Journal of Applied Mathematics & Bioinformatics, vol.3, no.2, 2013, 159-170
ISSN: 1792-6602 (print), 1792-6939 (online)
Scienpress Ltd, 2013

On Duality Principle in Exponentially Lévy Market

Olabisi O. Ugbebor[1] and Sunday Onos Edeki[2]

Abstract

This paper describes the effect of duality principle in option pricing driven by exponentially Lévy market model. This model is basically incomplete - that is; perfect replications or hedging strategies do not exist for all relevant contingent claims and we use the duality principle to show the coincidence of the associated underlying asset price process with its corresponding dual process.

The condition for the 'unboundedness' of the underlying asset price process and that of its dual is also established. The results are not only important in Financial Engineering but also from mathematical point of view.

Mathematics Subject Classification: 60G51, 62P05, 91B25, 91B26

Keywords: Option pricing, Lévy processes, incomplete market, option dual and primal

[1] Mathematics Department, University of Ibadan, Oyo State Nigeria,
e-mail: ugbebor1@yahoo.com

[2] Mathematics Department, University of Ibadan, Oyo State Nigeria,
e-mail: soedeki@yahoo.com

Article Info: *Received* : February 4, 2013. *Revised* : March 29, 2013
Published online : June 30, 2013

1 Introduction

Ever since the seminal article of Fisher Black, Myron Scholes, and R. Merton [4] on the analytical model that would determine the fair market value of options paying no dividends, and their reformation in terms of martingale theory by Harrison and Pliska [12], stochastic analysis has become indispensable in the study of modern finance. Stochastic analysis and martingale theory appear to be tailor made for their applications in mathematical finance; where the proceeds from the investment in an asset can be represented as a stochastic integral, while the rational price of an option on an asset equals its discounted expected payoff under a martingale measure.

Initially, the applications relied mainly on the use of Brownian motion as the driving process but empirical evidence showed that this assumption is too restrictive. Hence, one of the remedies was to consider Lévy processes as the driving force as pioneered be E. Eberlein, D. Madan and their co-workers in [6] and [7], since for any time increment Δt, any infinitely divisible distribution (i.d.d.) can be chosen as the increment distribution over periods of time Δt and also, they can describe the observed reality of financial markets in a more accurate way than models based on Brownian motions, since, in the real world, we observed that asset price processes have jumps or spikes and risk managers need to take them into consideration; these jumps are considered in Lévy processes.

Bachelier [2] proposed the exponential Brownian motion $expB_t$ as a stock price model. Few years later, an exponential Lévy process model with a non-stable distribution was proposed by Press in [8].

For investors and option traders, the price of the option is not the only thing of interest. The duality principle which establishes a call-put parity in option pricing plays a vital role. The duality principle as noted by Papapantoleon [25] and EPS in [7], demonstrates its full strength when considering exotic derivatives.

This paper is structured as follows: section 2 deals with preliminaries on Lévy processes and exponentially Lévy market model, section 3 takes care of the duality principle with a corresponding theorem. In section 4, the applications of the main results are presented, and a concluding remark is made in section 5.

With regard to our results, we can therefore say that the work is very effective in the option pricing theory from the theoretical point of view.

2 Preliminaries

Definition 2.1 Let $L = \{L_t : t \geq 0\}$ be a L$\acute{e}$vy processes defined on a filtered probability space $(\mu\,,\ \mathfrak{B},\ \Omega,\ \mathbb{F}(\mathfrak{B})\,)$, that is, L is an adapted stochastic, right continuous process starting from zero (0) with stationary and independent increments.

Under the market measure $\mathbb{P}$, the Le´vy process $\{L_t\}_t \geq 0$ is assumed to follow the process given by:

$$L_t = \mu t + \boldsymbol{\sigma} \mathrm{Wt} + \sum_{n=1}^{Nt} \mathrm{Yn} \tag{2.1}$$

where μ is a mean rate of return, W_t is a Brownian motion, $\boldsymbol{\sigma}$ is a volatility parameter, N_t is a Poisson process and Y_n is a jump size variable.

Equation (2.1) shows the three components of a L$\acute{e}$vy process: a purely deterministic linear part, a Brownian motion and a pure-jump process.

