{-n} d_i f^i(\xi) f'(\xi), \tag{2}$$

where $a_i = a_i(X)$, $b_i = b_i(X)$, $c_i = c_i(X)$, $d_i = d_i(X)$, $\xi = \xi(X)$ and $X = X(x,t)$ are all functions to be determined later. $f(\xi)$ satisfies the following auxiliary differential equation:

$$f'^2(\xi) = p f^4(\xi) + q f^2(\xi) + r, \tag{3}$$

and hence holds for $f(\xi)$ and $f'(\xi)$:

$$\begin{cases} f''(\xi) = 2pf^3(\xi) + qf(\xi), \\ f'''(\xi) = (6pf^2(\xi) + q)f'(\xi), \\ f^{(4)}(\xi) = 24p^2 f^5(\xi) + 20pqf^3(\xi) + (q^2 + 12pr)f(\xi), \\ f^{(5)}(\xi) = (120p^2 f^4(\xi) + 60pqf^2(\xi) + q^2 + 12pr)f'(\xi), \\ \vdots \end{cases} \tag{4}$$

where $' = \frac{\mathrm{d}}{\mathrm{d}\xi}$, p, q and r are all real parameters.

To determine $u = u(x, x_2, x_3, \cdots, t)$ explicitly, we take the following four steps:

Step 1. Determining the integer n by considering the homogeneous balancing between the highest order nonlinear term(s) and the highest order partial derivative of u in Eq.(1).

Step 2. With the aid of symbolic computation of the software Maple, substituting Eq.(2) along with (3) and (4) into Eq.(1), and collecting all the terms with the same order of $f'^l(\xi) f^j(\xi)(l = 0, 1; j = 0, \pm 1, \pm 2, \ldots)$ together, then the left- hand side of Eq.(1) is converted into a polynomial in $f'^l(\xi) f^j(\xi)(l = 0, 1; j = 0, \pm 1, \pm 2, \ldots)$. Setting each coefficient to zero yields a set of over-determined differential equations for a_0, a_i, b_i, c_i, $d_i(i = 1, 2, \ldots)$ and ξ.

Step 3. Solving the system of over-determined differential equations obtained in Step 2 for a_0, a_i, b_i, c_i, $d_i(i = 1, 2, \ldots)$ and ξ by use of Maple.

Step 4. Using the results obtained in above steps to derive a series of fundamental solutions of Eq.(1) which depend on the solution $f(\xi)$ of Eq.(3). For given different values of p, q, r, Eq.(3) has many kinds of Jacobi elliptic solutions, which are listed as follows:

$f(\xi)$	p	q	r
$\mathrm{sn}\xi, \mathrm{cd}\xi$	m^2	$-(1+m^2)$	1
$\mathrm{cn}\xi$	$-m^2$	$2m^2-1$	$1-m^2$
$\mathrm{dn}\xi$	-1	$2-m^2$	m^2-1

continued

$f(\xi)$	p	q	r
$\mathrm{ns}\xi, \mathrm{dc}\xi$	1	$-(1+m^2)$	m^2
$\mathrm{nc}\xi$	$1-m^2$	$2m^2-1$	$-m^2$
$\mathrm{nd}\xi$	m^2-1	$2-m^2$	-1
$\mathrm{sc}\xi$	$1-m^2$	$2-m^2$	1
$\mathrm{sd}\xi$	$m^2(m^2-1)$	$2m^2-1$	1
$\mathrm{cs}\xi$	1	$2-m^2$	$1-m^2$
$\mathrm{ds}\xi$	1	$2m^2-1$	$m^2(m^2-1)$
$m\mathrm{cn}\xi \pm \mathrm{dn}\xi$	$-\frac{1}{4}$	$\frac{1+m^2}{2}$	$-\frac{(1-m^2)^2}{4}$
$\mathrm{ns}\xi \pm \mathrm{cs}\xi$, $\frac{\mathrm{cn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi \pm \mathrm{dn}\xi}$	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$
$\mathrm{nc}\xi \pm \mathrm{sc}\xi$	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$
$\mathrm{ns}\xi \pm \mathrm{ds}\xi$	$\frac{1}{4}$	$\frac{m^2-2}{2}$	$\frac{m^4}{4}$
$\mathrm{sn}\xi \pm \mathrm{icn}\xi$, $\frac{\mathrm{dn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi \pm \mathrm{cn}\xi}$	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{m^2}{4}$
$m\mathrm{sn}\xi \pm \mathrm{idn}\xi$, $\frac{\mathrm{sn}\xi}{1\pm\mathrm{cn}\xi}$	$\frac{1}{4}$	$\frac{1-2m^2}{2}$	$\frac{1}{4}$
$\frac{\mathrm{sn}\xi}{1\pm\mathrm{dn}\xi}$	$\frac{m^2}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$
$\frac{\mathrm{dn}\xi}{1\pm m\mathrm{sn}\xi}$	$\frac{1-m^2}{4}$	$\frac{1+m^2}{2}$	$\frac{m^2-1}{4}$
$\frac{\mathrm{cn}\xi}{1\pm\mathrm{sn}\xi}$	$\frac{m^2-1}{4}$	$\frac{1+m^2}{2}$	$\frac{1-m^2}{4}$
$\frac{\mathrm{sn}\xi}{\mathrm{cn}\xi\pm\mathrm{dn}\xi}$	$\frac{(1-m^2)^2}{4}$	$\frac{1+m^2}{2}$	$\frac{1}{4}$
$\frac{\mathrm{cn}\xi}{\sqrt{1-m^2}\pm\mathrm{dn}\xi}$	$\frac{m^4}{4}$	$\frac{m^2-2}{2}$	$\frac{1}{4}$

