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Immunization Theory Revisited: Convex Hedges and Immunization Bounds between Bonds and Swaps

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Abstract

We construct and calculate static immunization bounds for hedging a single swap liability with two bonds in order to control the interest rate risk of these fixed income securities. These bounds are based on two kinds of duration and convexity measures, namely the traditional Fisher-Weil measures and the more recent stochastic measures of duration and convexity associated to affine models of the term structure of interest rates (e.g. the Vasicek and Cox-Ingersoll-Ross models). The immunization bounds are described for arbitrary portfolios that have deterministic future cash-flows with vanishing present value and can hitherto be used in this more general setting.

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1 Introduction

In the terminology of mathematical finance and portfolio theory, a (perfect) hedge refers to a self-financing portfolio that replicates some given financial claim at a future time point (e.g. [2]). Traditionally, in the field of fixed income securities, less stringent definitions of hedging have been considered, especially for the purpose of immunizing portfolios against changes in interest rates. In a partial hedge exact replication is relaxed to reduction of risk through minimization of risk with respect to some appropriate risk measure (e.g. mean-variance hedging) or through ordering of risk (e.g. variance order or convex order). Therefore, the considered hedges are throughout understood as partial hedges. In a fixed income framework, the goal of hedge optimization is the formulation and finding of good strategies that minimize (interest rate) risk as much as possible. Our goal is the construction of static immunization bounds for arbitrary portfolios of deterministic future cash-flows with vanishing present value in order to control the interest rate risk of fixed income securities. These bounds are based on two kinds of duration and convexity measures, namely the traditional Fisher-Weil measures and the more recent stochastic measures of duration and convexity associated to the Vasicek and Cox-Ingersoll-Ross affine models of the term structure of interest rates. The new main Theorem 4.4 for the affine risk measures has a more realistic and wider range of application than the previous Theorem 4.3, which has been initially derived in [14], Theorem 2.3.

We suppose the reader is familiar with the fundamentals of fixed income modelling as exposed in [25] or [1]. Our emphasis is on arbitrage free pricing (Section 2), interest rate risk measurement (Section 3) and interest rate risk management/optimal hedging (Section 4). The significance of the new formulation for hedge optimization is illustrated in Section 5, where static immunization bounds are calculated explicitly and numerically for hedging a single swap liability with two bonds. References that include further material on interest rate swaps and their hedging are [22] and [8].

2 Pricing Fixed Income Securities

The simplest fixed income securities are bonds, which are nothing but tradable loan agreements. We distinguish between a *zero-coupon bond* (single payment at a single future date, the maturity date of the bond) and other *coupon bonds* (more than one payment at some future dates). For simplicity, we assume that all fixed income securities have equally spaced *payment dates* $T_1,...,T_n$, where $T_{i+1} - T_i = \delta$. We refer to $T_0 = T_1 - \delta$ as the *starting date,* and $T = T_n$ as the *maturity date* of a given fixed income security. For derivative instruments like swaps we call $T_0, T_1,...,T_n$ the *reset dates*, and δ the frequency or *tenor*. If time is measured in years, then typical bonds and swaps have $\delta \in \{0.25, 0.5, 1\}$.

For all coupon bonds the payment at date T_i is denoted by Y_i. The size of each of the payments is determined by the *face value*, the *coupon rate*, and the *amortization principle* of the bond. The face value is also called *par value* or *principal* of the bond, and the coupon rate is also called *nominal rate* or stated interest rate. Often, the coupon rate is quoted as an annual rate denoted R, so that the corresponding periodic coupon rate is $\delta \cdot R$. For convenience, cash-flows of financial instruments are summarized into a vector denoted by $c = (C_{T_0}, C_{T_1},..., C_{T_n})$.

2.1 Zero-coupon bonds

By convention, the face value of any zero-coupon bond is 1 unit of account (say a "dollar"). In the arbitrage free pricing theory of fixed income securities, it is well-known that prices depend upon the *term structure of interest rates* (TSIR), which itself is determined by the *zero-coupon bond price structure* defined and denoted by

$P(t,s)$: price at time t of a bond with maturity $s \geq t$, $T_0 \leq t \leq s \leq T$.