2.1 Exponentially L$\acute{e}$vy Processes

Let S_t be the underlying asset price process and suppose that a filtered probability space $(\mu\,,\ \mathfrak{B},\ \Omega,\ \mathbb{F}(\mathfrak{B})\,)$ is given and that the asset price process $S_t = S_0 e^{rt + Lt}$ is defined on this space, where L_t is a L$\acute{e}$vy process, and r is the interest rate on a non-risky asset, we therefore call such process S_t an exponential Le´vy process (henceforth ELP). It is remarked that any L$\acute{e}$vy process has a specific form of its characteristic function given by

Lévy-Khintchine formula.

Thus, for $x \epsilon L_t$, $t \geq 0$, the characteristic form of $L = \{L_t : t \geq 0\}$ is:

$$\varphi_x(\theta) = \mathbb{E}\left(e^{i\langle\theta, L_t\rangle}\right) = \exp\left(tV^*(\theta)\right) \tag{2.2}$$

$$V^*(\theta) = i\langle b, \theta\rangle - \frac{1}{2}\langle a\theta; \theta\rangle + \int_{\mathbb{R}^d}[e^{i\langle\theta\,;\,x\rangle} - 1 - i\langle\theta; x\rangle . 1_{\{|x|\}\leq 1}]\upsilon(dx) \tag{2.3}$$

where $b \epsilon \mathbb{R}^d$, $a = a(d \times d)$ is a positive definite symmetric matrix, and $v = \upsilon(dx)$ is a Lévy measure on $\mathbb{R}^d$ such that $v(0) = 0$, $1_{\{|x|\}\leq 1}$ denotes an indicator function and

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\upsilon(dx) < \infty$$

It is noteworthy to point out that $\varphi_x(\theta)$ is specified by three characteristic triples a, b and v. For instance: if $X \sim N(m, \sigma^2)$, then, $b = m$ and $a = \sigma^2$ with $v = 0$ implies that

$$\varphi_x(\theta) = \mathbb{E}(exp i\langle\theta, \quad X_t\rangle) = exp\{t[i\langle m, \theta\rangle - \frac{1}{2}\langle\theta, \theta\sigma^2\rangle]\}$$

which is the characteristics function of a Gaussian random variable X_t.

If X is a random variable having a Poisson distribution with parameter λ, then $v(dx) = \lambda 1_{\{|x|\}\leq 1}(dx)$ is a measure concentrated at the point $x = 1$, $b = \lambda$, hence,

$$\varphi_x(\theta) = \exp\lambda[exp(i\theta) - 1] = e^{\lambda[e^{\theta i} - 1]}$$

3 Duality Principle

The duality principle plays a significant role in option pricing theory. This was carefully analyzed in [7] in terms of several assets whose price processes are driven by general semi-martingales.

The duality principle states that the calculation of a call option for a model with price process $S = exp(H)$ with respect to the measure $\mathbb{P}$ is equivalent to the calculation of the price of a put option for a suitable dual model with price process $S^* = \exp(H^*)$ with respect to a dual measure $\mathbb{P}^*$, where H in particular is a Lévy process.

In what follows, we shall assume that S is also a martingale on [0,T]; thus, the mathematical expectation of S at maturity is one (1), that is $\mathbb{E}(S_T) = 1$, this allows us to define on $(\mu, \mathfrak{B}, \Omega, \mathbb{F}(\mathfrak{B}))$ a new probability measure $\mathbb{P}^*$ with

$$\frac{d\mathbb{P}^*}{d\mathbb{P}} = S_T \tag{3.1}$$

Since S is a martingale and $0 \leq t \leq T$, then (3.1) gives:

$$\frac{d\mathbb{P}^*|^{\beta}{}_t}{d\mathbb{P}|^{\beta}{}_t} = S_t$$

and for $S > 0$ (p.a.s), we have that $\mathbb{P} \leq \mathbb{P}^*$ and $\frac{d\mathbb{P}}{d\mathbb{P}^*} = \frac{1}{S_T}$

Introducing the process:

$$S^* = \frac{1}{S} \tag{3.2}$$

and denoting H^* the dual of H with $H^* = -H$ gives:

$$S^* = \exp(H^*) \tag{3.3}$$

Remark 3.1 The following reveal the call-put duality in option pricing depending on the nature of option(s) that are involved.

