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Optimal Steady Motions for Oriented Vehicles

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Abstract

Motivated by the motion planning problem for oriented vehicles travelling in a 3-Dimensional space; Euclidean space $\mathbb{E}^3$, the sphere $\mathbb{S}^3$ and Hyperboloid $\mathbb{H}^3$. For such problems the orientation of the vehicle is naturally represented by an orthonormal frame over a point in the underlying manifold. The orthonormal frame bundles of the space forms $\mathbb{R}^3$, $\mathbb{S}^3$ and $\mathbb{H}^3$ correspond with their isometry groups and are the Euclidean group of motion $SE(3)$, the rotation group $SO(4)$ and the Lorentzian group $SO(1,3)$ respectively. Orthonormal frame bundles of space forms coincide with their isometry groups and therefore the focus shifts to left-invariant control systems defined on Lie groups. In this paper a method for integrating these systems is given where the controls are time-independent. For constant twist motions or helical motions, the corresponding curves $g(t) \in SE(3)$ are given in closed form by using the well known Rodrigues' formula. However, this formula is only applicable to the Euclidean case. This paper gives a method for computing the non-Euclidean screw/helical motions in closed form. This involves decoupling the system into two lower dimensional systems using the double cover properties of Lie groups, then the lower dimensional systems are solved explicitly in closed form.

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1 Introduction

This paper is motivated by the problem of motion planning for oriented vehicles travelling in a 3-Dimensional (3-D) space, such as the airplane landing problem [1], multi-vehicle formation control of Unmanned Air Vehicles (UAV) [2] and the underactuated Autonomous Underwater Vehicle (AUV) [3]. In each of these cases the oriented vehicles trace paths in Euclidean space $\mathbb{R}^3$. In this paper we generalize the Euclidean frame, simultaneously studying oriented vehicles travelling in either of the non-Euclidean 3-D space forms; spherical space $\mathbb{S}^3$ and Minkowski space $\mathbb{H}^3$, as in [4]. For such problems the orientation of the vehicle is naturally represented by an orthonormal frame over a point in the underlying manifold, that is, the configuration space of the vehicle can be taken as the orthonormal frame bundle of the manifold, and the motions of the vehicle are described by curves in this bundle. The orthonormal frame bundles of the space forms $\mathbb{S}^3$ and $\mathbb{H}^3$ correspond with their isometry groups and are the rotation group $SO(4)$ and the Lorentzian group $SO(1,3)$ respectively [5]. This generalization to framed curves in non-Euclidean space has applications in relativistic physics [6] and quantum control [7].

For the specific problem of oriented vehicles travelling at unit speed in a 3-D space, see [4], the authors illustrate that for a specific set of optimal controls (at singularities) the corresponding motions are helical motions. It is the aim of this paper to provide closed form expressions for these optimal steady motions. This problem amounts to integrating the left-invariant differential system:

$$\dot{g}(t) = g(t)A \tag{1}$$

where A is a constant element of the Lie algebra $\mathfrak{g}$. The solution $g(t) : \mathbb{R} \rightarrow \mathbb{R}^{4\times 4} \in G$ of the differential equation (1) is given by $g(t) = g_0 \exp(At)$ where $g(0) = g_0$. In the Euclidean case i.e. when $A \in \mathfrak{se}(3)$ the system (1) can be integrated using the well know **Rodrigues' Formula** [8], to obtain the motions $g(t) \in SE(3)$. However, this formula does not extend to the non-Euclidean cases. In this paper we describe a method for deriving closed form solutions of (1) when $A \in \mathfrak{so}(4)$ and $A \in \mathfrak{so}(1,3)$, and the corresponding motions are curves $g(t) \in SO(4)$ and $g(t) \in SO(1,3)$ respectively.

This method uses a natural isomorphism (described in [9]) to decouple the kinematic systems defined on them, into two trivial lower dimensional systems. These lower dimensional systems can then be solved in closed form using a similar procedure to that described in [5]. Finally, the solutions to the decoupled systems are projected back onto the original system to obtain the closed form solutions on $SO(4)$ and $SO(1,3)$. The method avoids using the computationally expensive methods of diagonalization [10] and the SN decomposition [11].

Finally, these explicit expressions are used to derive parametric expressions for helical motions for oriented vehicles travelling in a 3-D non-Euclidean space form.

2 Helical motions for oriented vehicles

In this section we state the kinematic equations of motion for oriented vehicles and relate these to the Serret-Frenet frame. Following this, controls that induce steady motions are identified and the problem of explicitly computing these motions is described.

2.1 Kinematics of oriented vehicles

The differential equations describing an oriented vehicle travelling at unit speed in a three dimensional space are described by the left-invariant differential system, see [4]:

$$\frac{dg}{dt}(t) = g(t) \begin{pmatrix} 0 & -\varepsilon & 0 & 0 \\ 1 & 0 & -u_3 & u_2 \\ 0 & u_3 & 0 & -u_1 \\ 0 & -u_2 & u_1 & 0 \end{pmatrix} \tag{2}$$

such that $g(t) \in G$ where G depends on ε and is $SE(3)$ for $\varepsilon = 0$, $SO(4)$ for $\varepsilon = 1$ and $SO(1,3)$ for $\varepsilon = -1$ and the controls u_i relate to the components of angular velocity of the vehicle. It follows that (2) describes the kinematic equations of the vehicle such that the vehicle traces out a trajectory $\gamma(t) \in M$, where M is the ambient space, and are related to $g(t) \in G$ via the projection $\gamma(t) = g(t)\vec{e}_1$ where $\vec{e}_1$ is a basis element in a standard orthonormal frame $\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4 \in \mathbb{R}^4$. The projected curves $\gamma(t) \in M$ describe the path that the vehicle traces in 3-D space, where $M = \mathbb{R}^3$ when $g(t) \in SE(3)$, $M = \mathbb{S}^3$ when $g(t) \in SO(4)$ and $M = \mathbb{H}^3$ when $g(t) \in SO(1,3)$. It follows from (2) that this particular vehicle is restricted to travel at unit speed $\left\|\frac{d\gamma(t)}{dt}\right\| = 1$ in a forward direction such that $\frac{d\gamma(t)}{dt}$ coincides with the first leg of the moving frame. To understand the path that the vehicle will trace in the ambient space the general frame (2) can be explicitly related to the Serret-Frenet Frame lifted to a left-invariant differential system [5]. The Serret-Frenet frame lifted to a left-invariant system on G is:

$$\frac{d\bar{g}(t)}{dt} = \bar{g}(t)\Lambda = \bar{g}(t)\begin{pmatrix} 0 & -\varepsilon & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & -\tau \\ 0 & 0 & \tau & 0 \end{pmatrix} \tag{3}$$

with $\gamma(t) = \bar{g}(t)\vec{e}_1, \vec{T}(t) = \bar{g}(t)\vec{e}_2, \vec{N}(t) = \bar{g}(t)\vec{e}_3$ and $\vec{B}(t) = \bar{g}(t)\vec{e}_4$, where $\vec{e}_i$ is the standard basis in $\mathbb{R}^4$ and $\vec{T}(t), \vec{N}(t), \vec{B}(t)$ are the tangent, normal and binormal vectors of the Serret-Frenet Frame [5]. Note that $\gamma(t) \in M$ where the associated base space is $M = \mathbb{R}^3$ when $G = SE(3)$, $M = \mathbb{S}^3$ when $G = SO(4)$ and $M = \mathbb{H}^3$ when $G = SO(1,3)$, see [5]. Assume that the vehicle described by the general frame (2) has attached to it a moving frame $\{M\} = \{x', y', z'\}$, then the general frame and Serret-Frenet frame are related by the following theorem:

Theorem 2.1. *The curvature κ and torsion τ of a path that the oriented vehicle traces are related to the steering controls u_1, u_2, u_3 by the following equations:*

$$\begin{aligned} \kappa &= u_3 \cos\beta - u_2 \sin\beta \\ \tau &= \frac{d\beta}{dt} + u_1 \\ \tan\beta &= -\frac{u_2}{u_3} \end{aligned} \tag{4}$$

where β is the angle between the normal vector $\vec{N}$ of the Serret-Frenet frame and the vector y' of the moving frame $\{M\}$.

Proof. See [5]. □

2.2 Helical Motions

In this paper the motions of interest are helical motions of oriented vehicles, which have been used to plan global manoeuvres for AUVs [12]. These steady motions provide feasible reference trajectories for oriented vehicles to track. Translating, circular and helical motions of oriented vehicles correspond to curves of constant curvature κ and constant torsion τ. Therefore, from the equations in (4) it is easily shown that kinematic controls that induce steady motions of the vehicle are of the form:

$$\begin{aligned} u_1 &= c_1 \\ u_2 &= c_2 \\ u_3 &= c_3 \end{aligned} \tag{5}$$

where c_1, c_2, c_3 are arbitrary constants or alternatively the controls can take the form:

$$\begin{aligned} u_1 &= c_1 \\ u_2 &= r \sin\theta(t) \\ u_3 &= r \cos\theta(t) \end{aligned} \tag{6}$$

where r is constant and $\theta(t)$ is linear in t will induce steady motions. Explicitly the constant angular velocity controls c_1, c_2, c_3 induce motions of the vehicle along curves of constant curvature and torsion. From equations (4) and (5) the curvature and torsion are given explicitly as:

$$\begin{aligned} \kappa &= \sqrt{c_2^2 + c_3^2} \\ \tau &= c_1 \end{aligned} \tag{7}$$

and the angular velocity controls (6) induce motions of the vehicle along curves of constant curvature and torsion. From equations (4) and (6) the curvature and torsion are given explicitly as:

$$\begin{aligned} \kappa &= r \\ \tau &= -\dot{\theta} + c_1 \end{aligned} \tag{8}$$

where $\dot{\theta}$ is constant and corresponds to the time differential of $\theta(t)$. The form of the controls (5) and (6) are of particular interest as they are shown to be optimal in [4].

2.3 Parametric equations for helical motions

In order to provide feasible and usable reference trajectories it is necessary to yield parametric expressions for the helical motions $\gamma(t) \in M$. The term global is used as the paths will be derived independently of a local coordinate chart. In the case of steady motions, the entries of Serret-Frenet frame (3) are constant and therefore the steady motions are described by

$$\gamma(t) = \bar{g}(t)\vec{e}_1 = \bar{g}(0)\exp(\Lambda t)\vec{e}_1.$$

The problem of computing these steady motions is generalized in the following problem statement.

Problem Statement 1. *Explicitly compute the solution $g(t) \in G$ of the following left-invariant differential equation:*

$$\frac{d\bar{g}(t)}{dt} = \bar{g}(t)A \tag{9}$$

where

$$A = \begin{pmatrix} 0 & -\varepsilon b_1 & -\varepsilon b_2 & -\varepsilon b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{pmatrix} \tag{10}$$

$a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ *are piecewise constant and where* $A \in \mathfrak{se}(3)$ *when* $\varepsilon = 0$, $A \in \mathfrak{so}(4)$ *when* $\varepsilon = -1$, *and* $A \in \mathfrak{so}(1,3)$ *when* $\varepsilon = 1$.

This paper describes methods to obtain the solution $g(t) : \mathbb{R} \to \mathbb{R}^{4\times 4} \in G$ of the differential equation (9). In general the solution is given by

$$g(t) = g_0 \exp(At),$$

where $g(0) = g_0$, where the matrix exponential $\exp(At)$ is (see [11] for detail):

$$\exp(At) = \sum_{k=0}^{+\infty} \frac{t^k A^k}{k!} \tag{11}$$

Therefore, the solution to (9) amounts to calculating the matrix exponential (11). In the Euclidean case $\varepsilon = 0$ the system (9) can be solved using the well know **Rodrigues' Formula** to obtain $g(t) \in SE(3)$ [8]. However, this formula does not extend to the non-Euclidean cases. In this paper we describe a method for deriving closed form solutions of equation (9), where $A \in \mathfrak{so}(4)$ and $A \in \mathfrak{so}(1,3)$, and the corresponding solutions are curves $g(t) \in SO(4)$ and $g(t) \in SO(1,3)$ respectively. The method here uses the double cover properties of these connected Lie Groups to decompose the system into trivial lower dimensional systems. The solutions of the decoupled systems are then computed in closed form. These decoupled solutions are then projected back onto the original system.