Where $\mathrm{i}^2=-1$. Selecting proper values of p, q, r and corresponding $f(\xi)$, then substituting them along with a_i, b_i, c_i, d_i and ξ into Eq.(2), we can obtain exact solutions of Eq.(1), from which hyperbolic solutions and trigonometric function solutions can be obtained in the limit cases when $m \to 1$ and $m \to 0$.

Here $\mathrm{sn}\xi = \mathrm{sn}(\xi, m)$, $\mathrm{cn}\xi = \mathrm{cn}(\xi, m)$ and $\mathrm{dn}\xi = \mathrm{dn}(\xi, m)$ are Jacobi elliptic sine function, Jacobi elliptic cosine and Jacobi elliptic function of the third kind respectively, m denotes the modulus of Jacobi elliptic functions. Other functions are derived from these three kinds of functions[20]:

$$\begin{aligned}
\mathrm{ns}\xi &= \frac{1}{\mathrm{sn}\xi}, & \mathrm{nc}\xi &= \frac{1}{\mathrm{cn}\xi}, & \mathrm{nd}\xi &= \frac{1}{\mathrm{dn}\xi},\\
\mathrm{sc}\xi &= \frac{\mathrm{sn}\xi}{\mathrm{cn}\xi}, & \mathrm{sd}\xi &= \frac{\mathrm{sn}\xi}{\mathrm{dn}\xi}, & \mathrm{cd}\xi &= \frac{\mathrm{cn}\xi}{\mathrm{dn}\xi},\\
\mathrm{cs}\xi &= \frac{\mathrm{cn}\xi}{\mathrm{sn}\xi}, & \mathrm{ds}\xi &= \frac{\mathrm{dn}\xi}{\mathrm{sn}\xi}, & \mathrm{dc}\xi &= \frac{\mathrm{dn}\xi}{\mathrm{cn}\xi}.
\end{aligned}$$

The Jacobi elliptic functions degenerate into hyperbolic functions when $m \to 1$[21]:

$$\begin{aligned}
&\lim_{m\to 1}\mathrm{sn}(\xi,m)=\tanh\xi, \quad \lim_{m\to 1}\mathrm{cn}(\xi,m)=\mathrm{sech}\,\xi, \quad \lim_{m\to 1}\mathrm{dn}(\xi,m)=\mathrm{sech}\,\xi,\\
&\lim_{m\to 1}\mathrm{ns}(\xi,m)=\coth\xi, \quad \lim_{m\to 1}\mathrm{nc}(\xi,m)=\cosh\xi, \quad \lim_{m\to 1}\mathrm{nd}(\xi,m)=\cosh\xi,\\
&\lim_{m\to 1}\mathrm{sc}(\xi,m)=\sinh\xi, \quad \lim_{m\to 1}\mathrm{sd}(\xi,m)=\sinh\xi, \quad \lim_{m\to 1}\mathrm{cd}(\xi,m)=1,\\
&\lim_{m\to 1}\mathrm{cs}(\xi,m)=\mathrm{csch}\,\xi, \quad \lim_{m\to 1}\mathrm{ds}(\xi,m)=\mathrm{csch}\,\xi, \quad \lim_{m\to 1}\mathrm{dc}(\xi,m)=1.
\end{aligned}$$