Suppose that many zero-coupon bonds with different maturities are traded on the financial market. Then, for a fixed time t the function $T \to P(t,T)$ is called the market discount function prevailing at time t. Clearly, the discount function should be decreasing, i.e.

$$1 \geq P(t,T) \geq P(t,S) \geq 0, \quad t \leq T \leq S. \tag{2.1}$$

In case the starting date coincides with the current date $T_0 = 0$ one often writes $P(s) = P(0,s)$.

2.2 Straight-coupon bonds

In a *straight-coupon bond* (or bullet bond) all payments before the final maturity date payment are identical and equal to the threefold product of the (annualized) coupon rate R, the payment frequency δ, and the principal H. To emphasize its defining parameters a straight-coupon bond will be denoted by $B = B(T,H,R,\delta)$. Clearly, a bond generates exactly $n = T \cdot \delta$ payments occurring at the dates $T_i = T_0 + i \cdot \delta,\ i = 1,...,n$, determined by

$$Y_i = \begin{cases} H \cdot R \cdot \delta, & i = 1,...,n-1 \\ H \cdot (1 + R \cdot \delta), & i = n \end{cases} \tag{2.2}$$

We note that the special case $H = 1, R = 0$ defines the zero-coupon bond. Since a coupon bond can be seen as a portfolio of zero-coupon bonds, namely a portfolio of Y_1 zero-coupon bonds maturing at T_1, Y_2 zero-coupon bonds maturing at T_2, and so on, and under the assumption that all zero-coupon bonds are tradable on the market, the *price* of the straight-coupon bond at any time t is determined by

$$B_t = \sum_{T_i > t} Y_i \cdot P(t,T_i), \tag{2.3}$$

where the sum runs over all future payment dates of the bond. We note that if (2.3) is not satisfied, there will be an arbitrage opportunity in the market. The absence of arbitrage is a cornerstone of financial asset pricing theory (for a short review consult [25], Chap. 4). Finally, the cash-flows of a bond can be summarized into the vector denoted by

$$c_B = (0, Y_1, \ldots, Y_n). \tag{2.4}$$

2.3 Floating rate bonds

Floating rate bonds have coupon rates that are reset periodically over the life of the bond. We assume that the coupon rate effective for the payment at the end of one period is set at the beginning of the period at the current market interest rate for that period. Therefore, the annualized coupon rate valid for the period $[T_{i-1}, T_i]$ is the δ-period market rate at date T_{i-1} computed with a compounded frequency of δ. This interest rate is defined and denoted by

$$R_i = R(T_{i-1}, T_{i-1}, T_i) = \frac{1}{\delta}\left(\frac{1}{P(T_{i-1}, T_i)} - 1\right), \quad i = 1, \ldots, n, \tag{2.5}$$

where $R(t, T, S)$ denotes the time t *forward LIBOR rate* for the period $[T, S]$ defined by

$$R(t, T, S) = \frac{1}{S - T}\left(\frac{P(t, T)}{P(t, S)} - 1\right). \tag{2.6}$$

Summarizing the variable interest rates (2.5) into a vector $r = (R_1, R_2, \ldots, R_n)$, a floating rate bond $B^{fl} = B^{fl}(T, H, r, \delta)$ generates variable payments determined by

$$Y_i = \begin{cases} H \cdot R_i \cdot \delta, & i = 1, \ldots, n-1 \\ H \cdot (1 + R_n \cdot \delta), & i = n \end{cases} \tag{2.7}$$

It can be shown that immediately after each reset date the value of the bond is equal to its face value, i.e. $B_{T_i^+}^{fl} = H$, and in this situation the floating rate bond is valued at par. More generally, the value of the floating rate bond at any time $t \in [T_0, T_n)$ is given by

$$B_t^{fl} = H \cdot \frac{P(t, T_{i(t)})}{P(T_{i(t)} - \delta, T_{i(t)})}, \tag{2.8}$$

where the time index

$$i(t) = \min\{i \in \{1,...,n\} : T_i > t\}, \tag{2.9}$$

indicates that $T_{i(t)}$ is the nearest following payment date after time t. The expression (2.9) also holds at payment dates $t = T_i$, where it results in H, which is the value excluding the payment at that date. Up to a straightforward rearrangement using (2.5), a proof of the relationship (2.8) is found in [25], Section 1.2.5. The cash-flows of a floating rate bond are summarized into the vector

