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Volatility of the European Stock Market Indices During the Global Financial Crisis - A New Proposal of Stochastic Volatility

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Abstract

This paper estimates the volatility of most important European stock market indices during the global financial crisis started in 2008, such as DAX, CAC_{40}, $FTSE_{100}$, among others. The estimation of volatility is made from a new family of stochastic volatility models proposed by Santos, Franco, Gamerman [33, 17] and extended to distributions of heavy tails by Pinho, Franco, Silva [32]. This new family of models denoted by non-Gaussian State Space Models (NGSSM) is a subclass of state space models where it is possible to compute the exact likelihood. It is also estimated volatility of the series by APARCH model and the results showed that NGSSM has a significantly better performance.

Mathematics Subject Classification: 62M10
Keywords: Volatility models, Heavy Tailed Distributions, NGSSM, Classical and Bayesian Inferences.

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1 Introduction

The last financial crises, which occurred all over the world, have been characterized by the speed in with which volatility spreads around the global stock markets. One of the main concerns, among economists and the population in general, is how long this crisis will last. Therefore, there has been recently an outburst of researches focusing on the study and modeling of volatility.

Relying on the fact that the unconditional distribution of daily returns has fatter tails than the normal distribution, the usual time series models, such as VAR and ARIMA, which assume the normality and homoscedasticity assumptions, are not appropriate for modeling volatility. Thus, more adequate procedures, whose conditional variance varies in time, have been proposed.

In the literature there are many works about heteroscedastic models, such as ARCH proposed by Engle [13], GARCH proposed by Bollerslev [6], multivariate GARCH proposed by Bauwens, Laurent, Rombouts [4], EGARCH proposed by Nelson [30], TGARCH proposed by Zakoian [42], stochastic volatility model proposed by Taylor [39] and multivariate stochastic volatility model proposed by Harvey, Ruiz, Shephard [23].

West, Harrison, Migon [40] proposed the Dynamic Generalized Linear Models (DGLM) that attracted an immense interest due to great applicability in diverse areas of the knowledge, but the analytical form is easily lost, even using very simple components.

Santos, Franco, Gamerman [33, 17] proposed a Non Gaussian State Space Models - NGSSM, a generalization of the Smith-Miller result [37]. This procedure comprises a dynamic model with exact evolution equation, as well as transformations one by one of the series, allowing the analytical integration of the states and the obtaining of the exact likelihood function, as well the predictive distributions one step ahead.

Pinho, Franco, Silva [32] proposed other distributions (all are heavy tail distributions) that are special cases of the NGSSM, that are the Log-normal, Log-gamma, Fréchet, Lévy and Skew GED. Santos, Franco, Gamerma [33, 17] presented two distributions heavy tails, the Pareto and Weibull.

The paper is organized as follows. Section 2 defines the NGSSM. Section 3 shows the GARCH and APARCH models. Section 4 discusses about the inferential procedure in NGSSM. Section 5 shows the results of the models fitted to the real series and Section 6 concludes the work.

2 Non Gaussian State Space Model - NGSSM

In this section, the models are introduced. They are given in this formulation in [33, 17]. The main advantage of these models compared to the DGLM is that exact inference can be performed.

A time series $\{y_t\}$ is in this class of models if it satisfies the following assumptions:

A0 Its probability (density) function can be written in the form:

$$p(y_t|\mu_t, \boldsymbol{\varphi}) = q(y_t, \boldsymbol{\varphi})\mu_t^{r(y_t,\boldsymbol{\varphi})} \exp\left(-\mu_t s(y_t, \boldsymbol{\varphi})\right), \text{ for } y_t \in H(\boldsymbol{\varphi}) \subset \Re \quad (1)$$

and $p(y_t|\mu_t, \boldsymbol{\varphi}) = 0$, otherwise. Functions $q(\cdot)$, $r(\cdot)$, $s(\cdot)$ and $H(\cdot)$ are such that $p(y_t|\mu_t, \boldsymbol{\varphi}) \geq 0$ and therefore $\mu_t > 0$, for all $t > 0$. It is also assumed that $\boldsymbol{\varphi}$ varies in the p-dimensional parameter space Φ.

A1 If x_t is a covariate vector, the link function g relates the predictor to the parameter μ_t through the relation $\mu_t = \lambda_t g(x_t, \boldsymbol{\beta})$, where $\boldsymbol{\beta}$ are the regression coefficients (one of the components of $\boldsymbol{\varphi}$) and λ_t is the latent state variable related to the description of the dynamic level. If the predictor is linear, then $g(x_t, \boldsymbol{\beta}) = g(x_t'\boldsymbol{\beta})$.

A2 The dynamic level λ_t evolves according to the system equation $\lambda_{t+1} = \omega^{-1}\lambda_t\varsigma_{t+1}$, where $\varsigma_{t+1}|\boldsymbol{Y}_t \sim Beta\,(\omega a_t, (1-\omega)a_t)$, $0 < \omega \leq 1$, $t = 1, 2, ...$, that is, $\omega\frac{\lambda_{t+1}}{\lambda_t} \mid \lambda_t, \boldsymbol{Y}_t \sim Beta\,(\omega a_t, (1-\omega)a_t)$, $\boldsymbol{Y}_t = \{Y_0, y_1, \ldots, y_t\}$ and Y_0 represents previously available information.

A3 The dynamic level λ_t is initialized with prior distribution $\lambda_0|Y_0 \sim Gamma(a_0, b_0)$.

There is a wide range of distributions that belong to this class of models. It includes many commonly known discrete and continuous distributions such as Poisson, Gamma and Normal (with static mean) but also includes many other distributions that are not so commons.