The Jacobi elliptic functions degenerate into trigonometric functions when $m \to 0$:

$$\begin{aligned}
&\lim_{m\to 0}\mathrm{sn}(\xi,m)=\sin\xi, \quad \lim_{m\to 0}\mathrm{cn}(\xi,m)=\cos\xi, \quad \lim_{m\to 0}\mathrm{dn}(\xi,m)=1,\\
&\lim_{m\to 0}\mathrm{ns}(\xi,m)=\csc\xi, \quad \lim_{m\to 0}\mathrm{nc}(\xi,m)=\sec\xi, \quad \lim_{m\to 0}\mathrm{nd}(\xi,m)=1,\\
&\lim_{m\to 0}\mathrm{sc}(\xi,m)=\tan\xi, \quad \lim_{m\to 0}\mathrm{sd}(\xi,m)=\sin\xi, \quad \lim_{m\to 0}\mathrm{cd}(\xi,m)=\cos\xi,\\
&\lim_{m\to 0}\mathrm{cs}(\xi,m)=\cot\xi, \quad \lim_{m\to 0}\mathrm{ds}(\xi,m)=\csc\xi, \quad \lim_{m\to 0}\mathrm{dc}(\xi,m)=\sec\xi.
\end{aligned}$$

In this paper, we consider the following coupled KdV equation with variable coefficients

$$\begin{cases} F_1(t,u,v,u_t,v_t,u_x,v_x,u_{xx},v_{xx},u_{xxx},\ldots)=0,\\ F_2(t,u,v,u_t,v_t,u_x,v_x,u_{xx},v_{xx},v_{xxx},\ldots)=0. \end{cases} \tag{5}$$

In order to search for explicit solutions of Eq.(5), we suppose that the solutions of Eqs.(5) can be expressed as

$$\begin{cases} u = a_0 + \sum_{i=1}^{n_1} a_i f^i(\xi) + \sum_{i=-1}^{-n_1} b_i f^i(\xi) + \sum_{i=2}^{n_1} c_i f^{i-2}(\xi) f'(\xi) + \sum_{i=-1}^{-n_1} d_i f^i(\xi) f'(\xi),\\ v = A_0 + \sum_{i=1}^{n_2} A_i f^i(\xi) + \sum_{i=-1}^{-n_2} B_i f^i(\xi) + \sum_{i=2}^{n_2} C_i f^{i-2}(\xi) f'(\xi)\\ \qquad + \sum_{i=-1}^{-n_2} D_i f^i(\xi) f'(\xi). \end{cases} \tag{6}$$

where $a_i = a_i(X)$, $b_i = b_i(X)$, $c_i = c_i(X)$, $d_i = d_i(X)$, $A_i = A_i(X)$, $B_i = B_i(X)$, $C_i = C_i(X)$, $D_i = D_i(X)$, $\xi = \xi(X)$, $X = X(x,t)$ are all functions to be determined later. By balancing the highest order nonlinear term(s) and the highest order partial derivative in Eqs.(6), we can get the values of n_1, n_2.

3 Application

General variable coefficient KdV equation is given as

$$u_t + \alpha(t)uu_x + \beta(t)u_{xxx} = 0, \tag{7}$$

where $\alpha(t)$ and $\beta(t)$ are arbitrary functions of t,the equation first was introduced by Grimshaw.

In this paper,a new auto-coupled KdV equations with variable coefficients is as follows:

$$\begin{cases} u_t + \alpha(t)uv_x + \beta(t)u_{xxx} = \gamma(t)v_x, \\ v_t + \alpha(t)vu_x + \beta(t)v_{xxx} = \gamma(t)u_x. \end{cases} \tag{8}$$

where $\beta(t) = \delta\alpha(t)$, δ is a constant.

By balancing u_{xxx} and uv_x, v_{xxx} and vu_x, we get $n_1 = 2$, $n_2 = 2$. In order to search for explicit solutions, we assume that Eqs.(8) has the following solutions of the form:

$$\begin{cases} u(\xi) = a_0 + a_1 f(\xi) + a_2 f^2(\xi) + b_1 \frac{1}{f(\xi)} + b_2 \frac{1}{f^2(\xi)} + c_2 f'(\xi) + d_1 \frac{f'(\xi)}{f(\xi)} + d_2 \frac{f'(\xi)}{f^2(\xi)}, \\ v(\xi) = A_0 + A_1 f(\xi) + A_2 f^2(\xi) + B_1 \frac{1}{f(\xi)} + B_2 \frac{1}{f^2(\xi)} + C_2 f'(\xi) + D_1 \frac{f'(\xi)}{f(\xi)} \\ \qquad +D_2 \frac{f'(\xi)}{f^2(\xi)}. \end{cases} \tag{9}$$

where $a_0 = a_0(t)$, $a_1 = a_1(t)$, $a_2 = a_2(t)$, $b_1 = b_1(t)$, $b_2 = b_2(t)$, $c_2 = c_2(t)$, $d_1 = d_1(t)$, $d_2 = d_2(t)$, $A_0 = A_0(t)$, $A_1 = A_1(t)$, $A_2 = A_2(t)$, $B_1 = B_1(t)$, $B_2 = B_2(t)$, $C_2 = C_2(t)$, $D_1 = D_1(t)$, $D_2 = D_2(t)$, $\xi = \kappa x + \eta$, $\kappa = \kappa(t)$, $\eta = \eta(t)$.