3 Decoupling the system

In this section the system described by equations (9) defined on $SO(4)$ and $SO(1,3)$ are decoupled into two lower dimensional systems. This decoupling then allows us to compute the solutions of the decoupled systems using a simple technique. The solutions of the decoupled systems can then be projected back onto the original manifold to yield the solution to the original system (9) on $SO(4)$ and $SO(1,3)$. We begin here by describing the decoupling of the system defined on $SO(4)$.

3.1 Decoupling the system on $SO(4)$

The system defined by the differential equations (9) on $SO(4)$ can be decoupled into two lower dimensional systems. The decoupling is possible as the Lie algebra $\mathfrak{so}(4)$ is isomorphic to $\mathfrak{su}(2) \times \mathfrak{su}(2)$ and an element $A \in \mathfrak{so}(4)$ can be identified with the elements $(V_1, V_2) \in \mathfrak{su}(2) \times \mathfrak{su}(2)$ using the following theorem:

Theorem 3.1. $\mathfrak{so}(4)$ *is isomorphic to* $\mathfrak{su}(2) \times \mathfrak{su}(2)$ *where an element* $A \in \mathfrak{so}(4)$ *is associated with the elements* $(V_1, V_2) \in \mathfrak{su}(2) \times \mathfrak{su}(2)$ *via the following* *mapping:*

$$
\begin{aligned}
A \mapsto (V_1, V_2) = \\
&\begin{pmatrix} 0 & -b_1 & -b_2 & -b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{pmatrix} \\
&\mapsto \frac{1}{2}\begin{pmatrix} (a_1+b_1)i & (a_2+b_2)+(a_3+b_3)i \\ -(a_2+b_2)+(a_3+b_3)i & -(a_1+b_1)i \end{pmatrix} \\
&, \frac{1}{2}\begin{pmatrix} (a_1-b_1)i & (a_2-b_2)+(a_3-b_3)i \\ -(a_2-b_2)+(a_3-b_3)i & -(a_1-b_1)i \end{pmatrix}
\end{aligned} \tag{12}
$$

where $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$

Proof. See [13]. □

Using Theorem 3.1, the system (9) on $SO(4)$ can be decoupled into a system on $SU(2) \times SU(2)$:

$$
\begin{aligned}
\dot{g}_1(t) &= g_1(t)V_1 \\
\dot{g}_2(t) &= g_2(t)V_2
\end{aligned} \tag{13}
$$

it follows that the solutions of these differential equations are:

$$
\begin{aligned}
g_1(t) &= g_1(0)\exp(tV_1) \\
g_2(t) &= g_2(0)\exp(tV_2)
\end{aligned} \tag{14}
$$

This is useful as the exponential of an element in $\mathfrak{su}(2)$ can be expressed in closed form, which is detailed in Section 4.

3.2 Decoupling systems on $SO(1,3)$

In this section the system described by equation (9) is decoupled into two lower dimensional systems when $\varepsilon = -1$, where the corresponding frame bundle is $SO(1,3)$. Therefore, elements of the Lie algebra are of the form:

$$
A = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{pmatrix} \tag{15}
$$

In a similar manner to Section 3.1 the system described by equation (9) on $SO(1,3)$ is decoupled into two lower dimensional systems. This decoupling is performed by using the following theorem:

Theorem 3.2. $\mathfrak{so}(1,3)$ *is isomorphic to* $\mathfrak{sl}_2(\mathbb{C}) \times \mathfrak{sl}_2(\mathbb{C})$ *where an element* $A \in \mathfrak{so}(1,3)$ *is identified with the elements* (U, U^*) *where* $U, U^* \in \mathfrak{sl}_2(\mathbb{C})$ *via the following mapping:*

$$\begin{aligned} &A \mapsto (U, U^*) = \\ &\begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & -a_3 & a_2 \\ b_2 & a_3 & 0 & -a_1 \\ b_3 & -a_2 & a_1 & 0 \end{pmatrix} \\ &\mapsto \frac{1}{2}\begin{pmatrix} (ia_1 + b_1) & (a_2 + b_3) + i(a_3 - b_2) \\ (b_3 - a_2) + i(a_3 + b_2) & -(ia_1 + b_1) \end{pmatrix} \\ &\quad , \frac{1}{2}\begin{pmatrix} (b_1 - ia_1) & (b_3 - a_2) - i(a_3 + b_2) \\ (b_3 + a_2) - i(a_3 - b_2) & -(b_1 - ia_1) \end{pmatrix} \end{aligned} \tag{16}$$

where $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ *and the* $*$ *notation denotes the conjugate transpose.*

Proof. See [13]. □

Following Theorem 3.2 the differential system (9) with $A \in \mathfrak{so}(1,3)$ can be written as a decoupled system:

$$\begin{aligned} \dot{g}_1(t) &= g_1(t)U \\ \dot{g}_1^*(t) &= g_1^*(t)U^* \end{aligned} \tag{17}$$

where $g_1(t), g_1^*(t) \in SL_2(\mathbb{C})$ and the solutions to differential equations (17) are:

$$\begin{aligned} g_1(t) &= g_1(0)\exp(tU) \\ g_1^*(t) &= g_1^*(0)\exp(tU^*) \end{aligned} \tag{18}$$

the solutions $g_1(t), g_1^*(t) \in SL_2(\mathbb{C})$ in equation (18) can be solved in closed form as shown in the following section.

4 Solving the decoupled systems

In the previous section the kinematic system defined on the six dimensional Lie groups $SO(4)$ and $SO(1,3)$ were decoupled into two lower dimensional systems defined on $SU(2)$ and $SL_2(\mathbb{C})$. In this section we explicitly solve these lower dimensional systems. Calculating the solutions $g(t) \in G$ amounts to computing the exponential maps of their Lie algebras. We begin with kinematic systems defined on the Lie group $SU(2)$.

4.1 Exponential map of the matrices V_1 and V_2

It is necessary to calculate the matrix exponentials of the matrices $(V_1, V_2) \in \mathfrak{su}(2) \times \mathfrak{su}(2)$ to obtain closed form solutions for $(g_1(t), g_2(t)) \in SU(2) \times SU(2)$.