Santos, Franco, Gamerman [33, 17] and Pinho, Franco, Silva [32] found some special cases of the NGSSM as follow in the Table 1, which provides the form of functions $q(\cdot)$, $r(\cdot)$, $s(\cdot)$ and $H(\cdot)$ for distributions in this family. The more common cases such as Poisson and Exponential were previously singled out in the literature. Several other cases of this family are introduced here and they include continuous and discrete distributions. Some of them are well known such as Normal and Pareto, but the family includes also the Borel-Tanner and the Rayleigh distributions, for example.

Table 1: Cases of the NGSSM

Model	Tails	$q\left(y_t,\, \boldsymbol{\varphi}\right)$	$r\left(y_t,\, \boldsymbol{\varphi}\right)$	$s\left(y_t,\, \boldsymbol{\varphi}\right)$	$H\left(\boldsymbol{\varphi}\right)$
Log-Normal	heavy	$\left[\left(y_t-\gamma\right)\sqrt{2\pi}\right]^{-1}$	$\frac{1}{2}$	$\frac{[\ln(y_t-\gamma)-\delta]^2}{2}$	(γ,∞)
Log-Gamma	heavy	$\frac{\alpha^{\alpha}[ln(y_t)]^{\alpha-1}}{[\Gamma(\alpha)y_t]}$	α	$\alpha\ln\left(y_t\right)$	$(1,\infty)$
Fréchet	heavy	$\alpha\left(y_t-\gamma\right)^{-\alpha-1}$	1	$\left(y_t-\gamma\right)^{-\alpha}$	(γ,∞)
Lévy	heavy	$\left[2\pi\left(y_t-\gamma\right)\right]^{-\frac{3}{2}}$	$\frac{1}{2}$	$\left[2\left(y_t-\gamma\right)\right]^{-1}$	(γ,∞)
Skew GED	heavy/light	$\frac{\kappa}{\Gamma\left(\alpha^{-1}\right)\left(1+\kappa^2\right)}$	$\frac{1}{\alpha}$	$\left[\frac{(y_t-\delta)^+}{k^{-\alpha}}\right]^{\alpha}+\left[\frac{(y_t-\delta)^-}{k^{\alpha}}\right]^{\alpha}$	$(-\infty,\infty)$
Pareto	heavy	y_t^{-1}	1	$\ln\left(y_t\right)$	$(1,\infty)$
Weibull	heavy/light	$\upsilon y_t^{\upsilon-1}$	1	y_t^{υ}	$(0,\infty)$
Poisson	light	$\left(y_t!\right)^{-1}$	y_t	1	$\{0,1,\ldots\}$
Borel-Tanner	light	$\frac{\gamma}{(y_t-\gamma)!}y_t^{y_t-\gamma-1}$	$y_t-\gamma$	y_t	$\{\gamma,\gamma+1,\ldots\}$
Gamma	light	$\frac{\alpha^{\alpha}y_t^{\alpha-1}}{\Gamma(\alpha)}$	α	αy_t	$(0,\infty)$
Normal	light	$[2\pi]^{-\frac{1}{2}}$	$\frac{1}{2}$	$\frac{(y_t-\gamma)^{-2}}{2}$	$(-\infty,\infty)$
Laplace	light	$\frac{1}{\sqrt{2}}$	1	$\sqrt{2}\left\|y_t-\gamma\right\|$	$(-\infty,\infty)$
Inverse Gaussian	light	$\frac{1}{\sqrt{2\pi y_t^3}}$	$\frac{1}{2}$	$\frac{(y_t-\gamma)^{-2}}{2y_t\gamma^2}$	$(0,\infty)$
Rayleigh	light	y_t	1	$\frac{1}{2}y_t^2$	$(0,\infty)$
Generalized Gamma	light	$\frac{\upsilon y_t^{\alpha-1}}{\Gamma\left(\frac{\alpha}{\upsilon}\right)}$	1	y_t^{υ}	$(0,\infty)$

Theorem 2.1 in [33, 17] below provides basic results of these models for sequential or on-line inference for the level λ_t (filtering results) and the predictive distribution.

Theorem 2.1. *If the model is defined in* ***A0-A3****, the following results, for t=1, 2, ..., can be obtained:*

1. *The prior distribution* $\lambda_t|\boldsymbol{Y}_{t-1},\boldsymbol{\varphi}$ *follows a Gamma(*$a_{t|t-1}$*,* $b_{t|t-1}$*) distribution such that*

$$a_{t|t-1} = \omega a_{t-1}, \tag{2}$$

$$b_{t|t-1} = \omega b_{t-1}. \tag{3}$$

2. *The on-line or updated distribution of* $\lambda_t|\boldsymbol{Y}_t,\boldsymbol{\varphi}$ *is Gamma (*a_t*,*b_t*), where*

$$a_t = a_{t|t-1} + r(y_t,\boldsymbol{\varphi}), \tag{4}$$

$$b_t = b_{t|t-1} + s(y_t,\boldsymbol{\varphi}). \tag{5}$$

3. The one step ahead predictive density function is given by

$$p(y_t|\boldsymbol{Y}_{t-1},\boldsymbol{\varphi}) = \frac{\Gamma(r(y_t,\boldsymbol{\varphi}) + a_{t|t-1})q(y_t,\boldsymbol{\varphi})(b_{t|t-1})^{a_{t|t-1}}}{\Gamma(a_{t|t-1})[s(y_t,\boldsymbol{\varphi}) + b_{t|t-1}]^{r(y_t,\boldsymbol{\varphi})+a_{t|t-1}}}, \quad y_t \in H(\boldsymbol{\varphi}), \tag{6}$$

$\forall t \in N$ *and* $\Gamma(\cdot)$ *is the gamma function.*

For more details and the proof of Theorem 2.1 can be found in [33, 17].