Definition 3.1 *European call and put options* In the case of a standard a call and a put option, the payoff function on them is defined and denoted as:

$$\boldsymbol{f_T} = \begin{cases} (S_T - K)^+, & \text{for call options} \quad (3.4) \\ (K - S_T)^+, & \text{for put options} \quad (3.5) \end{cases}$$

where $K > 0$ is the strike price, and the associated option prices are given by the formulae:

$$\mathbb{E}[\boldsymbol{f_T}] = \begin{cases} C_T(S;K) = \mathbb{E}[(S_T - K)^+], & \text{for call options} \quad (3.6) \\ P_T(K;S) = \mathbb{E}[(K - S_T)^+], & \text{for put options} \quad (3.7) \end{cases}$$

for $\mathbb{E}$ the expectation operator with respect to the martingale measure $\mathbb{P}$.

Using $S = exp(H)$ in equation (3.6) with $\mathbb{E}[(\mathrm{S_T})]^+ = 1, KK^* = 1,$ and $S_T S^*{}_T = 1$ gives the below relation:

$$\mathbb{E}[\boldsymbol{f_T}] = \mathrm{C_T}(\mathrm{S}\,;\,\mathrm{K}) = \mathbb{E}[(\mathrm{S_T} - \mathrm{K})^+] = \mathbb{E}[\{\mathrm{K}S_T\left(\frac{1}{\mathrm{K}} - \frac{1}{\mathrm{S_T}}\right)\}^+$$

$$= \mathrm{K}\,\mathbb{E}[\{S_T\left(\frac{1}{\mathrm{K}} - \frac{1}{\mathrm{S_T}}\right)\}^+] = \mathrm{K}[\mathbb{E}^*(\mathrm{K}^* - \mathrm{S}^*)^+]$$

$$= \mathrm{K}\,\mathrm{P^*}_{\mathrm{T}}(\mathrm{K}^*, \mathrm{S}^*) \tag{3.8}$$

Similarly, using (3.7) yields the following relation:

$$\mathbb{E}\left[\boldsymbol{f_T}\right] = P_T(K\,;S) = \mathbb{E}\,[(\mathrm{K} - \mathrm{S_T})^+] = \mathrm{K}\,\mathrm{C^*}_{\mathrm{T}}(\mathrm{S}^*\,;\mathrm{K}^*) \tag{3.9}$$

Remark 3.2 Comparing (3.6) with (3.8) and (3.7) with (3.9) gives the following results:

Theorem 3 For standardized call and put options, the option prices satisfy the following duality relations

$$\frac{1}{\mathrm{K}}\mathrm{C_T}\,(\mathrm{S}\,;\mathrm{K}) = \mathrm{P^*}_{\mathrm{T}}(\mathrm{K}^*, \mathrm{S}^*)$$

and

$$\frac{1}{\mathrm{K}}\mathrm{P_T}\,(\mathrm{K}\,;\mathrm{S}) = \mathrm{C^*}_{\mathrm{T}}(\mathrm{S}^*\,;\mathrm{K}^*),$$

where $\mathrm{P^*}_{\mathrm{T}}(\mathrm{K}^*, \mathrm{S}^*)$ and $\mathrm{C^*}_{\mathrm{T}}(\mathrm{S}^*\,;\mathrm{K}^*)$ are the corresponding prices for put and call options respectively, with S^* as the price process computed with respect to the dual measure $\mathbb{P}^*$.

The proof of Theorem 3 is immediate from Remark 3.1 above.

4 Applications of Results

4.1 Options Parity

Suppose S is the price process of an underlying asset and $\mathbb{P}$ is the associated probability measure, then the call and put prices in markets $(S\,;\,\mathbb{P})$ and $(S^*;\mathbb{P}^*)$ satisfying the duality relations are connected by the following:

- Call-call parity:

$$C_T\,(S\,;K) = K\mathrm{C}^*{}_{\mathrm{T}}(\mathrm{S}^*\,;\mathrm{K}^*) + 1 - K \tag{4.1}$$

and

- Put - put parity:

$$P_T\,(K\,;S) = K\mathrm{P}^*{}_{\mathrm{T}}(\mathrm{K}^*\,;\mathrm{S}^*) + K\, - 1 \tag{4.2}$$