Theorem 4.1. *The closed form solutions $g(t) \in SU(2)$ of the differential equation $\dot{g}(t) = g(t)V$ where $V \in \mathfrak{su}(2)$ consisting of all complex skew hermitian matrices:*

$$V = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \tag{19}$$

with $\alpha + \bar{\alpha} = 0$ is given by

$$g(t) = g_0 \begin{pmatrix} \left(\cos t\lambda + \frac{\alpha}{\lambda}\sin t\lambda\right) & \left(\frac{\beta}{\lambda}\sin t\lambda\right) \\ \left(-\frac{\bar{\beta}}{\lambda}\sin t\lambda\right) & \left(\cos t\lambda + \frac{\bar{\alpha}}{\lambda}\sin t\lambda\right) \end{pmatrix} \tag{20}$$

where $\lambda^2 = |\alpha|^2 + |\beta|^2$

Proof. Note that $V^2 = -\lambda^2 I$ where I is the identity element in $SU(2)$. Then using formula (11) and splitting the power series into odd and even powers gives:

$$\exp(tV) = \sum_{k=0}^{\infty} \frac{t^k}{k!} V^k = \sum_{n=0}^{\infty} \frac{t^{2n} V^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{t^{2n+1} V^{2n+1}}{(2n+1)!} \tag{21}$$

substituting V^2 into (21) and rearranging yields:

$$\exp(tV) = I \sum_{n=0}^{\infty} \frac{(-1)^n (t\lambda)^{2n}}{(2n)!} + \frac{V}{\lambda} \sum_{n=0}^{\infty} (-1)^n \frac{(t\lambda)^{2n+1}}{(2n+1)!} \tag{22}$$

simplifying (22) using the Taylor Series for the sine and cosine functions the matrix exponential is stated in closed form:

$$\begin{aligned} \exp(tV) &= (\cos t\lambda)I + \left(\frac{1}{\lambda}\sin t\lambda\right)V \\ &= \begin{pmatrix} \left(\cos t\lambda + \frac{\alpha}{\lambda}\sin t\lambda\right) & \left(\frac{\beta}{\lambda}\sin t\lambda\right) \\ \left(-\frac{\bar{\beta}}{\lambda}\sin t\lambda\right) & \left(\cos t\lambda + \frac{\bar{\alpha}}{\lambda}\sin t\lambda\right) \end{pmatrix} \end{aligned} \tag{23}$$

then substituting (20) into $g(t) = g(0)\exp(tV)$ yields (20). □

Using Theorem 4.1 to evaluate $\exp(tV_1)$ and $\exp(tV_2)$ and assuming for simplicity of exposition that $g_1(0) = I$ and $g_2(0) = I$, where I is the identity matrix, yields $g_1(t)$ and $g_2(t)$ in closed form as:

$$g_1(t) = \exp(V_1 t) = (\cos t\lambda_1)I + \left(\frac{1}{\lambda_1}\sin t\lambda_1\right)V_1 \tag{24}$$

where

$$\lambda_1^2 = \frac{(a_1+b_1)^2}{4} + \frac{(a_2+b_2)^2}{4} + \frac{(a_3+b_3)^2}{4} \tag{25}$$

and

$$g_2(t) = \exp(V_2 t) = (\cos t\lambda_2)I + \left(\frac{1}{\lambda_2}\sin t\lambda_2\right)V_2 \tag{26}$$

such that

$$\lambda_2^2 = \frac{(a_1-b_1)^2}{4} + \frac{(a_2-b_2)^2}{4} + \frac{(a_3-b_3)^2}{4} \tag{27}$$

where V_1 and V_2 are the matrices defined in (12). Therefore, the solutions of the decoupled systems (14) can be expressed simply in closed form.

4.2 Exponential map of the matrices U and U^*

The computation of the matrix exponentials of the matrices $U, U^* \in \mathfrak{sl}_2(\mathbb{C})$ given in equations (4.2), is analogous to that used for $V_1, V_2 \in \mathfrak{su}(2)$, therefore we state the results in the following Theorems:

Theorem 4.2. *The closed form solutions $g_1(t) \in SL_2(\mathbb{C})$ of the differential equation $\dot{g}_1(t) = g_1(t)U$ where $U \in \mathfrak{sl}_2(\mathbb{C})$ consists of all matrices of the form:*

$$U = \frac{1}{2}\begin{pmatrix} (ia_1+b_1) & (a_2+b_3)+i(a_3-b_2) \\ (b_3-a_2)+i(a_3+b_2) & -(ia_1+b_1) \end{pmatrix} \tag{28}$$

is given by

$$g_1(t) = g_0((\cos t\lambda_1)I + \left(\frac{1}{\lambda_1}\sin t\lambda_1\right)U) \tag{29}$$

where

$$\lambda_1^2 = \frac{(a_1 - ib_1)^2}{4} + \frac{(a_2 - ib_2)^2}{4} + \frac{(a_3 - ib_3)^2}{4} \tag{30}$$

And the following:

Theorem 4.3. *The closed form solutions $g_1^*(t) \in SL_2(\mathbb{C})$ of the differential equation $\dot{g}_1^*(t) = g_1^*(t)U^*$ where $U^* \in \mathfrak{sl}_2(\mathbb{C})$ consists of all matrices of the form:*

$$U^* = \frac{1}{2}\begin{pmatrix} (b_1 - ia_1) & (b_3 - a_2) - i(a_3 + b_2) \\ (b_3 + a_2) - i(a_3 - b_2) & -(b_1 - ia_1) \end{pmatrix} \tag{31}$$

is given by

$$g_1^*(t) = g_0((\cos t\lambda_2)I + \left(\frac{1}{\lambda_2}\sin t\lambda_2\right)U^*) \tag{32}$$

where

$$\lambda_2^2 = \frac{(a_1 + ib_1)^2}{4} + \frac{(a_2 + ib_2)^2}{4} + \frac{(a_3 + ib_3)^2}{4}. \tag{33}$$

The decoupled systems can be solved explicitly using Theorem 4.1, Theorem 4.2 and Theorem 4.3.