Substituting Eqs.(9),(3) and (4) into Eqs.(8), collecting the coefficients with the same power $x^\mu f'^l(\xi) f^j(\xi)(\mu = 0, 1; l = 0, 1; j = 0, \pm1, \pm2, \cdots)$ and setting each of the obtained coefficients to be zero, we get a set of over-determined nonlinear differential algebraic equations, which is omitted here. Solving the set of over-determined nonlinear differential algebraic equations by using symbolic computation of software Maple, we have results in the following cases:

Case 1

$$\begin{aligned} &a_0 = c,\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = -12\delta rk^2,\ c_2 = 0,\ d_1 = 0,\ d_2 = 0, \\ &A_0 = c,\ A_1 = 0,\ A_2 = 0,\ B_1 = 0,\ B_2 = -12\delta rk^2,\ C_2 = 0,\ D_1 = 0,\ D_2 = 0, \\ &\kappa = k,\ \eta = (\gamma(\tau)k - \alpha(\tau)kc - 4\beta(\tau)qk^3)\mathrm{d}t. \end{aligned} \tag{10}$$

where c and k are arbitrary constants.

Therefore, the first solution of Eqs.(8) are given as follows:

$$u = v = c - 12\delta r k^2 \frac{1}{f^2(\xi)}, \tag{11}$$

We can obtain many new and more types of exact solutions of Eqs.(8). For example, selecting $p = -m^2$, $q = 2m^2 - 1$, $r = 1 - m^2$ and $f(\xi) = \mathrm{cn}\xi$ yields Jacobi elliptic function solutions as

$$u = v = c - 12\delta(1 - m^2)k^2 \mathrm{nc}^2\xi, \tag{12}$$

In the limit cases when $m \to 0$, from Eq.(12) we can obtain trigonometric function solutions as

$$u = v = c - 12\delta k^2 \sec^2 \xi, \tag{13}$$

Selecting $p = \frac{1}{4}$, $q = \frac{1-2m^2}{2}$, $r = \frac{1}{4}$ and $f(\xi) = \frac{\mathrm{cn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi \pm \mathrm{dn}\xi}$ yields

$$u = v = c - 3\delta k^2 \frac{(\sqrt{1 - m^2}\mathrm{sn}\xi \pm \mathrm{dn}\xi)^2}{\mathrm{cn}^2\xi}, \tag{14}$$

when $m \to 0$, from Eq.(14)we can obtain trigonometric function solutions as

$$u = v = c - 3\delta k^2 \frac{(\sin \xi \pm 1)^2}{\cos^2 \xi}, \tag{15}$$

Selecting $p = \frac{1}{4}$, $q = \frac{1-2m^2}{2}$, $r = \frac{1}{4}$ and $f(\xi) = \frac{\mathrm{sn}\xi}{1 \pm \mathrm{cn}\xi}$ yields

$$u = v = c - 3\delta k^2 \frac{(1 \pm \mathrm{cn}\xi)^2}{\mathrm{sn}^2\xi}, \tag{16}$$

When $m \to 1$ from Eq.(16) we can obtain hyperbolic function solutions

$$u = v = c - 3\delta k^2 \frac{(1 \pm \mathrm{sech}\, \xi)^2}{\tanh^2 \xi}, \tag{17}$$

When $m \to 0$ from Eq.(16) we can obtain trigonometric function solutions

$$u = v = c - 3\delta k^2 \frac{(1 \pm \cos \xi)^2}{\sin^2 \xi}, \tag{18}$$

where $\xi = kx - (\gamma(\tau)k - \alpha(\tau)kc - 4\beta(\tau)qk^3)\mathrm{d}t$.

Case 2

$$a_0 = c,\ a_1 = 0,\ a_2 = -12\delta pk^2,\ b_1 = 0,\ b_2 = 0,\ c_2 = 0,\ d_1 = 0,\ d_2 = 0,$$
$$A_0 = c,\ A_1 = 0,\ A_2 = -12\delta pk^2,\ B_1 = 0,\ B_2 = 0,\ C_2 = 0,\ D_1 = 0,\ D_2 = 0,$$
$$\kappa = k,\ \eta = (\gamma(\tau)k - \alpha(\tau)kc - 4\beta(\tau)qk^3)\mathrm{d}t. \tag{19}$$

where c and k are arbitrary constants.