$$c_{B^{fl}} = (0, Y_1, ..., Y_n). \tag{2.10}$$

2.4 Interest rate swaps

In general, an *(interest rate) swap* is an exchange of two cash-flow streams that are determined by certain interest rates. In the most common form, a *plain vanilla swap*, two parties exchange a stream of fixed interest rate payments, called *fixed leg*, and a stream of floating interest rate payments, called *floating leg*. The payments are in the same currency, and are computed from the same (hypothetical, i.e. not swapped) face value or *notional principal*, denoted by H. The floating rate is usually a money market rate, e.g. a LIBOR rate, possibly augmented or reduced by a fixed margin. The fixed interest rate, denoted by K, is (usually) set such that the swap has zero net present value at contract agreement, a condition assumed throughout. In a *payer swap*, or fixed-for-floating swap, the owner party pays a stream of fixed rate payments and receives a stream of floating rate payments. The *receiver swap*, or floating-for-fixed swap, is the counterpart, where the owner party pays a stream of floating rate payments and receives a stream of fixed rate payments.

Consider now a (plain vanilla) swap $SW = SW(T, H, K, r, \delta)$, where the floating interest rate vector $r = (R_1, R_2, ..., R_n)$ is determined by the money market LIBOR rates (2.5). Without loss of generality one assumes that there is no fixed extra margin on these floating rates (such an extra charge can be treated in the same manner as the value of the fixed rate payments of the swaps, as done below). Combining the payments of the straight-coupon bond with those of the floating rate bond, the cash-flow vector of a payer swap can be summarized into the vector

$$c_{SW^p} = (0, H \cdot (R_i - K) \cdot \delta, ..., H \cdot (R_n - K) \cdot \delta) . \tag{2.11}$$

The (market) value of a swap is determined by the value of the fixed rate payments (V^{fix}) and the value of the floating rate payments (V^{fl}). Clearly, the value at time t of the fixed rate payments is determined by the value of the remaining fixed payments and is given by

$$V_t^{fix} = H \cdot K \cdot \delta \cdot \sum_{i=i(t)}^{n} P(t, T_i) . \tag{2.12}$$

Note that this coincides with (2.3) when omitting the final face value payment in (2.2). Similarly, the value at time t of the floating rate payments is determined by the value of the remaining floating payments and is given by

$$V_t^{fl} = H \cdot \left\{ \frac{P(t, T_{i(t)})}{P(T_{i(t)} - \delta, T_{i(t)})} - P(t, T_n) \right\}, \quad T_0 < t < T_n . \tag{2.13}$$

This is obtained from (2.8) by subtracting the value of the final repayment face value, which does not occur in a swap. Clearly, previously to or at the starting date we have

$$V_t^{fl} = H \cdot \{P(t,T_0) - P(t,T_n)\}, \quad t \le T_0. \tag{2.14}$$

It is worthwhile to mention the alternative expressions in [25], Section 6.5.1:

$$\begin{aligned} &V_t^{fl} = H\delta \left\{ R(T_{i(t)} - \delta, T_{i(t)} - \delta, T_{i(t)}) \cdot P(t, T_{i(t)}) + \sum_{i=i(t)+1}^{n} R(t, T_i - \delta, T_i) \cdot P(t, T_i) \right\}, \\ &T_0 < t < T_n, \quad V_t^{fl} = H \cdot \delta \cdot \sum_{i=1}^{n} R(t, T_i - \delta, T_i) \cdot P(t, T_i) \quad t \le T_0. \end{aligned} \tag{2.15}$$

Through combination the values of a payer swap and receiver swap are respectively given by

$$SW_t^p = V_t^{fl} - V_t^{fix}, \quad SW_t^r = V_t^{fix} - V_t^{fl}. \tag{2.16}$$

3 Interest rate risk measurement

The values of bonds and other fixed income securities vary over time due to changes in the term structure of interest rates. To measure and compare the sensitivities of different securities to term structure movements, one uses various interest rate risk measures, which constitute an important input to portfolio management decisions.