5 Projecting the decoupled system back onto the original system

The previous section solves the decoupled solutions explicitly in closed form, however it is necessary to reconstruct the solutions on the original Lie group $SO(4)$ and $SO(1,3)$. This projection is performed by using the mapping described in the following subsection:

5.1 Projecting back onto $SO(4)$

Firstly, define the set:

$$X = \left\{ \begin{pmatrix} x_0 + ix_1 & x_2 + ix_3 \\ -x_2 + ix_3 & x_0 - ix_1 \end{pmatrix} : x_0, x_1, x_2, x_3 \in \mathbb{R} \right\} \tag{34}$$

then for any element $\hat{z} \in \mathbb{R}^4$ associate an element $Z \in X$ via the mapping:

$$\hat{z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \to Z = \begin{pmatrix} z_0 + iz_1 & z_2 + iz_3 \\ -z_2 + iz_3 & z_0 - iz_1 \end{pmatrix} \tag{35}$$

where $z_0, z_1, z_2, z_3 \in \mathbb{R}$ and define a second element, for simplicity of exposition, as $\hat{w} \in \mathbb{R}^4$ associated to $W \in X$ in the same way as equation (35):

$$\hat{w} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} \to W = \begin{pmatrix} w_0 + iw_1 & w_2 + iw_3 \\ -w_2 + iw_3 & w_0 - iw_1 \end{pmatrix} \tag{36}$$

where $w_0, w_1, w_2, w_3 \in \mathbb{R}$ then recall from [9] the homomorphism $\Phi : SU(2) \times SU(2) \to SO(4)$ is defined through the following equivalent group actions:

$$g\hat{z} = \hat{w} \tag{37}$$

for $g \in SO(4)$ if and only if

$$g_1 Z g_2^{-1} = W \tag{38}$$

where $g_1, g_2 \in SU(2)$. Using the homomorphism Φ we can construct a closed form solution $g \in SO(4)$ from the closed form solutions $g_1, g_2 \in SU(2)$. Firstly, note that $g_1, g_2 \in SU(2)$ can be projected onto $\mathbb{R}^4$ following the equations (38) and (36).

Expressing these two equations as one projection yields:

$$g_1 Z g_2^{-1} = W \mapsto \hat{w} \in \mathbb{R}^4 \tag{39}$$

using the projection (39) and the equivalence of the group actions (37) and (38) implies that the solution $g \in SO(4)$ can be constructed by associating the first column of $g \in SO(4)$ defined by

$$\hat{w}_1 = g \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^T$$

with

$$g_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_2^{-1} = W_1 \to \hat{w}_1$$

it follows that the remaining columns of $SO(4)$ are identified with:

$$\begin{aligned} g_1 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} g_2^{-1} &= W_2 \to \hat{w}_2 \\ g_1 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g_2^{-1} &= W_3 \to \hat{w}_3 \\ g_1 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} g_2^{-1} &= W_3 \to \hat{w}_4 \end{aligned} \tag{40}$$

then it follows that

$$g = \begin{pmatrix} \hat{w}_1 & \hat{w}_2 & \hat{w}_3 & \hat{w}_4 \end{pmatrix} \tag{41}$$

The solution (41) is a particular solution of the system (9) for $G = SO(4)$. However, through the trivial properties of left-invariance of the system (9) and the existence of a group inverse in G, this can be expressed as a general solution. Denote the general solution of equation (9) for $G = SO(4)$ as $g_{gen} \in SO(4)$ subject to the initial condition $g_{\text{int}} \in SO(4)$, then the general solution is:

$$g_{gen} = g_{\text{int}} g(0)^{-1} g \tag{42}$$

where g is defined by (41) and $g(0)$ is g at $t = 0$.

5.2 Projecting back onto $SO(1,3)$

In a similar manner to Section 5.1 define a set of matrices X such that:

$$X = \left\{ \begin{pmatrix} x_0 + x_1 & x_3 - ix_2 \\ x_3 + ix_2 & x_0 - x_1 \end{pmatrix} : x_0, x_1, x_2, x_3 \in \mathbb{C} \right\} \tag{43}$$

this is the real vector space of Hermitian 2×2 matrices, in addition let V_{-1} denote the real linear space spanned by the basis ie_0, e_1, e_2, e_3. For any element $\hat{z} = z_0 ie_0 + z_1 e_1 + z_2 e_2 + z_3 e_3$ in V_{-1}, with $z_0, z_1, z_2, z_3 \in \mathbb{C}$ is associated to

$Z \in X$ via the mapping:

$$\hat{z} = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{bmatrix} \to Z = \begin{pmatrix} z_0 + z_1 & z_3 - iz_2 \\ z_3 + iz_2 & z_0 - z_1 \end{pmatrix} \tag{44}$$

and for simplicity of exposition define a second element $\hat{w} \in V_{-1}$ associated to $W \in X$ via the mapping:

$$\hat{w} = \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \end{bmatrix} \to W = \begin{pmatrix} w_0 + w_1 & w_3 - iw_2 \\ w_3 + iw_2 & w_0 - w_1 \end{pmatrix} \tag{45}$$

then the homomorphism $\Phi : SL_2(\mathbb{C}) \to SO(1,3)$, see [9], is defined through the following equivalent group actions:

$$g\hat{z} = \hat{w} \tag{46}$$

for $g \in SO(1,3)$ whenever

$$g_3 Z g_3^* = W \tag{47}$$

for $g_3 \in SL_2(\mathbb{C})$ and where g_3^* is the conjugate transpose of g_3. Therefore, we can obtain the solution $g(t) \in SO(1,3)$ by using the homomorphism Φ defined by equations (46) and (47). Then each column of $SO(1,3)$ is identified with:

$$\begin{gathered} g_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} {g_3}^* = W_1 \to \hat{w}_1 \\ g_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} {g_3}^* = W_2 \to \hat{w}_2 \\ g_3 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} {g_3}^* = W_3 \to \hat{w}_3 \\ g_3 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} {g_3}^* = W_3 \to \hat{w}_4 \end{gathered} \tag{48}$$

where $g \in SO(1,3)$ is defined by:

$$g = \begin{pmatrix} \hat{w}_1 & \hat{w}_2 & \hat{w}_3 & \hat{w}_4 \end{pmatrix} \tag{49}$$

The general solution of equation (9) for $G = SO(1,3)$ denoted $g_{gen} \in SO(1,3)$ subject to the initial condition $g_{\text{int}} \in SO(1,3)$, is then given by:

$$g_{gen} = g_{\text{int}} g(0)^{-1} g \tag{50}$$

where g is defined by (49) and $g(0)$ is g at $t = 0$.