Therefore, the second solution of Eqs.(8) are given as follows:

$$u = v = c - 12\delta pk^2 f^2(\xi), \tag{20}$$

Selecting $p = 1$, $q = -(1+m^2)$, $r = m^2$ and $f = \mathrm{dc}\xi$ yields

$$u = v = c - 12\delta k^2 \mathrm{dc}^2\xi, \tag{21}$$

When $m \to 0$, from Eq.(21) we can obtain trigonometric function solutions as

$$u = v = c - 12\delta k^2 \sec^2\xi, \tag{22}$$

Selecting $p = \frac{1}{4}$, $q = \frac{1-2m^2}{2}$, $r = \frac{1}{4}$ and $f = \frac{\mathrm{sn}\xi}{1+\mathrm{cn}\xi}$ yields

$$u = v = c - 3\delta k^2 \frac{\mathrm{sn}^2\xi}{(1+\mathrm{cn}\xi)^2}, \tag{23}$$

When $m \to 1$, from Eq.(23) we can obtain hyperbolic function solutions as

$$u = v = c - 3\delta k^2 \frac{\tanh^2\xi}{(1+\mathrm{sech}\,\xi)^2}, \tag{24}$$

When $m \to 0$, from Eq.(23) we can obtain trigonometric function solutions as

$$u = v = c - 3\delta k^2 \frac{\sin^2\xi}{(1+\cos\xi)^2}, \tag{25}$$

where $\xi = kx - (\gamma(\tau)k - \alpha(\tau)kc - 4\beta(\tau)qk^3)\mathrm{d}t$.

Case 3

$$a_0 = c,\ a_1 = 0,\ a_2 = -12\delta pk^2,\ b_1 = 0,\ b_2 = -12\delta rk^2,\ c_2 = 0,\ d_1 = 0,$$
$$d_2 = 0, A_0 = c,\ A_1 = 0,\ A_2 = -12\delta pk^2,\ B_1 = 0,\ B_2 = -12\delta rk^2,$$
$$C_2 = 0,\ D_1 = 0,\ D_2 = 0, \kappa = k,\ \eta = (\gamma(\tau)k - \alpha(\tau)kc - 4\beta(\tau)qk^3)\mathrm{d}t. \tag{26}$$

where c and k are arbitrary constants.

Therefore, the third solutions of Eqs.(8) are given as follows:

$$u = v = c - 12\delta p k^2 f^2(\xi) - 12\delta r k^2 \frac{1}{f^2(\xi)}, \tag{27}$$

Selecting $p = \frac{1}{4}$, $q = \frac{1-2m^2}{2}$, $r = \frac{1}{4}$ and $f(\xi) = \frac{\mathrm{cn}\xi}{\sqrt{1-m^2}\mathrm{sn}\xi \pm \mathrm{dn}\xi}$ yields

$$u = v = c - 3\delta k^2 \frac{\mathrm{cn}^2\xi}{(\sqrt{1-m^2}\mathrm{sn}\xi \pm \mathrm{dn}\xi)^2} - 3\delta k^2 \frac{(\sqrt{1-m^2}\mathrm{sn}\xi \pm \mathrm{dn}\xi)^2}{\mathrm{cn}^2\xi}, \tag{28}$$

When $m \to 0$, from Eq.(28) we can obtain trigonometric function solutions as

$$u = v = c - 3\delta k^2 \frac{\cos^2\xi}{(\sin\xi \pm 1)^2} - 3\delta k^2 \frac{(\sin\xi \pm 1)^2}{\cos^2\xi}, \tag{29}$$

Selecting $p = \frac{1}{4}$, $q = \frac{1-2m^2}{2}$, $r = \frac{1}{4}$ and $f(\xi) = \frac{\mathrm{sn}\xi}{1 \pm \mathrm{cn}\xi}$ yields

$$u = v = c - 3\delta p k^2 \frac{\mathrm{sn}^2\xi}{(1 \pm \mathrm{cn}\xi)^2} - 3\delta k^2 \frac{(1 \pm \mathrm{cn}\xi)^2}{\mathrm{sn}^2\xi}, \tag{30}$$

When $m \to 1$, from Eq.(30) we can obtain hyperbolic function solutions as

$$u = v = c - 3\delta p k^2 \frac{\tanh^2\xi}{(1 \pm \operatorname{sech}\xi)^2} - 3\delta k^2 \frac{(1 \pm \operatorname{sech}\xi)^2}{\tanh^2\xi}, \tag{31}$$