3 APARCH models

The Autoregressive Conditional Heteroscedasticity (ARCH) models were proposed by Engle [13] and the Generalized Autoregressive Conditional Heteroscedasticity (GARCH) models were proposed by Bollerslev [6, 7] and [8]. The GARCH models are defined as follows.

$$y_t = \sigma_t \epsilon_t \tag{7}$$

$$\sigma_t^2 = \theta_0 + \sum_{i=1}^{p} \theta_i \epsilon_{t-1}^2 + \sum_{j=1}^{q} \phi_j \sigma_{t-j}^2 \tag{8}$$

where $\theta_0 > 0$, $\theta_i \geq 0$, $\phi_j \geq 0$ and $\sum_{k=1}^{r} (\theta_k + \phi_k) < 1$ with $i = 1,\ldots,p$, $j = 1,\ldots,q$ and $r = max\,(p,q)$.

In the literature, many extensions have been proposed for the ARCH-GARCH. These extensions take into account the asymmetry, different distributions, regime change, among others. There may be mentioned Logarithmic GARCH (Log-GARCH) proposed independently in slightly different forms by [18, 31] and [29], the Exponential GARCH proposed by [30], Nonlinear GARCH (NGARCH) proposed by [24], GJR-GARCH proposed by Glosten *et al.* (1993), Asymmetric Power ARCH (APARCH) proposed by Ding *et al.* (1993), Threshold GARCH (TGARCH) proposed by [42], Fractionally Integrated GARCH (FIGARCH) proposed by [3], the Asymmetric Threshold GARCH (ATGARCH) proposed by [11], among many other works.

In this paper the NGSSM are compared with the APARCH models, because Laurent shows that the APARCH models are extensions of the GARCH models [27], as it was mentioned above. The APARCH(p,q) models proposed by Ding, Granger, Engle [12] are defined as follows:

$$y_t = \sigma_t \epsilon_t \tag{9}$$

$$\sigma_t^\delta = \theta_0 + \sum_{i=1}^{p} \theta_i k\left(\epsilon_{t-i}\right)^\delta + \sum_{j=1}^{q} \phi_j \sigma_{t-j}^\delta \tag{10}$$

$$k\left(\epsilon_{t-i}\right) = |\epsilon_{t-i}| - \gamma_i \epsilon_{t-i} \tag{11}$$

where $\theta_0 > 0$, $\theta_i \geq 0$, $\delta \geq 0$, $-1 < \gamma_i < 1$, $\phi_j \geq 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$.

Then, the proposal of this model imposes a Box Cox power transformation of the conditional standard deviation process and the asymmetric absolute residuals. Thus, this proposal is aligned with [10], who shows that the asymmetric response of volatility to positive and negative shocks produces the leverage effect of the stock market returns. Laurent shows that the APARCH includes seven other ARCH extensions as special cases [27], that follow:

1. ARCH by Engle [13], when $\delta = 2$, $\gamma_i = 0$ and $\phi_j = 0$ ($i = 1, \ldots, p$; $j = 1, \ldots, q$);

2. GARCH by Bollerslev [6], when $\delta = 2$ and $\gamma_i = 0$ ($i = 1, \ldots, p$);

3. GARCH by Taylor [38] and Schwert:1990, when $\delta = 1$ and $\gamma_i = 0$ ($i = 1, \ldots, p$);

4. GARCH-GJR by Glosten *et al.* (1993), when $\delta = 2$;

5. TARCH by Zakoian [42], when $\delta = 2$;

6. NARCH by Higgins and Bera [24], when $\gamma_i = 0$ and $\phi_j = 0$ ($i = 1, \ldots, p$; $j = 1, \ldots, q$);

7. Log-ARCH by Geweke [18], Pantalu [31] and MILHA [29], when $\delta \to \infty$.

4 Inference for hyperparameters of the NGSSM models

The model parameters are divided into the latent states $\{\lambda_t\}$ and fixed parameters $\boldsymbol{\varphi}$, usually called hyperparameters. The *on-line* and smoothed inference for the state parameters are presented and the knowledge of the hyperparameters is assumed in both cases.

In this section, inference for the hyperparameters and the latent states is discussed.

4.1 Classical Inference

One way of making classical inference about the parameter vector $\boldsymbol{\varphi}$ is through the marginal likelihood function, whose form is given by

$$L(\boldsymbol{\varphi};\boldsymbol{Y}_n)=\prod_{t=1}^{n}p(y_t|\boldsymbol{Y}_{t-1},\boldsymbol{\varphi})=\prod_{t=1}^{n}\frac{\Gamma(r(y_t,\boldsymbol{\varphi})+a_{t|t-1})q(y_t,\boldsymbol{\varphi})(b_{t|t-1})^{a_{t|t-1}}}{\Gamma(a_{t|t-1})[s(y_t,\boldsymbol{\varphi})+b_{t|t-1}]^{r(y_t,\boldsymbol{\varphi})+a_{t|t-1}}}, \tag{12}$$

where $y_t \in H(\boldsymbol{\varphi})$ and $\boldsymbol{\varphi}$ is composed by ω, β and by parameters of the specific model. Maximization of the marginal likelihood function (12) is typically performed numerically.