Recall from (3.4) and (3.5) that

$$f_T = \max(\mathrm{S_T} - \mathrm{K},\ 0) = (\mathrm{S_T} - \mathrm{K})^+ ,$$

hence, the following identity holds:

$$(\mathrm{S_T} - \mathrm{K})^+ = (\mathrm{K} - \mathrm{S_T})^+ + \mathrm{S_T} - \mathrm{K} \tag{4.3}$$

Taking mathematical expectation of (4.3) with respect to the measure $\mathbb{P}$ gives:

$$\mathbb{E}(S_\mathrm{T} - \mathrm{K})^+ = \mathbb{E}\,(\mathrm{K} - \mathrm{S_T})^+ + \mathbb{E}(\mathrm{S_T}) - \mathbb{E}(\mathrm{K})$$

$$\Rightarrow \qquad \mathrm{C_T}(\mathrm{S}\,;\ \mathrm{K}) = \mathrm{P_T}(\mathrm{K}\,;\ \mathrm{S}) + 1 - \mathrm{K} \tag{4.4}$$

Applying Theorem 3 and (4.1) on (4.4) gives:

$$\mathrm{C_T}(\mathrm{S}\,;\ \mathrm{K}) = K\mathrm{C}^*{}_{\mathrm{T}}(S^*\,;\mathrm{K}^*) + 1 - K \qquad (\textit{call - call parity})$$

Similarly, for put-put parity:

$$f_T = \max\big(\mathrm{K} - \mathrm{S_T}\ ,\ 0\big) = \big(\ \mathrm{K} - \mathrm{S_T}\big)^+ , \tag{4.5}$$

We recall from (4.3) and (4.4) that:

$$\mathrm{C_T}(\mathrm{S}\,;\ \mathrm{K}) = \mathrm{P_T}(\mathrm{K}\,;\ \mathrm{S}) + 1 - \mathrm{K} \quad \text{and} \quad \mathrm{C_T}\,(S\,;\mathrm{K}) = \mathrm{KP}^*{}_{\mathrm{T}}(\mathrm{K}^*, \mathrm{S}^*)$$

(From Theorem 3)

Hence, relating (4.5) and Theorem 3 with (4.4) gives:

$$\mathrm{C_T}(\mathrm{S}\,;\ \mathrm{K}) = \mathrm{P_T}(\mathrm{K}\,;\ \mathrm{S}) + 1 - \mathrm{K}$$

$$\mathrm{KP}^*{}_{\mathrm{T}}(\mathrm{K}^*, \mathrm{S}^*) = \mathrm{P_T}(\mathrm{K}\,;\ \mathrm{S}) + 1 - \mathrm{K} \tag{4.6}$$

Showing that:

$$\mathrm{P_T}(\mathrm{K}\,;\ \mathrm{S}) = \mathrm{KP}^*{}_{\mathrm{T}}(\mathrm{K}^*, S^*) + \mathrm{K} - 1 \qquad (\textit{put - put parity})$$

Whence, the markets $(S\,;\,\mathbb{P})$ and $(S^*;\mathbb{P}^*)$ satisfied the duality relation and are connected via the *call - call parity* and *put - put parity* Q.E.D.

4.2 The dual dynamics

Consider the stock price process of a risky asset defined by

$$S_t = S_0 e^{rt + L_t} \tag{4.7}$$

For a Lévy process L and r the interest rate, letting $L_t = \left(\lambda - r - \frac{\sigma^2}{2}\right) t + \sigma W_t$ in (4.7) gives:

$$S_t = S_0 e^{\left(\lambda - \frac{\sigma^2}{2}\right)t + \sigma W_t} \tag{4.8}$$

where λ is the drift parameter (rate of return), σ is the volatility rate and $W = \{W_t : t \epsilon \mathbb{R}_+\}$ is a standard Brownian motion. Therefore, the associated dynamics of the process is:

$$dS_t = S_t(\lambda dt + \sigma dW_t) \tag{4.9}$$

Suppose the (financial) market is purely volatile such that:

$$S = exp(H) \epsilon M(\mathbb{P})$$

i.e. S is a $\mathbb{P}$-martingale, such that (4.8) and (4.9) become:

$$S_t = exp\left\{\sigma W_t - \frac{\sigma^2}{2} t\right\} \tag{4.10}$$

and

$$dS_t = \sigma S_t dW_t \tag{4.11}$$

respectively, where $H = \left\{\sigma W_t - \frac{\sigma^2}{2} t\right\}$, then, the dual price process $S^* = e^{H^*} = e^{-H}$ under the dual measure P^* has stochastic differential