This section has described methods for integrating kinematic systems with piecewise constant controls defined on the 6-D Lie groups $SO(4)$ and $SO(1,3)$. These methods are now applied to integrate the generalized Serret-Frenet frame that describe steady motions of oriented vehicles. Therefore, we are able to derive parametric equations describing generalized helical, circular and geodesic motions of vehicles in 3-D space. The term natural is used as the equations describing the path that the vehicle traces, are expressed solely in terms of its curvature and torsion.

6 Analytic expressions for Helical Motions

In this section closed form analytic expressions are derived that describe the steady motions of an oriented vehicle travelling at unit speed given the controls (5) or (6). Using the lifted Serret-Frenet Frame (3) and the integration methods outlined in this paper, we are able to describe this motion in terms of the paths curvature and torsion.

6.1 Euclidean case

It is well known that when the Lie algebra is $\mathfrak{se}(3)$ i.e when $\varepsilon = 0$ in equation (9), the **Rodrigues** formula can be used to obtain the solutions $\bar{g}(t) \in SE(3)$ (see [8]). It follows that when the curvature and torsion are constant in (3) the solutions $\bar{g}(t) \in SE(3)$ of the Serret-Frenet Frame can be given in closed form and then projected down to $\mathbb{R}^3$, the equations for helices through the identity

($\bar{g}(0) = I$) are given by $\gamma(t) = [x(t), y(t), z(t)]^T$:

$$
\begin{aligned}
x(t) &= \frac{t\tau^2}{\kappa^2+\tau^2} + \frac{\kappa^2 \sin(\sqrt{\kappa^2+\tau^2}t)}{(\kappa^2+\tau^2)^{3/2}} \\
y(t) &= \frac{\kappa}{\kappa^2+\tau^2} - \frac{\kappa \cos(\sqrt{\kappa^2+\tau^2}t)}{\kappa^2+\tau^2} \\
z(t) &= \kappa\tau \left(\frac{t}{\kappa^2+\tau^2} - \frac{\sin(\sqrt{\kappa^2+\tau^2}t)}{(\kappa^2+\tau^2)^{3/2}} \right)
\end{aligned}
\tag{51}
$$

these helices are explicitly defined in terms of the geometric invariants κ and τ and are therefore in their most natural parametric form, where κ and τ are dependent on the controls (5) and (6).

6.2 Spherical Case

When $\bar{g}(t) \in SO(4)$, the differential equation describing the vehicles motion is:

$$
\frac{d\bar{g}(t)}{dt} = \bar{g}(t) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & -\tau \\ 0 & 0 & \tau & 0 \end{pmatrix} \tag{52}
$$

Following **Theorem** 3.1, (52) can be decoupled into the differential equations:

$$
\begin{aligned}
\dot{g}_1(t) &= g_1(t)\frac{1}{2} \begin{pmatrix} (\tau+1)i & \kappa i \\ \kappa i & -(\tau+1)i \end{pmatrix} \\
\dot{g}_2(t) &= g_2(t)\frac{1}{2} \begin{pmatrix} (\tau-1)i & \kappa i \\ \kappa i & -(\tau-1)i \end{pmatrix}
\end{aligned}
\tag{53}
$$

where $g_1(t), g_2(t) \in SU(2)$. Assuming that the curves are through the identity and using equations (24) and (26) obtain the solutions to the differential equations (53) as:

$$
\begin{aligned}
g_1(t) &= \begin{pmatrix} \cos t\lambda_1 & 0 \\ 0 & \cos t\lambda_1 \end{pmatrix} \\
&+ \frac{\sin t\lambda_1}{2\lambda_1} \begin{pmatrix} (\tau+1)i & \kappa i \\ \kappa i & -(\tau+1)i \end{pmatrix}
\end{aligned}
\tag{54}
$$

where:

$$\lambda_1^2 = \frac{(\tau+1)^2}{4} + \frac{\kappa^2}{4} \tag{55}$$

and

$$\begin{aligned} g_2(t) = & \begin{pmatrix} \cos t\lambda_2 & 0 \\ 0 & \cos t\lambda_2 \end{pmatrix} \\ & + \frac{\sin t\lambda_2}{2\lambda_2} \begin{pmatrix} (\tau-1)i & \kappa i \\ \kappa i & -(\tau-1)i \end{pmatrix} \end{aligned} \tag{56}$$

with

$$\lambda_2^2 = \frac{(\tau-1)^2}{4} + \frac{\kappa^2}{4} \tag{57}$$

the projection of (54) and (56) onto the base space $\mathbb{S}^3$ is given by:

$$g_1(t) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} g_2^{-1}(t) = W_1 \to \hat{w}_1 = \gamma(t) \tag{58}$$

substituting (54) and (56) into (58) and simplifying we obtain that $\gamma(t) = [x_0, x_1, x_2, x_3]^T$:

$$\begin{aligned} x_0 &= \cos\lambda_1 t \cos\lambda_2 t + \sin\lambda_1 t \sin\lambda_2 t \left(\frac{\tau^2+\kappa^2-1}{4\lambda_1\lambda_2}\right) \\ x_1 &= \frac{(\tau+1)}{2\lambda_1} \cos\lambda_2 t \sin\lambda_1 t + \frac{(1-\tau)}{2\lambda_2} \cos\lambda_1 t \sin\lambda_2 t \\ x_2 &= \frac{\kappa}{2\lambda_1\lambda_2} \sin\lambda_1 t \sin\lambda_2 t \\ x_3 &= \frac{\kappa}{2\lambda_1} \sin\lambda_1 t \cos\lambda_2 t - \frac{\kappa}{2\lambda_2} \cos\lambda_1 t \sin\lambda_2 t \end{aligned} \tag{59}$$

this gives an equation for a Helix in $\mathbb{S}^3$ through the identity expressed explicitly in terms of the geometric invariants κ and τ. Then for any initial configuration $g_0 \in SO(4)$, $g_0\gamma(t)$ describes all helices in $\mathbb{S}^3$.