We consider a portfolio of fixed income securities, typically a portfolio constituted of bonds (as assets) and swaps (as liabilities). The net positions between assets and liabilities generate a vector of cash-flows denoted by $c = (C_0, C_1, ..., C_n)$, where in contrast to the preceding Section 2, the time unit is now the tenor δ. Therefore, the maturity date of the portfolio is $T = n \cdot \delta$. The non-negative net positions generate a vector $c^+ = (C_0^+, C_1^+, ..., C_n^+)$ and the negative net positions a vector of positive numbers $c^- = (C_0^-, C_1^-, ..., C_n^-)$ such that $c = c^+ - c^-$. Following Section 2.1, the market discount function prevailing at the current time $t = 0$ is denoted $P_i = P(0, \delta \cdot i), i = 0, ..., n,$ with $P_0 = 1$. The *current price* of a cash-flow is given by

$$P_c = \sum_{i=0}^{n} P_i C_i = P_{c^+} - P_{c^-} \, , \, P_{c^\pm} = \sum_{i=0}^{n} P_i C_i^\pm \, .$$

For simplicity, we assume that the cash-flows are independent of interest rate movements.

3.1 Traditional risk measures as probabilistic risk measures

Originally, the Macaulay [19] duration of a bond was defined as a weighted average of the time distance to the payment of the bond, that is as an “effective time-to-maturity”. As shown by [11], the Macaulay duration measures the sensitivity of the bond value with respect to changes in its own yield. Macaulay [19] also defined an alternative duration measure based on the zero-coupon yield curve rather than the bond’s own yield. After decades of neglect, the latter duration measure found a revival in [7], who demonstrated its relevance for the construction of immunization strategies. Following the modern approach, it is possible to define these risk measures for arbitrary portfolios of fixed income securities, and simplify their use by considering them as probabilistic risk measures.

3.1.1 Macaulay duration and convexity of portfolio future cash-flows

Usually these sensitivity measures are defined for non-negative cash-flows only. Their use is extended to arbitrary portfolios of future cash-flows $c = c^+ - c^-$ by defining them separately for the non-negative and negative components as follows. As only future cash-flows are involved we assume that $C_0 = 0$. Let $y_\pm$ be the yields to maturity of the cash-flows $c^\pm$, i.e. the unique solutions of the equations

$$\sum_{i=1}^{n} e^{-y_\pm i\delta} C_i = P_{c^\pm} \, . \qquad (3.1)$$

Then the (modified) *Macaulay duration* and *convexity* of the cash-flow vectors $c^{\pm}$ are defined by

$$D_{c^{\pm}}^{M} = -\frac{1}{P_{c^{\pm}}} \cdot \frac{\partial P_{c^{\pm}}}{\partial y_{\pm}}, \qquad C_{c^{\pm}}^{M} = \frac{1}{P_{c^{\pm}}} \cdot \frac{\partial^{2} P_{c^{\pm}}}{\partial y_{\pm}{}^{2}}. \tag{3.2}$$

Given a shift in the term structure of interest rates, that is the zero-coupon bond price curve changes from P_i to say P_i^*, one is interested in approximations to the current shifted arbitrage-free price $P_c^* = \sum_{i=1}^{n} P_i^* C_i$, which only depend on the initial term structure and the changes in cash flow yields $\Delta y_{\pm} = y_{\pm}^* - y_{\pm}$, where $y_{\pm}^*$ are the shifted yields (theoretical solutions of the equations $\sum_{i=0}^{n} e^{-y_{\pm}^* \cdot i \cdot \delta} C_i = P_{c^{\pm}}^*$). One considers the following first and second order (Macaulay) approximations to P_c^* defined and denoted by

$$\begin{aligned} P_c^{M,(1)} &= \left[1 - \frac{D_{c^+}^{M} \Delta y_+ P_{c^+} - D_{c^-}^{M} \Delta y_- P_{c^-}}{P_c} \right] \cdot P_c, \\ P_c^{M,(2)} &= \left[1 - \frac{(D_{c^+}^{M} \Delta y_+ + \frac{1}{2} C_{c^+}^{M} (\Delta y_+)^2) P_{c^+} - (D_{c^-}^{M} \Delta y_- + \frac{1}{2} C_{c^-}^{M} (\Delta y_-)^2) P_{c^-}}{P_c} \right] \cdot P_c. \end{aligned} \tag{3.3}$$

Mathematically, these formulas are just the first and second order Taylor approximations of the price-yield function of the cash-flows $c^{\pm}$ (straightforward generalization of [16], Section 4, equation (4.2)). Some comments about this extended definition of the Macaulay duration and convexity follow in the Remarks 3.1 of the next Section.