$$dS_t{}^* = -\sigma S_t{}^*(dW_t - \sigma dt), \text{ with } S_0 = 1$$

Proof:

$$S_t = exp\left\{\sigma W_t - \frac{\sigma^2}{2} t\right\}, \text{ with } S_0 = 1 \text{ shows that:}$$

$$S_t = exp(H^*) \ , \ H = exp\{\sigma W_t - \frac{\sigma^2}{2} t\}$$

$$\therefore \quad S_t = exp(H^*) = exp(-H)$$

$$= exp\left(\frac{\sigma^2}{2} t - \sigma W_t\right) \tag{4.12}$$

Applying one-dimensional version of Itô formula on (4.12), with $X = S^* = W$

implies that:

$$dS_t^{\,*} = dW_t, \quad \text{for}$$

g ≡ 0, f ≡ 0, and $\quad u(t,S^*) = exp\left(\frac{\sigma^2}{2}t - \sigma S_t^{\,*}\right)$

hence,

$$\frac{\partial u}{\partial t} = \frac{\sigma^2}{2}S_t^{\,*}, \quad \frac{\partial u}{\partial S^*} = -\sigma S_t^{\,*}, \qquad \frac{\partial^2 u}{\partial S^{*2}} = \sigma^2 S_t^{\,*}$$

But

$$du(t,S^*) = \left(\frac{\partial u}{\partial t} + g\frac{\partial u}{\partial S^*} + \frac{1}{2}f^2\frac{\partial^2 u}{\partial S^{*2}}\right)dt + f\frac{\partial u}{\partial S^*}dW_t$$

$$\therefore \quad d\,S^* = \left[\frac{\sigma^2}{2}S_t^{\,*} + \frac{1}{2}\sigma^2 S_t^{\,*}\right]dt + -\sigma S_t^{\,*}dW_t$$

$$= \sigma^2 S_t^{\,*}dt - \sigma S_t^{\,*}\,dW_t$$

$$= -\sigma S_t^{\,*}\,(dW_t - \sigma dt) \qquad \square$$

4.3 Remark

(i) Considering (4.10) and (4.11) above, where the market model is purely volatile, if the volatility rate is so small; (i.e. $\sigma \to 0$), then the associated price process $S_t = exp(H_t)$ and its dual process $S_t^{\,*} = epx(H_t^{\,*}) = epx(-H_t)$ become a constant say (S_0). That is;

$$S_t = S_t^{\,*} = S_0$$

with dynamics process $\quad dS_t^{\,*} = dW_t = 0, \quad$ where $\mathbb{E}(\mathrm{S_T}) = \mathbb{E}(\mathrm{S_0}) = 1.$

This shows the coincidence of the associated underlying asset price process with its corresponding dual process.

(ii) In (4.8), if the volatility rate is again so small (i.e. $\sigma \to 0$), then the asset price process becomes $S_t = S_0 exp(\lambda t)$ and the corresponding dynamics of the price process is purely deterministic. In this case, the asset process grows without bound, where $dS_t = \lambda S_t dt$.

Showing that the underlying asset is virtually riskless. At this time, the payoff of a European call option with this underlying asset is:

$$f_T = (\mathrm{S_T} - \mathrm{K})^+ = (S_0 e^{\lambda T} - K)^+ \quad .$$

5 Concluding Remarks

The previous sections with regard to the corresponding remarks reveal the strength of the duality principle from the theoretical point of view and its usefulness in Mathematical finance. The models are quite powerful candidates for an incomplete market. Our next target is to verify the immense usefulness of this work in the actual world in terms of computational analysis.

References

[1] D. Applebaum, *Lévy Processes and Calculus*, Second edition, Cambridge University Press, 2009.

[2] L. Bachelier, *Theoriede laspeculation*, Paris, Gauthier–villars, 1900, Translated in Cootner 1964.

[3] D.S. Bates, The Skewness Premium: Option pricing under asymmetric process, *Advances in Futures and Options Research*, *Elsevier*, **9**, (1997), 51-82.

[4] F. Black, and M. Scholes, The Pricing of Options and Corporate Liabilities,. *Journal of Political Economy*, **81**(3), (1973), 637-654.