When $m \to 0$, from Eq.(30) we can obtain trigonometric function solutions as

$$u = v = c - 3\delta p k^2 \frac{\sin^2\xi}{(1 \pm \cos\xi)^2} - 3\delta k^2 \frac{(1 \pm \cos\xi)^2}{\sin^2\xi}, \tag{32}$$

Selecting $p = \frac{1}{4}$, $q = \frac{1-2m^2}{2}$, $r = \frac{1}{4}$, $f(\xi) = \mathrm{ns}\xi \pm \mathrm{cs}\xi$ yields

$$u = v = c - 3\delta k^2 (\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2 - 3\delta k^2 \frac{1}{(\mathrm{ns}\xi \pm \mathrm{cs}\xi)^2}, \tag{33}$$

When $m \to 1$, from Eq.(33) we can obtain hyperbolic function solutions as

$$u = v = c - 3\delta k^2 (\coth\xi \pm \operatorname{csch}\xi)^2 - 3\delta k^2 \frac{1}{(\coth\xi \pm \operatorname{csch}\xi)^2}, \tag{34}$$

When $m \to 0$, from Eq.(33) we can obtain trigonometric function solutions as

$$u = v = c - 3\delta k^2 (\csc\xi \pm \cot\xi)^2 - 3\delta k^2 \frac{1}{(\csc\xi \pm \cot\xi)^2}. \tag{35}$$

where $\xi = kx - (\gamma(\tau)k - \alpha(\tau)kc - 4\beta(\tau)qk^3)\mathrm{d}t$.

Case 4

$$
\begin{aligned}
&a_0 = c,\ a_1 = 0,\ a_2 = -6\delta p k^2,\ b_1 = 0,\ b_2 = 0,\ c_2 = \pm 6\delta k^2\sqrt{p},\ d_1 = 0,\\
&d_2 = 0, A_0 = c,\ A_1 = 0,\ A_2 = -6\delta p k^2,\ B_1 = 0,\ B_2 = 0,\ C_2 = \pm 6\delta k^2\sqrt{p},\\
&D_1 = 0,\ D_2 = 0, \kappa = k,\ \eta = (\gamma(\tau)k - \alpha(\tau)kc - \beta(\tau)qk^3)\mathrm{d}t.
\end{aligned} \tag{36}
$$

where c and k are arbitrary constants.

Therefore, the fourth solutions of Eqs.(8) are given as follows:

$$u = v = c - 12\delta p k^2 f^2(\xi) \pm 6\delta k^2\sqrt{p} f'(\xi), \tag{37}$$

Selecting $p = 1$, $q = 2 - m^2$, $r = 1 - m^2$ and $f = \mathrm{cs}\xi$ yields

$$
\begin{aligned}
u = v &= c - 12\delta k^2 \mathrm{cs}^2\xi \pm 6\delta k^2 \mathrm{cs}'\xi\\
&= c - 12\delta k^2 \mathrm{cs}^2\xi \mp 6\delta k^2 \mathrm{ns}\xi\mathrm{ds}\xi,
\end{aligned} \tag{38}
$$

When $m \to 1$, from Eq.(38) we can obtain hyperbolic function solutions as

$$u = v = c - 12\delta k^2 \operatorname{csch}^2\xi \mp 6\delta k^2 \coth\xi \operatorname{csch}\xi, \tag{39}$$

When $m \to 0$, from Eq.(38) we can obtain trigonometric function solutions as

$$u = v = c - 12\delta k^2 \cot^2\xi \mp 6\delta k^2 \csc^2\xi, \tag{40}$$

where $\xi = kx - (\gamma(\tau)k - \alpha(\tau)kc - \beta(\tau)qk^3)\mathrm{d}t$.

Cases 5

$$
\begin{aligned}
&a_0 = c,\ a_1 = 0,\ a_2 = 0,\ b_1 = 0,\ b_2 = -6\delta r k^2,\ c_2 = 0,\ d_1 = 0,\\
&d_2 = \pm 6\delta k^2\sqrt{r}, A_0 = c,\ A_1 = 0,\ A_2 = 0,\ B_1 = 0,\ B_2 = -6\delta r k^2,\\
&C_2 = 0,\ D_1 = 0,\ D_2 = \pm 6\delta k^2\sqrt{r},\\
&\kappa = k,\ \eta = (\gamma(\tau)k - \alpha(\tau)kc - \beta(\tau)qk^3)\mathrm{d}t.
\end{aligned} \tag{41}
$$

where c and k are arbitrary constants.