3.1.2 Fisher-Weil duration, convexity, and M-square index

We use the modern probabilistic definitions of the Fisher-Weil sensitivity measures (of non-negative future cash-flows) as originally found in [33] and followed-up by [34], [26], Section 3.5, and [14]-[16]. Our (slight) extension to arbitrary portfolios $c = c^+ - c^-$ along the line of [14] is straightforward. Let

$\alpha_k = P_k C_k^+$ and $\ell_k = P_k C_k^-$ be the current prices of the future cash-flows $C_k^{\pm}$, $k = 1,...,n$.

Definitions 3.1. The random variable S_c^+ with support $\{\delta,...,n\delta\}$ and probabilities $\{q_1,...,q_n\}$, where $q_k = \alpha_k \cdot \left(\sum_{i=1}^n \alpha_i\right)^{-1}$ is the normalized future cash inflow at time k, is called *positive cash-flow risk*. Similarly, the random variable S_c^- with support $\{\delta,...,n\delta\}$ and probabilities $\{p_1,..., p_n\}$, where $p_k = \ell_k \cdot \left(\sum_{i=1}^n \ell_i\right)^{-1}$ is the normalized future cash outflow at time k, is called *negative cash-flow risk*.

The *Fisher-Weil duration*, *convexity* and *M-square index* of the future cash-flow vectors $c^{\pm}$ are defined as first and second order expected values, respectively variances, associated to the positive and negative cash-flow risks:

$$D_{c^{\pm}} = E\left[S_c^{\pm}\right], \quad C_{c^{\pm}} = E\left[(S_c^{\pm})^2\right], \quad M_{c^{\pm}}^2 = Var\left[S_c^{\pm}\right]. \tag{3.4}$$

With this the approximations (3.3) are replaced by

$$\begin{aligned} P_c^{(1)} &= \left[1 - \frac{D_{c^+}\Delta y_+ P_{c^+} - D_{c^-}\Delta y_- P_{c^-}}{P_c}\right] \cdot P_c, \\ P_c^{(2)} &= \left[1 - \frac{(D_{c^+}\Delta y_+ + \frac{1}{2}C_{c^+}(\Delta y_+)^2)P_{c^+} - (D_{c^-}\Delta y_- + \frac{1}{2}C_{c^-}(\Delta y_-)^2)P_{c^-}}{P_c}\right] \cdot P_c. \end{aligned} \tag{3.5}$$

Remarks 3.1.

(i) The Macaulay duration and convexity of the future cash-flows can also be interpreted in probabilistic terms. Let $\alpha_k^M = e^{-y_+ k\delta} C_k^+$ and $\ell_k^M = e^{-y_- k\delta} C_k^-$ be the current yield to maturity discounted values of the future cash-flows $C_k^{\pm}$, $k = 1,...,n$. The random variable $S_c^{M,+}$ with support $\{\delta,...,n\delta\}$ and probabilities $\{q_1^M,...,q_n^M\}$, with $q_k^M = \alpha_k^M \cdot \left(\sum_{i=1}^n \alpha_i^M\right)^{-1}$, is called *Macaulay* *positive cash-flow risk*. Similarly, the random variable $S_c^{M,-}$ with support $\{\delta,...,n\delta\}$ and probabilities $\{p_1^M,...,p_n^M\}$, with $p_k^M = \ell_k^M \cdot \left(\sum_{i=1}^n \ell_i^M\right)^{-1}$, is called *Macaulay negative cash-flow risk*. Then, similarly to (3.4) we define the probabilistic notions of *Macaulay duration*, *convexity* and *M-square index* of the future cash-flow vectors $c^{\pm}$ as

$$D_{c^{\pm}} = E\left[S_c^{M,\pm}\right],\quad C_{c^{\pm}}^M = E\left[(S_c^{M,\pm})^2\right],\quad M_{c^{\pm}}^{M,2} = Var\left[S_c^{M,\pm}\right]. \tag{3.6}$$