[5] F. Bolshuizen, A.W. van der Vaar, H. van Zanteen et. al., *Stochastic Processes for Finance-Risk Management Tools.* www.math.sc.edu/Bolshuizen/, 2006.

[6] E. Eberlein, and K. Prause, *The General Hyperbolic Model: Financial Derivatives and Risk Measures*, FDM preprint 56, University of Freiburg, 1998.

[7] E. Eberlein, A. Papapantoleean and A.N. Shirryaev, *Esscher Transform and Duality Principle for Multidimensional Semimartingale,* www.math.sc.edu/EPS/, 2009.

[8] S.O. Edeki, *Applications of Lévy Processes in Finance- duality principle approach,* Unpublished, Msc Thesis, Mathematics Department, University of Ibadan, Nigeria, 2010.

[9] J. Fajardo and E. Mordecki, Skewness Premium with Lévy Processes, *Working Paper*, IBMEC, (2006).

[10] J. Fajardo, and E. Mordecki, Symmetry and Duality in Levy Markets, *Quant. Finance* , **6**, (2006), 219-227.

[11] A. Friedman, *Stochastic Differential Equations and Applications,* Academic Press, New York, San Francisco, London, 1975.

[12] J.M. Harrison and S. R. Pliska, Martingales and Stochastic Integrals in the Theory of Continuous Trading, *Stochastic Processes and their Applications*, **11**(3), (1981), 215-260.

[13] R.V. Ivanov, On the Pricing of American Options in Exponentially *Lévy* Markets, *J. Applied Prob.*, **44**, (2007), 407-414.

[14] J. Kallsen and A.N. Shiryaev, The Cumulant Process and Esscher's Change of Measure, *Finance and Statistics*, **6**(4), (2002), 397-428.

[15] P.E. Klodden and E. Plateu, *Numerical Solutions of Stochastic Differential Equations,* Springer-Verlag, 1992.

[16] B.K. Leonib and G. Yakov, *Theory of Probability and Random Processes,* Second edition Springer-Verlag, New York, 2007.

[17] D.B. Madan and E. Senata, Chebyshev Polynomial Approximations and Characteristic Function Estimation, *Journal of the Royal Statistical Society* Series B, **49**(2), (1987), 163-169.

[18] B. Mandelbrot, The Variation of certain speculative prices, *Journal of Business*, **36**, (1963), 394-169.

[19] K. Massaki, *Stochastic Processes with Applications to Finance*, www.math./ Massaki/ , 2001.

[20] W. Margrabe, The Value of an Option to exchange one asset for another. *Journal of Finance*, **33**, (1978), 177-186.

[21] Y. Miyahara, *A Note on Esscher Transformed Martingale Measures for Geometric Lévy processes.* E-mail: y-miya@econ.nagoya-cu.ac.jp, 2004.

[22] D. Nualart and W. Schoutens, *BSDE's and Feynman-Kac Formula for Lévy Processes with Applications in Finance.* www.math.sc.edu, 2001.

[23] B. Oksanda, *Stochastic Differential Equations: An introduction with Applications*, Sixth edition, Springer-Verlag, 2000.

[24] M.F.M, Osborne, Rational Theory of Warrant pricing, *Industrial Management review*, **6**, (1965), 13-32.

[25] A. Papapantoleon, *Applications of semimartingales and Lévy processes in Finance-duality and valuation*, PhD Thesis, Univ. Freiburg, 2007.

[26] K. Prause, *The Generalized Hyperbolic Model: Estimation, Financial Derivatives, and Risk Measures.* Dissertation. Mathematische Fakultät der Albert-Ludwigs-Universität Freiburg im Breisgau, 1999.

[27] S. Raible, *Lévy processes in Finance- Theory, Numerics, and Emperical Facts*, www.math.sc.edu/Raible/ , 2000.

[28] A.A.F. Saib and M. Bhuiruth, *Option Pricing of jump Diffusion Models under Exponential Lévy models using Mathematica*, 2008.

[29] P. Samuelson, Economics Theory and Mathematics – An appraisal, Cowles Foundation Paper 61, Reprinted from *American Economic Review*, **42**, (1952), 56-69.

[30] A.N. Shiryaev, *Essentials of Stochastic Finance – Facts, Models, Theory*, World Scientific, 1999.