6.3 Hyperbolic Case

For the hyperbolic case where $\bar{g}(t) \in SO(1,3)$, we use the method outlined in Section 5 to derive the equations steady motions for an oriented vehicle travelling in $\mathbb{H}^3$ where the differential equation describing the vehicles motion is:

$$\frac{d\bar{g}(t)}{dt} = \bar{g}(t)\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -\kappa & 0 \\ 0 & \kappa & 0 & -\tau \\ 0 & 0 & \tau & 0 \end{pmatrix} \tag{60}$$

Following Theorem 4.1, this can be decoupled into the differential equations:

$$\begin{aligned} \dot{g}_1(t) &= g_1(t)\frac{1}{2}\begin{pmatrix} i\tau+1 & \kappa i \\ \kappa i & -(i\tau+1) \end{pmatrix} \\ \dot{g}_1^*(t) &= g_1^*(t)\frac{1}{2}\begin{pmatrix} 1-i\tau & -\kappa i \\ -\kappa i & -(1-i\tau) \end{pmatrix} \end{aligned} \tag{61}$$

where $g_1(t), g_1^*(t) \in SL_2(\mathbb{C})$. Assuming that the curves are through the identity, then using equations (29) and (32) obtain the solutions:

$$\begin{aligned} g_1(t) &= \begin{pmatrix} \cos t\lambda_1 & 0 \\ 0 & \cos t\lambda_1 \end{pmatrix} \\ &+ \frac{\sin t\lambda_1}{2\lambda_1}\begin{pmatrix} i\tau+1 & \kappa i \\ \kappa i & -(i\tau+1) \end{pmatrix} \\ g_1^*(t) &= \begin{pmatrix} \cos t\lambda_2 & 0 \\ 0 & \cos t\lambda_2 \end{pmatrix} \\ &+ \frac{\sin t\lambda_2}{2\lambda_2}\begin{pmatrix} 1-i\tau & -\kappa i \\ -\kappa i & -(1-i\tau) \end{pmatrix} \end{aligned} \tag{62}$$

where

$$\begin{aligned} \lambda_1^2 &= \frac{(\tau-i)^2}{4} + \frac{\kappa^2}{4} \\ \lambda_2^2 &= \frac{(\tau+i)^2}{4} + \frac{\kappa^2}{4} \end{aligned} \tag{63}$$

Then the projection of $g_1(t), g_1^*(t)$ onto $\mathbb{H}^3 \subset \mathbb{R}^4$ is given by equation (47) explicitly expressed as:

$$g_1(t)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} {g_1}^*(t) = W_1 \to \hat{w}_1 = \gamma(t) \tag{64}$$

substituting (62) into (64), yields $\gamma(t) = [ix_0, x_1, x_2, x_3]^T$:

$$\begin{aligned}
x_0 &= \cos\lambda_1 t \cos\lambda_2 t + \frac{(1+\kappa^2+\tau^2)}{4\lambda_1\lambda_2} \sin\lambda_1 t \sin\lambda_2 t \\
x_1 &= \frac{(1+i\tau)}{2\lambda_1} \cos\lambda_2 t \sin\lambda_1 t + \frac{(1-i\tau)}{2\lambda_2} \cos\lambda_1 t \sin\lambda_2 t \\
x_2 &= -\frac{\kappa \sin\lambda_1 t \sin\lambda_2 t}{2\lambda_1\lambda_2} \\
x_3 &= \frac{i\kappa}{2\lambda_1\lambda_2} (\lambda_2 \cos\lambda_2 t \sin\lambda_1 t - \lambda_1 \cos\lambda_1 t \sin\lambda_2 t)
\end{aligned} \tag{65}$$

Therefore, we have derived parametric equations for helical motions of vehicles travelling in a 3-D space explicitly in terms of their curvature and torsion.

7 Inducing geodesic and circular motions

The method proposed in this paper has been used to derive analytic steady helical motions in a 3-dimensional space form. In this section we use these equations to yield equations for geodesic motions by setting $\kappa = 0$ and circular motions by setting $\tau = 0$.

Definition 7.1. *A vehicle travelling at constant speed in a 3-D space form, is said to trace a geodesic, if the path traced has curvature equal to zero (no acceleration).*

Therefore, using our equations for helical steady motions and setting $\kappa = 0$, we obtain particular unit speed parameterizations through the group identity. The controls that induce this motion are:

$$\begin{aligned}
u_1 &= c_1 \\
u_2 &= u_3 = 0
\end{aligned} \tag{66}$$

these controls induce the vehicle to trace the following paths:

Theorem 7.2. *A vehicle travelling with unit speed, that traces a geodesic curve through the group identity $id \in G$ embedded in either $\mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{H}^3$, traces a curve that parameterizes $\mathbb{R}, \mathbb{S}$ or $\mathbb{H}$ respectively.*

Proof. Proceeding with the Euclidean case, to obtain a geodesic in $\mathbb{R}^3$ through the identity, substitute $\kappa = 0$ into the natural parameterization for a Helix in $\mathbb{R}^3$ (51) yields:

$$\begin{aligned} x_1 &= t \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned} \tag{67}$$

which is a straight line (geodesic) in $\mathbb{R}^3$ and parameterizes $\mathbb{R}$. In addition a geodesic in $\mathbb{S}^3$ corresponds to a curve of zero curvature, put $\kappa = 0$ into (55) and (57) then

$$\begin{aligned} \lambda_1 &= \frac{\tau + 1}{4} \\ \lambda_2 &= \frac{\tau - 1}{4} \end{aligned} \tag{68}$$

it follows on substituting (68) and $\kappa = 0$ into (59) and simplifying using well known trigonometric identities gives:

$$\begin{aligned} x_0 &= \cos t \\ x_1 &= \sin t \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned} \tag{69}$$

equation (69) describes a geodesic in $\mathbb{S}^3$ and for this particular case parameterizes the unit circle $\mathbb{S} \subset \mathbb{R}^2 \subset \mathbb{R}^4$ for $t \in [-\infty, \infty]$ and satisfy's the equation:

$$x_0^2 + x_1^2 = 1 \tag{70}$$

To obtain an equation for a geodesic in $\mathbb{H}^3$ put $\kappa = 0$ into (65) to give

$$\begin{aligned} \lambda_1 &= \frac{\tau - i}{2} \\ \lambda_2 &= \frac{\tau + i}{2} \end{aligned} \tag{71}$$

then on substituting (71) and $\kappa = 0$ into equations (65) and simplifying using well known trigonometric identities and hyperbolic properties obtain:

$$\begin{aligned} x_0 &= \cosh t \\ x_1 &= \sinh t \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned} \tag{72}$$

these equations parameterize a standard hyperbola $\mathbb{H} \subset \mathbb{R}^2 \subset \mathbb{R}^4$ for $t \in [-\infty, \infty]$:

$$-(x_1)^2 + x_0^2 = 1 \tag{73}$$

□

As well as geodesic motions the controls can be manipulated to induce circular motions:

Definition 7.3. *A vehicle travelling at constant speed in a 3-D space form is said to trace a Riemannian circle, if the path traced has constant curvature and zero torsion.*

From (8) and (7) it is easily shown that controls that induce circular motion can be of the form:

$$\begin{aligned} u_1 &= 0 \\ u_2 &= c_2 \\ u_3 &= c_3 \end{aligned} \tag{74}$$

and additionally

$$\begin{aligned} u_1 &= c_1 \\ u_2 &= r \sin{(c_1 t + \gamma)} \\ u_3 &= r \cos{(c_1 t + \gamma)} \end{aligned} \tag{75}$$

where γ is a phase constant. These controls induce the vehicle to trace the following paths:

Theorem 7.4. *A vehicle travelling with unit speed tracing a Riemannian circular path through the group identity $id \in G$ embedded in $\mathbb{R}^3, \mathbb{S}^3$ or $\mathbb{H}^3$ trace a path that parameterizes $\mathbb{S}, \mathbb{S}^2$ or $\mathbb{H}^2$ respectively.*

Proof. Using the equations for the helices, we derive particular Riemannian circles through the identity by putting $\tau = 0$ and $\kappa \neq 0$. Proceeding with the

Euclidean case first. When $\tau = 0$ and $\kappa \neq 0$ the equations (51) simplify to:

$$\begin{aligned} x_1 &= \frac{\sin \kappa t}{\kappa} \\ x_2 &= \frac{1}{\kappa} - \frac{\cos \kappa t}{\kappa} \\ x_3 &= 0 \end{aligned} \tag{76}$$

which parameterize the unit circle $\mathbb{S} \subset \mathbb{R}^2 \subset \mathbb{R}^3$ for $t \in [-\infty, \infty]$ in the plane of x_3 and satisfy the equation:

$$x_1^2 + (x_2 - \frac{1}{\kappa})^2 = 1 \tag{77}$$

Therefore, giving the classic result that a curve of constant curvature and zero torsion in $\mathbb{R}^3$ parameterizes the unit circle $\mathbb{S}$.
For the spherical case Substituting $\tau = 0$ into (55) and (57) gives

$$\lambda_1 = \lambda_2 = \frac{\sqrt{1+\kappa^2}}{2} \tag{78}$$

it follows on substituting (78) and $\tau = 0$ into (59) and simplifying using well known trigonometric identities gives:

$$\begin{aligned} x_0 &= \frac{\kappa^2 + \cos t\sqrt{1+\kappa^2}}{1+\kappa^2} \\ x_1 &= \frac{\sin\left(t\sqrt{1+\kappa^2}\right)}{\sqrt{1+\kappa^2}} \\ x_2 &= \frac{\kappa}{1+\kappa^2} - \frac{\kappa\cos\left(t\sqrt{1+\kappa^2}\right)}{1+\kappa^2} \\ x_3 &= 0 \end{aligned} \tag{79}$$

these equations parameterize the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3 \subset \mathbb{R}^4$ for $t \in [-\infty, \infty]$ satisfying the equation:

$$x_0^2 + x_1^2 + x_2^2 = 1 \tag{80}$$

Therefore, the vehicle traces a particular Riemannian circle in $\mathbb{S}^3$ that parameterizes the unit sphere $\mathbb{S}^2$. For the Hyperbolic case substitute $\tau = 0$ into equation (63) gives:

$$\lambda_1 = \lambda_2 = \frac{\sqrt{\kappa^2 - 1}}{2} \tag{81}$$

on substituting (81) and $\tau = 0$ into (65) yields the following equations:

$$\begin{aligned} x_0 &= \frac{\kappa^2 - \cos t\sqrt{\kappa^2 - 1}}{\kappa^2 - 1} \\ x_1 &= \frac{\sin t\sqrt{\kappa^2 - 1}}{\sqrt{\kappa^2 - 1}} \\ x_2 &= \frac{\kappa \cos t\sqrt{\kappa^2 - 1}}{\kappa^2 - 1} - \frac{\kappa}{\kappa^2 - 1} \\ x_3 &= 0 \end{aligned} \tag{82}$$

it follows that on substituting $\sqrt{\kappa^2 - 1} \equiv i\sqrt{1 - \kappa^2}$ into (82) the equations can be expressed in terms of hyperbolic functions

$$\begin{aligned} x_0 &= \frac{\kappa^2 - \cosh\left(t\sqrt{1 - \kappa^2}\right)}{\kappa^2 - 1} \\ x_1 &= \frac{\sinh\left(t\sqrt{1 - \kappa^2}\right)}{\sqrt{\kappa^2 - 1}} \\ x_2 &= \frac{\kappa \cosh\left(t\sqrt{1 - \kappa^2}\right)}{\kappa^2 - 1} - \frac{\kappa}{\kappa^2 - 1} \\ x_3 &= 0 \end{aligned} \tag{83}$$

these equations parameterize the standard unit hyperboloid $\mathbb{H}^2 \subset \mathbb{R}^3 \subset \mathbb{R}^4$ for $t \in [-\infty, \infty]$ that satisfy the equation:

$$x_0^2 - x_1^2 - x_2^2 = 1 \tag{84}$$

therefore, the vehicle traces a path in $\mathbb{H}^3$ that parameterizes the standard unit hyperboloid of one surface $\mathbb{H}^2$. $\square$

8 Conclusion

This paper gives a method for obtaining closed form solutions for control systems defined on the frame bundles of the 3-dimensional space forms where the controls are piece-wise constant. The method uses the universal cover property of connected Lie groups to decouple the system into lower dimensional systems. These lower dimensional control systems are then solved explicitly in closed form. These solutions are then projected back onto the original Lie group to obtain a closed form solution for the original system.

This method is applied to the problem of controlling oriented vehicles along steady motion trajectories. The integration method is used to derive closed form expressions for these steady motions. These closed form expressions are expressed explicitly in terms of the geometric invariants κ and τ. An extension of this work to the time-dependent case is an area of current research.

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