Therefore, the fifth solutions of Eqs.(8) are given as follows:

$$u = v = c - 6\delta r k^2 \frac{1}{f^2(\xi)} \pm 6\delta k^2\sqrt{r}\frac{f'(\xi)}{f^2(\xi)}, \tag{42}$$

Selecting $p=m^2$, $q=-(1+m^2)$, $r=1$, $f=\mathrm{sn}\xi$ yields

$$\begin{aligned} u=v&=c-6\delta k^2\frac{1}{\mathrm{sn}^2\xi}\pm 6\delta k^2\frac{\mathrm{sn}'\xi}{\mathrm{sn}^2\xi}\\ &=c-6\delta k^2\frac{1}{\mathrm{sn}^2\xi}\pm 6\delta k^2\frac{\mathrm{cn}\xi\mathrm{dn}\xi}{\mathrm{sn}^2\xi}, \end{aligned}\tag{43}$$

When $m\to 1$, from Eq.(43) we can obtain hyperbolic function solutions as

$$\begin{aligned} u=v&=c-6\delta k^2\frac{1}{\tanh^2\xi}\pm 6\delta k^2\frac{\operatorname{sech}^2\xi}{\tanh^2\xi}\\ &=c-6\delta k^2\frac{1}{\tanh^2\xi}\pm 6\delta k^2\frac{1}{\sinh^2\xi}, \end{aligned}\tag{44}$$

When $m\to 0$, from Eq.(43) we can obtain trigonometric function solutions as

$$u=v=c-6\delta k^2\frac{1}{\sin^2\xi}\pm 6\delta k^2\frac{\cos\xi}{\sin^2\xi},\tag{45}$$

Selecting $p=\frac{(1-m^2)^2}{4}$, $q=\frac{1+m^2}{2}$, $r=\frac{1}{4}$ and $f=\frac{\mathrm{sn}\xi}{\mathrm{cn}\xi\pm\mathrm{dn}\xi}$ yields

$$\begin{aligned} u=v&=c-\frac{3}{2}\delta k^2\frac{(\mathrm{cn}\xi\pm\mathrm{dn}\xi)^2}{\mathrm{sn}^2\xi}\pm 3\delta k^2\frac{\mathrm{sn}'\xi(\mathrm{cn}\xi\pm\mathrm{dn}\xi)-\mathrm{sn}\xi(\mathrm{cn}'\xi\pm\mathrm{dn}'\xi)}{\mathrm{sn}^2\xi}\\ &=c-\frac{3}{2}\delta k^2\frac{(\mathrm{cn}\xi\pm\mathrm{dn}\xi)^2}{\mathrm{sn}^2\xi}\pm 3\delta k^2\frac{\mathrm{cn}^2\xi\mathrm{dn}\xi+\mathrm{sn}^2\xi\mathrm{dn}\xi+\mathrm{cn}\xi}{\mathrm{sn}^2\xi}, \end{aligned}\tag{46}$$

When $m\to 1$, from Eq.(46) we can obtain hyperbolic function solutions as

$$u=v=c-\frac{3}{2}\delta k^2\frac{(\operatorname{sech}\xi\pm\operatorname{sech}\xi)^2}{\tanh^2\xi}\pm 3\delta k^2\frac{\operatorname{sech}^3\xi+\tanh^2\xi\operatorname{sech}\xi+\operatorname{sech}\xi}{\tanh^2\xi},\tag{47}$$

so

$$u=v=c\mp 3\delta k^2\frac{\operatorname{sech}^3\xi+\tanh^2\xi\operatorname{sech}\xi+\operatorname{sech}\xi}{\tanh^2\xi},\tag{48}$$

or

$$\begin{aligned} u=v&=c-6\delta k^2\frac{\operatorname{sech}^2\xi}{\tanh^2\xi}\pm 3\delta k^2\frac{\operatorname{sech}^3\xi+\tanh^2\xi\operatorname{sech}\xi+\operatorname{sech}\xi}{\tanh^2\xi}\\ &=c-6\delta k^2\frac{1}{\sinh^2\xi}\pm 3\delta k^2\frac{\operatorname{sech}^3\xi+\tanh^2\xi\operatorname{sech}\xi+\operatorname{sech}\xi}{\tanh^2\xi}, \end{aligned}\tag{49}$$

When $m\to 0$, from Eq.(46) we can obtain trigonometric function solutions as

$$\begin{aligned} u=v&=c-\frac{3}{2}\delta k^2\frac{(\cos\xi\pm 1)^2}{\sin^2\xi}\pm 3\delta k^2\frac{\cos^2\xi+\sin^2\xi+\cos\xi}{\sin^2\xi}\\ &=c-\frac{3}{2}\delta k^2\frac{(\cos\xi\pm 1)^2}{\sin^2\xi}\pm 3\delta k^2\frac{1+\cos\xi}{\sin^2\xi}. \end{aligned}\tag{50}$$

where $\xi = kx - (\gamma(\tau)k - \alpha(\tau)kc - \beta(\tau)qk^3)\mathrm{d}t$.