The asymptotic confidence interval for $\boldsymbol{\varphi}$ is built based on a numerical approximation for the Fisher information matrix $I_n(\boldsymbol{\varphi})$, using $I_n(\boldsymbol{\varphi}) \cong -G(\boldsymbol{\varphi})$, where $-G(\boldsymbol{\varphi})$ is the matrix of second derivatives of the log-likelihood function with respect to the parameters. As the computation of the derivatives is not an easy task, numerical derivatives are used (see [15]).

Let φ_i, $i=1,\ldots,p$, be any component of $\boldsymbol{\varphi}$. Then, an asymptotic confidence interval of $100(1-\kappa)\%$ for φ_i is given by

$$\hat{\varphi}_i \pm z_{\kappa/2}\sqrt{\widehat{Var}(\hat{\varphi}_i)},$$

where $z_{\kappa/2}$ is the $\kappa/2$ percentile of the standard normal distribution and $\widehat{Var}(\hat{\varphi}_i)$ is obtained from the diagonal elements of the Fisher information matrix.

Pinho, Franco, Silva [32] presented Monte Carlo results comparing Bayesian and classical inferences methods in the estimation of the NGSSM parameters for the heavy tailed distributions (see Table 1). In that work an important classical inference problem is appointed, the parameter ω (known as the discount factor) presented, for small series, a large bias. This problem do not affect the inferential process here, once all series have more than 1.000 observations.

4.2 Bayesian Inference

Bayesian inference for $\boldsymbol{\varphi}$ can be performed using MCMC algorithms proposed by Gamerman and Lopes [16], since the posterior distribution of the hyperparameter is not analytically tractable. The marginal posterior distribution of parameter vector $\boldsymbol{\varphi}$ is given by

$$\pi(\boldsymbol{\varphi}|\boldsymbol{Y}_n) \propto L(\boldsymbol{\varphi}; \boldsymbol{Y}_n)\pi(\boldsymbol{\varphi}), \tag{13}$$

where $L(\boldsymbol{\varphi}; \boldsymbol{Y}_n)$ is the likelihood function defined in (12) and $\pi(\boldsymbol{\varphi})$ is the prior distribution for $\boldsymbol{\varphi}$. In this work, proper uniform priors are used for $\boldsymbol{\varphi}$.

Credibility intervals for φ_i, $i = 1, ..., p$ are built as follows. Given a value $0 < \kappa < 1$, the interval $[c_1, c_2]$ satisfying

$$\int_{c_1}^{c_2} \pi(\varphi_i \mid \boldsymbol{Y}_n)\, d\varphi_i = 1 - \kappa$$

is a credibility interval for φ_i with level $100(1-\kappa)\%$.

4.3 Stochastic volatility smoothed

Inference for the latent variables (volatility) can be made with the output from the MCMC algorithm proposed by Santos, Franco and Gamerman [33]. Once a sample $\boldsymbol{\varphi}^{(1)}, ..., \boldsymbol{\varphi}^{(M)}$ is available, posterior samples $\lambda^{(1)}, ..., \lambda^{(M)}$ from the latent variables are obtained as follows:

1. set $j = 1$;
2. sample the hyperparameter $\boldsymbol{\varphi}^{(j)}$ from the MCMC algorithm;
3. sample the set $\lambda^{(j)}$ of latent variables from $p(\lambda|\boldsymbol{\varphi}^{(j)}, \boldsymbol{Y}_n)$ using Theorem 3;

4. increase $j \to j+1$ and return to step 1, until $j = M$.

Again, similar comments are valid for the predictive distributions. Note that

$$p(y_{t+h}|\boldsymbol{Y}_t) = \int p(y_{t+h}|\boldsymbol{Y}_t, \boldsymbol{\varphi})\ \pi(\boldsymbol{\varphi}|\boldsymbol{Y}_n)\ d\boldsymbol{\varphi}.$$

Thus, h-step-ahead predictive distributions can be approximated by

$$\frac{1}{M}\sum_{j=1}^{M} p(y_{t+h}|\boldsymbol{Y}_t, \boldsymbol{\varphi}^{(j)})$$

from which summaries such as means, variances and credibility intervals can be obtained.

4.4 Methods of adequacy and choice of models

After fitting the model to the data, it is necessary to verify the adequacy model. In literature, there are many suggested methods of diagnosis and below are described two proposals.

Harvey and Fernandes [22] suggest some methods of diagnosis, based on standardized residuals, also known as residuals of the Pearson. These residuals are defined by:

$$r_t^p = \frac{y_t - E\left(y_t\left|\boldsymbol{Y_{t-1}}, \boldsymbol{\varphi}\right.\right)}{\sqrt{Var\left(y_t\left|\boldsymbol{Y_{t-1}}, \boldsymbol{\varphi}\right.\right)}}$$

The another alternative is to use the residuals called by deviance proposed by McCulagh and Nelder [28], which can be expressed as:

$$r_t^d = \left\{2ln\left[\frac{p\left(y_t\left|y_t, \boldsymbol{\varphi}\right.\right)}{p\left(y_t\left|\hat{\phi}_t, \boldsymbol{\varphi}\right.\right)}\right]\right\}^{\frac{1}{2}},$$

where $\hat{\phi}_t = E\left(y_t\left|\boldsymbol{Y_{t-1}}, \boldsymbol{\varphi}\right.\right)$.

The authors propose the following analysis of the residuals:

1. Examining the plot of the residuals vs. time and vs. an estimate component level.

2. Verify that the sample variance of the standardized residuals is close 1. A value greater than 1 indicates overdispersion relative to the model which is fitted to the data.