(ii) On the other hand, as observed in [16], Section 5, it is not difficult to see that in general $D_c \neq D_c^M$, $C_c \neq C_c^M$, hence $P_c^{(1)} \neq P_c^{M,(1)}$, $P_c^{(2)} \neq P_c^{M,(2)}$, but the differences are usually negligible (see also [25], Section 12.3.3, p.333). Note that equality holds for flat term structures. Furthermore, simulation examples suggest that the approximations (3.5) outperform in accuracy the traditional ones (3.3). For these reasons, only the Fisher-Weil measures are retained for further analysis in Section 4.

(iii) From a more advanced point of view, we note that [3] has obtained simple composition formulas for the Macaulay sensitivity measures (3.2) applied to (economic) cash-flow sums and products. The Fisher-Weil probabilistic counterparts of them have been derived in [16], Theorems 5.1 and 5.2. It is also possible to use a multivariate model of so-called directional duration and convexity (consult [17] for a recent account).

(iv) Like [25], p.332, we like to emphasise that the Macaulay and Fisher-Weil duration and convexity risk measures are only meaningful in the context of cash-flows that are independent of the interest rate movements (e.g. portfolios of bonds), an assumption made at the beginning of Section 3.1. Of course, if a financial instrument can be reduced to interest rate independent cash-flows, then the traditional risk measures still apply. As shown later in Section 3.2.4, this is the case for swaps. For a more general use that includes interest rate derivatives (e.g. caps/floors and swaptions), one has to consider also stochastic risk measures of duration and convexity as those defined in the next Section.

3.2 Stochastic risk measures in one-factor diffusion models

It is known that the Macaulay risk measures are *not consistent* with any arbitrage-free dynamic term structure model. Similarly, the Fisher-Weil measures are *only consistent* with the model by [20], which is a very unrealistic model (e.g. [25], Section 12.2.3). To obtain measures of interest rate risk that are more in line with a realistic evolution of the TSIR, it is natural to consider uncertain price movements in reasonable dynamic term structure models.

3.2.1 Definitions and relationships

We focus on the sensitivity of the prices with respect to a change in the state variable(s). For simplicity, we restrict the analysis to one-factor diffusion models, for which the instantaneous interest rate or short rate follows a stochastic process of the type

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t)dW_t, \tag{3.7}$$

with W_t the standard Wiener process. In applications, we consider a mean-reverting short rate with drift $\mu(t, r_t) = \kappa(\theta - r_t)$, and the instantaneous standard deviation is either constant $\sigma(t, r_t) = \sigma$ (model of Vasicek [35]) or of square-root type $\sigma(t, r_t) = \sigma\sqrt{r_t}$ (model of Cox-Ingersoll-Ross [5] or CIR model). The condition $2ab > \sigma^2$ for the CIR model guarantees that the process never touches zero and implies a stationary gamma distribution. Similar calculations can be done for the Hull-White model with $\mu(t, r_t) = a(b(t) - r_t)$ and $\sigma(t, r_t) = \sigma$ following the specification and calibration in [12], (2003), Chap. 23.

For a non-negative cash-flow c with price process $P_t^c = P^c(t, r_t)$ Itô's Lemma implies that

$$\frac{dP_t^c}{P_t^c} = \left(\frac{1}{P^c(t,r_t)} \frac{\partial P^c(t,r_t)}{\partial t} + \frac{\mu(t,r_t)}{P^c(t,r_t)} \frac{\partial P^c(t,r_t)}{\partial r} + \frac{1}{2} \frac{\sigma(t,r_t)^2}{P^c(t,r_t)} \frac{\partial^2 P^c(t,r_t)}{\partial r^2} \right) dt + \frac{1}{P^c(t,r_t)} \frac{\partial P^c(t,r_t)}{\partial r} \sigma(t,r_t) dW_t . \quad (3.8)$$