4 Conclusion

In this paper, a new kind of auto-coupled KdV equation with variable coefficients is proposed. We apply the extended variable coefficient mapping method to the coupled model, and obtain many exact solutions which include Jacobi elliptic function solutions, hyperbolic function solutions and trigonometric function solutions of auto-coupled KdV equation with variable coefficients.

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References

[1] J.S. Russel, Report on waves, Rep. 14th Meet, *Brit. Assoc. Adv. Sci.*, (1844), 311–390.

[2] M. Wang, Y. Wang and Y. Zhou, An auto-Bäcklund transformation and exact solutions to a generalized KdV equation with variable coefficients and their applications, *Physics Letters A*, **303**(1), (2002), 45–51.

[3] B. Li, Y. Chen and H. Zhang, Auto-Bäcklund transformation and exact solutions for compound KdV-type and compound KdV-Burgers-type equations with nonlinear terms of any order, *Physics Letters A*, **305**(6), (2002), 377–382.

[4] A. Bekir, New solitons and periodic wave solutions for some nonlinear physical models by using the sine–cosine method, *Physica Scripta*, **77**(4), (2008), 501.

[5] M. Wang, Exact solutions for a compound KdV-Burgers equation, *Physics Letters A*, **213**(5-6), (1996), 279–287.

[6] W. Malfliet and W. Hereman, The tanh method: I. Exact solutions of nonlinear evolution and wave equations, *Physica Scripta*, **54**, (1996), 563.

[7] W. Malfliet and W. Hereman, The tanh method: II. Perturbation technique for conservative systems, *Physica Scripta*, **54**, (1996), 569.

[8] Z. Lü, H. Zhang, On a further extended tanh method, *Physics Letters A*, **307**(5-6), (2003), 269–273.

[9] A.M. Wazwaz, The extended tanh method for abundant solitary wave solutions of nonlinear wave equations, *Applied Mathematics and Computation*, **187**(2), (2007), 1131–1142.

[10] S.Liu, Z. Fu, S. Liu and Q. Zhao, Expansion Method about the Jacobi Elliptic Function and Its Applications to Nonlinear Wave Equations, *Acta Physica Sinica*, **50**(11), (2001), 2068–2073.

[11] E. Fan and J. Zhang, Applications of the Jacobi elliptic function method to special-type nonlinear equations, *Physics Letters A*, **305**(6), (2002), 383–392.

[12] Gegenna Taogetusang, Constructing New Exact Solutions to Several Nonlinear Evolution Equations by Riccati Equation, *Journal of Inner Mongolia Normal University (Natural Science)*, **40**(5), (2011), 467–473.

[13] S. Liu, Z. Fu, S. Liu and Q. Zhao, Jacobi elliptic function expansion solution to the variable coefficient nonlinear equations, *Acta Physica Sinica*, **51**(9), (2002), 1923–1926.

[14] E. Fan, Auto-Bäcklund transformation and similarity reductions for general variable coefficient KdV equations, *Physics Letters A*, **294**(1), (2002), 26–30.

[15] W. Hong and Y.D. Jung, Auto-Bäcklund transformation and analytic solutions for general variable-coefficient KdV equation, *Physics Letters A*, **257**(3-4), (1999), 149–152.

[16] D. Li and H. Zhang, Improved tanh-function method and the new exact solutions for the general variable coefficient KdV equation and MKdV equation, *Acta Physica Sinica*, **52**(7), (2003), 1569–1573.

[17] C.L. Zheng, J.P. Fang and L.Q. Chen, New variable separation excitations of (2+ 1)-dimensional dispersive long-water wave system obtained by an extended mapping approach, *Chaos, Solitons* & *Fractals*, **23**(5), (2005), 1741–1748.

[18] M.A. Abdou, Exact periodic wave solutions to some nonlinear evolution equations, *International Journal of Nonlinear Science*, **6**(2), (2008), 145–153.

[19] S. Zhang and T. Xia, Variable-coefficient extended mapping method for nonlinear evolution equations, *Physics Letters A*, **372**(11), (2008), 1741–1749.

[20] Ahmad T. Ali, New generalized Jacobi elliptic function rational expansion method, *Journal of computational and applied mathematics*, **235**(14), (2011), 4117–4127.

[21] W. Hong and Y.D. Jung, A Generalized F-expansion Method and its Application to (2+ 1)-dimensional Breaking Soliton Equations, *Internatioal Journal of Nonlinear Science*, **5**(1), (2008), 25–32.