More details about these methods of adequacy can be seen in [33].

There are not rare times when you get a more appropriate model for the data and when this occurs is necessary a criterion for determining the best model. According to Harvey [21] the AIC and BIC criteria proposed by Akaike [1] and Schwarz [34], respectively, are suitable. The AIC and BIC criteria are defined by:

$$AIC = -2l\left(\hat{\boldsymbol{\varphi}}\right) + 2k$$

and

$$BIC = -2l\left(\hat{\boldsymbol{\varphi}}\right) + 2k\ln\left(n\right),$$

where $l\left(\cdot\right)$ is the log-likelihood function, k number of parameters and n the number of observations.

Hurvich and Tsai [25] proposed AICc which is a correction of the AIC. Burnham and Anderson [5] strongly recommend using AICc, rather than AIC, if n is small or k is large. The AICc criterion is defined by:

$$AICc = AIC + \frac{2k\left(k+1\right)}{n-k-1}$$

In this work, to verify the adequacy model is used the deviance and for comparison of models is used AICc and BIC criteria.

5 Application to real time series

In this subsection, some the NGSSM stochastic volatility were fitted to some of main stock market indexes in the world, such as DAX (Germany), CAC_{40} (France), $FTSE_{100}$ (United Kingdom), $FTSE_{MIB}$ (Italy), $IBEX_{35}$ (Spain) and PSI_{20} (Portugal). The period of the series between 01/02/2007 to 12/06/2011 and each series has, respectively, 1261, 1264, 1247, 1244, 1256 and 1090 observations.

Figure 1 presents the indexes and the log-returns of the six series. It can be observed, in all cases, two volatility clusters, the first and biggest is observed around the instants 400 and 500, which correspond to the second semester of 2008, period of the global financial crisis in 2008. The second volatility clusters the instants observed around the instants 1200, which are in the second semester of 2011, period that the crisis intensified in Europe. In this, it is easy still to observe that the biggest volatility clusters are in Germany and France, because these countries are more exposed to the European crisis.

Table 2: Likelihood, AIC and BIC values for the NGSSM and APARCH models fitted to the DAX and CAC_{40} series.

SERIES	NGSSM	AICc	BIC	LN LIKE	APARCH(1,1)	AICc	BIC	LN LIKE
	LOGNORMAL	-15.44	-15.43	9737.39	NORMAL	1.55	1.57	-973.50
	LOGGAMMA	-15.77	-15.76	9949.14	SKEW NORMAL	-13.54	-13.52	8539.39
	PARETO	-15.16	-15.15	9558.32	t-STUDENT	-13.99	-13.98	8826.46
DAX	**WEIBULL**	**-15.79**	**-15.78**	**9959.21**	SKEW t-STUDENT	-11.81	-11.80	7452.30
	FRÉCHET	-14.79	-14.77	9325.42	GED	-14.34	-14.32	9044.14
	LÉVY	-14.04	-14.03	8855.78	**SKEW GED**	**-15.26**	**-15.24**	**9622.73**
	LAPLACE	-13.94	-13.93	8791.30				
	SKEW LAPLACE	-15.02	-15.00	9471.75				
	GED	-14.51	-14.50	9154.58				
	SKEW GED	-15.49	-15.48	9771.27				
	LOGNORMAL	-15.08	-15.07	9536.28	NORMAL	1.53	1.55	-962.77
	LOGGAMMA	-15.39	-15.38	9732.31	SKEW NORMAL	-13.41	-13.39	8476.82
	PARETO	-14.91	-14.91	9428.08	t-STUDENT	-13.79	-13.78	8721.14
CAC_{40}	**WEIBULL**	**-15.42**	**-15.41**	**9748.88**	SKEW t-STUDENT	-14.04	-14.03	8880.02
	FRÉCHET	-14.67	-14.65	9272.11	GED	-14.01	-14.00	8859.66
	LÉVY	-14.14	-14.13	8938.04	**SKEW GED**	**-14.87**	**-14.85**	**9401.88**
	LAPLACE	-13.73	-13.71	8677.39				
	SKEW LAPLACE	-14.78	-14.77	9347.10				
	GED	-14.29	-14.28	9033.88				
	SKEW GED	-15.27	-15.25	9653.61				

Table 3: Likelihood, AIC and BIC values for the NGSSM and APARCH models fitted to the $FTSE_{100}$ and $FTSE_{MIB}$ series.

SERIES	NGSSM	AICc	BIC	LN LIKE	APARCH(1.1)	AICc	BIC	LN LIKE
	LOGNORMAL	-15.64	-15.63	9757.34	NORMAL	-13.24	-13.22	8259.48
	LOGGAMMA	-15.94	-15.93	9942.63	SKEW NORMAL	-14.08	-14.07	8784.03
	PARETO	-15.48	-15.48	9656.82	t-STUDENT	-14.41	-14.40	8989.56
$FTSE_{100}$	**WEIBULL**	**-15.97**	**-15.96**	**9958.95**	SKEW t-STUDENT	-14.93	-14.91	9312.05
	FRÉCHET	-15.19	-15.17	9471.33	GED	-14.61	-14.59	9112.19
	LÉVY	-14.70	-14.69	9167.93	**SKEW GED**	**-15.47**	**-15.46**	**9650.76**
	NORMAL	-12.46	-12.45	7770.50				
	SKEW NORMAL	-7.45	-7.43	4649.11				
	LAPLACE	-14.32	-14.30	8929.05				
	SKEW LAPLACE	-15.37	-15.35	9586.66				
	GED	-14.88	-14.87	9279.78				
	SKEW GED	-15.80	-15.78	9854.99				
	LOGNORMAL	-14.90	-14.89	9271.11	NORMAL	-12.60	-12.58	7840.07
	LOGGAMMA	-15.24	-15.22	9479.46	SKEW NORMAL	-13.34	-13.32	8298.45
	PARETO	-14.77	-14.76	9188.63	t-STUDENT	-13.63	-13.61	8479.02
$FTSE_{MIB}$	**WEIBULL**	**-15.27**	**-15.25**	**9498.85**	SKEW t-STUDENT	-13.95	-13.93	8678.42
	FRÉCHET	-14.45	-14.44	8991.81	GED	-13.89	-13.87	8640.99
	LÉVY	-13.92	-13.91	8660.91	**SKEW GED**	**-14.45**	**-14.44**	**8993.03**
	NORMAL	-11.72	-11.70	7290.19				
	SKEW NORMAL	-7.42	-7.40	4616.71				
	LAPLACE	-13.56	-13.55	8436.73				
	SKEW LAPLACE	-14.63	-14.62	9105.18				
	GED	-14.13	-14.11	8788.87				
	SKEW GED	-15.11	-15.10	9405.34				

The models in the tables were adjusted, using both Bayesian and classical inferences, to the time series with an one-day delay as a covariate, using the logarithm link function, that is, $\mu_t = \lambda_t exp\left\{\beta X_t\right\}$, where X_t is one day delay log-return. The irregularity of the data due to the holidays and weekends

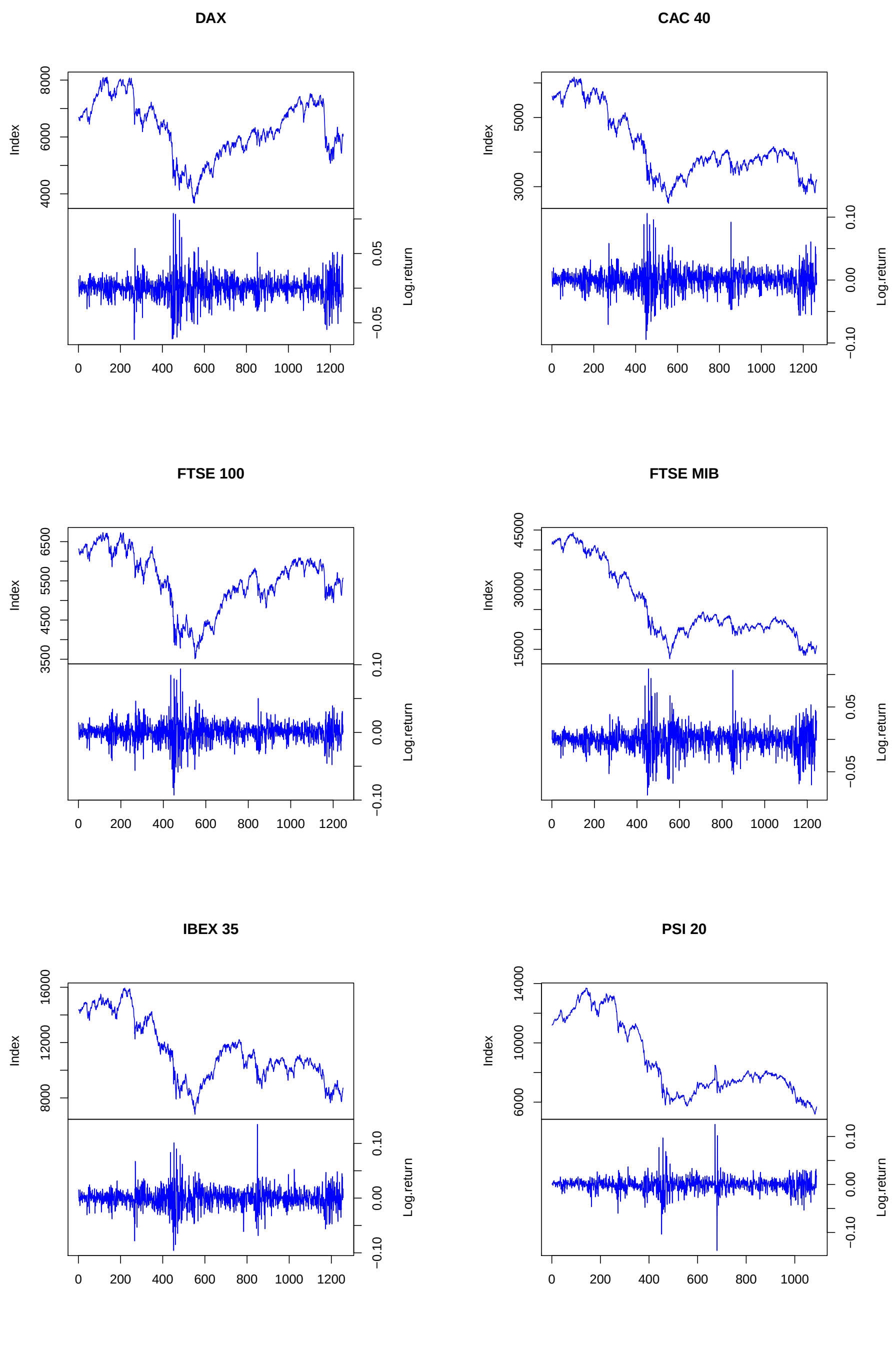


Figure 1: The index and the log-return of DAX, CAC_{40}, $FTSE_{100}$, $FTSE_{MIB}$, $IBEX_{35}$ and PSI_{20}, in the period from 01/02/2007 to 12/06/2011.

Table 4: Likelihood, AIC and BIC values for the NGSSM and APARCH models fitted to the PSI_{20} and $IBEX_{35}$ series.

SERIES	NGSSM	AICc	BIC	LN LIKE	APARCH(1.1)	AICc	BIC	LN LIKE
	LOGNORMAL	-15.73	-15.72	8575.23	NORMAL	-12.76	-12.74	6955.76
	LOGGAMMA	-16.12	-16.11	8790.45	SKEW NORMAL	-12.88	-12.86	7024.52
	PARETO	-15.65	-15.64	8533.18	t-STUDENT	-14.54	-14.52	7929.09
PSI_{20}	**WEIBULL**	**-16.16**	**-16.15**	**8810.77**	SKEW t-STUDENT	-15.27	-15.25	8326.68
	FRÉCHET	-15.32	-15.30	8350.79	GED	-14.80	-14.78	8067.46
	LÉVY	-14.70	-14.69	8013.44	**SKEW GED**	**-15.71**	**-15.70**	**8568.12**
	NORMAL	-12.60	-12.59	6871.99				
	SKEW NORMAL	-13.93	-13.91	7593.20				
	LAPLACE	-13.35	-13.33	7278.00				
	SKEW LAPLACE	-15.54	-15.52	8473.96				
	GED	-15.03	-15.02	8195.98				
	SKEW GED	-15.96	-15.94	8703.42				
	LOGNORMAL	-14.95	-14.94	9393.84	NORMAL	-12.48	-12.47	7843.85
	LOGGAMMA	-15.21	-15.19	9552.15	SKEW NORMAL	-13.21	-13.19	8298.79
	PARETO	-14.76	-14.75	9271.39	t-STUDENT	-13.63	-13.61	8560.56
$IBEX_{35}$	**WEIBULL**	**-15.24**	**-15.23**	**9576.33**	SKEW t-STUDENT	-13.63	-13.61	8560.56
	FRÉCHET	-14.52	-14.51	9120.81	GED	-13.91	-13.90	8742.33
	LÉVY	-14.05	-14.04	8824.65	**SKEW GED**	**-14.87**	**-14.85**	**9341.72**
	NORMAL	-11.71	-11.70	7359.32				
	SKEW NORMAL	-7.43	-7.41	4669.95				
	LAPLACE	-13.57	-13.56	8526.36				
	SKEW LAPLACE	-14.63	-14.62	9193.43				
	GED	-14.14	-14.13	8881.32				
	SKEW GED	-15.10	-15.08	9487.20				

was ignored. For implementation of the Bayesian inference, the Metropolis-Hastings algorithm is used with generation of two chains. In general, vague and proper priors are assumed for the hyperparameters and latent parameters as uniform with suitably large limits and Gamma with small parameters values, respectively.

For comparing the NGSSM with some known procedures in the literature, APARCH models with Gaussian, Skew Gaussian, t-Student, Skew t-Student, GED and Skew GED errors were also fitted to the series. All models for both approachs were estimated using the square of the log-return of the stock market indexes.

The programs, developed in Ox Metrics by authors, is used to estimate the NGSSM-SV. The fGARCH package in software R is used to estimate the APARCH models and the Coda package is used for diagnostic methods, checking the chains convergence through graphic methods such as the autocorrelogram, time series and trace plots. The method of Quasi-Maximum Likelihood Estimation is used for estimating the parameters of APARCH models in the package fGARCH. For more details see [9].

In the Tables 2, 3 and 4 can be observed that the comparison of the models were performed using the AICc, BIC and log-likelihood (LN LIKE) criterion. For more details about AICc and BIC see [5].

According to the three criterion, the Weibull model is the best one within the NGSSM models and the APARCH (1,1) with Skew GED errors is the best one in the GARCH family. Comparing the two approaches (NGSSM and APARCH) it is worth to note that the Log-normal, Log-gamma, Weibull presented better results than the APARCH models for all series, with the Weibull model being the best one, followed closely by the Log-gamma model.

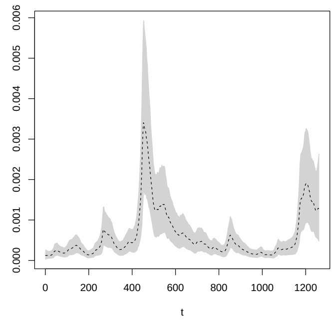
For assessing the fit of the models, the pearson residuals were utilized according to the description and suggestion in [33]. The pearson residuals analysis did not give evidence about any misspecification of the model fitted to the real series. The Box-Pierce test was also used for verifying the hypothesis of no autocorrelation in the residuals and their square, which was not violated.

Table 5 presents the estimates of maximum likelihood (MLE) and Bayesian estimates, posterior mean (BE Mean) and posterior median (BE Median) for parameters of the Weibull model fitted to the volatility series of all indexes. In addition, 95% asymptotic confidence (Conf I) and credibility (Cred I) intervals are also built. It is verified that all parameters are significant to the 5% level.

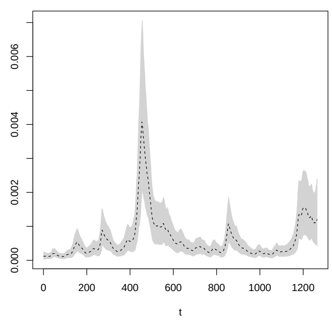
Table 5: Parameter estimates of the Weibull models for the volatility of the indexes.

SERIES	φ	MLE	BE-Mean	BE-Median	Conf I	Cred I
	ω	0.9450	0.9428	0.9434	[0.9238 ; 0.9605]	[0.9222; 0.9597]
DAX	β	8.3022	8.2390	8.2384	[4.7684 ; 11.8359]	[4.6592 ; 11.7813]
	υ	0.5585	0.5596	0.5598	[0.5331 ; 0.5839]	[0.5342 ; 0.5853]
	ω	0.9366	0.9345	0.9349	[0.9131 ; 0.9539]	[0.9124 ; 0.9542]
$\mathbf{CAC_{40}}$	β	6.9278	6.8905	6.8557	[3.6311 ; 10.2244]	[3.7835 ; 10.2181]
	υ	0.5901	0.5909	0.5912	[0.5633 ; 0.6169]	[0.5623 ; 0.6166]
	ω	0.9324	0.9303	0.9306	[0.9087 ; 0.9501]	[0.9092 ; 0.9497]
$\mathbf{FTSE_{100}}$	β	6.4875	6.3674	6.4090	[2.7243 ; 10.2508]	[2.5262 ; 10.1761]
	υ	0.5968	0.5977	0.5976	[0.5696 ; 0.6240]	[0.5711 ; 0.6249]
	ω	0.9366	0.9349	0.9355	[0.9148 ; 0.9529]	[0.9141 ; 0.9523]
$\mathbf{FTSE_{MIB}}$	β	5.6876	5.6596	5.6437	[2.4538 ; 8.9214]	[2.4974 ; 8.9660]
	υ	0.5958	0.5968	0.5966	[0.5688 ; 0.6228]	[0.5707 ; 0.6240]
	ω	0.9320	0.9302	0.9307	[0.9085 ; 0.9497]	[0.9088 ; 0.9490]
$\mathbf{IBEX_{35}}$	β	4.9623	4.9590	4.9278	[1.9046 ; 8.0200]	[1.9477 ; 7.9854]
	υ	0.5938	0.5948	0.5947	[0.5669 ; 0.6207]	[0.5682 ; 0.6215]
	ω	0.9100	0.9078	0.9082	[0.8807 ; 0.9323]	[0.8806 ; 0.9325]
$\mathbf{PSI_{20}}$	β	6.2642	6.2645	6.3132	[2.3792 ; 10.1493]	[2.3486 ; 10.1653]
	υ	0.5803	0.5814	0.5811	[0.5515 ; 0.6090]	[0.5531 ; 0.6092]

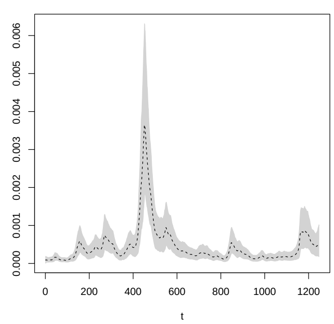
In the Figures 2 the smoothed estimate of the stochastic volatility and the 95% credibility intervals can be observed and it was obtained by the fit of the Weibull model under the Bayesian approach. The peak around the instants 400 and 500 is referent to, in the second semester of 2008, the Imobiliary crisis



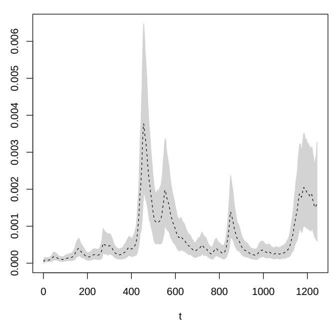
(a) DAX



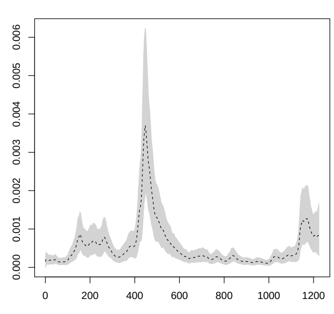
(b) CAC_{40}



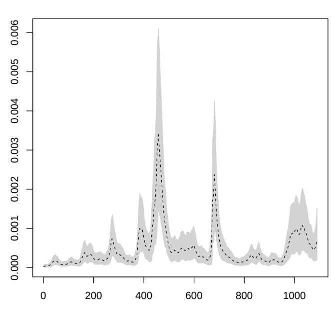
(c) $FTSE_{100}$



(d) $FTSE_{MIB}$



(e) $IBEX_{35}$



(f) PSI_{20}

Figure 2: The dashed line represents the smoothed estimate of the stochastic volatility, obtained by the fit of the Weibull model under the Bayesian approach. The grey area indicates the 95% credibility intervals.

in US and Lehmann-Brothers crisis, the period of the global financial crisis in 2008.

6 Conclusion

This work presents a comparative study between the NGSSM models proposed by Santos, Franco and Gamerman [33] and APARCH models proposed by Ding, Granger and Engle [12] for important stock market indexes of America, Europe and Asia, such as DAX, CAC_{40}, $FTSE_{100}$, $FTSE_{MIB}$, $IBEX_{35}$ and PSI_{20}, in the period between 01/02/2007 to 12/06/2011.

For all series the volatility fitted by the NGSSM models was better than the APARCH models. In particular, the best NGSSM for all series was the Weibull model.

Three future works additional to the study presented here are in progress. The first is the evaluation of the maximization methods obtaining the maximum likelihood estimatives that produce results faster and with less bias. In this work was used the BFGS, algorithm proposed by [2, 14, 20] and [36]. The second is the comparison between NGSSM and APARCH models for the forecasting of the stock market index. The third is the comparison between NGSSM and other stochastic volatility models proposed in the literature.

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