Typically, for a bond, the derivative $\frac{\partial}{\partial r} P^c(t,r_t)$ is negative for the above models, and the volatility of the bond is $-(\frac{\partial}{\partial r} P^c(t,r_t) / P^c(t,r_t)) \cdot \sigma(t,r_t)$. Since it is natural to use the cash-flow specific part of the volatility as a risk measure, we define the (stochastic) *cash-flow duration* as (note the similarity with Macaulay duration)

$$D(t,r_t) = \frac{-1}{P^c(t,r_t)} \frac{\partial P^c(t,r_t)}{\partial r} . \quad (3.9)$$

According to (3.8) the unexpected relative return on the cash-flow is $-D(t,r_t)\sigma(t,r_t)dW_t$. Furthermore, we define the (stochastic) *cash-flow convexity* as

$$C(t,r_t) = \frac{1}{P^c(t,r_t)} \frac{\partial^2 P^c(t,r_t)}{\partial r^2} , \quad (3.10)$$

and the (stochastic) *cash-flow time value* as

$$\Theta(t,r_t) = \frac{1}{P^c(t,r_t)} \frac{\partial P^c(t,r_t)}{\partial t} . \quad (3.11)$$

It follows that the rate of return (3.8) on the cash-flow over the next infinitesimal period of time can be rewritten as

$$\frac{dP_t^c}{P_t^c} = \left(\Theta(t,r_t) - \mu(t,r_t) D(t,r_t) + \frac{1}{2} \sigma(t,r_t)^2 C(t,r_t) \right) dt - D(t,r_t) \sigma(t,r_t) dW_t . \quad (3.12)$$

Next, consider the *market price of risk* of the cash-flow, denoted $\lambda(t,r_t)$ and also called *Sharpe ratio*, which is defined as excess expected return (above the risk-free rate) per unit of risk and with (3.12) is given by

$$\lambda(t,r_t)=\frac{\Theta(t,r_t)-\mu(t,r_t)D(t,r_t)+\frac{1}{2}\sigma(t,r_t)^2C(t,r_t)-r_t}{-D(t,r_t)\sigma(t,r_t)}. \tag{3.13}$$

One sees that (3.13) is equivalent with the relationship

$$\Theta(t,r_t)-\hat{\mu}(t,r_t)D(t,r_t)+\frac{1}{2}\sigma(t,r_t)^2C(t,r_t)=r_t, \tag{3.14}$$

where $\hat{\mu}(t,r_t)=\mu(t,r_t)-\sigma(t,r_t)\lambda(t,r_t)$ is the *risk-neutral drift* of the short rate. Note that (3.14) also follows by substituting the definitions of duration, convexity and time value into the partial differential equation that the price process is known to satisfy, that is (e.g. [25], Section 4.8, Theorem 4.10)

$$\frac{\partial P^c(t,r_t)}{\partial t}+\hat{\mu}(t,r_t)\frac{\partial P^c(t,r_t)}{\partial r}+\frac{1}{2}\sigma(t,r_t)^2\frac{\partial^2P^c(t,r_t)}{\partial r^2}-r_tP^c(t,r_t)=0. \tag{3.15}$$

Remarks 3.2. According to [25], p.330, the importance of (3.14) for the construction of interest rate risk hedging strategies has been first noticed by [6]. Within the context of the Black-Scholes-Merton return model, the time value and the Δ (delta) and Γ (gamma) Greeks are related in a way similar to (3.14) (e.g. [12], (2009), Section 17.7).

Further, we note that (3.12) can also be rewritten as

$$\frac{dP_t^c}{P_t^c}=\left(r_t-\lambda(t,r_t)\sigma(t,r_t)\right)D(t,r_t)dt-D(t,r_t)\sigma(t,r_t)dW_t, \tag{3.16}$$

which only involves the duration, and not the convexity nor the time value. Through differentiation of the duration one obtains (e.g. [25], Exercise 12.1)

$$\frac{\partial D(t,r_t)}{\partial r}=D(t,r_t)^2-C(t,r_t), \tag{3.17}$$

which shows that the convexity can be interpreted as a measure of the interest rate sensitivity of the duration.

In contrast to the Macauly and Fisher-Weil durations, the stochastic duration (3.9) is not measured in time units, but it can be transformed into such a new measure, called *time-denominated duration* and denoted by $D^*(t, r_t)$. For example, [4] define the time-denominated duration of a coupon-bond with price process $B(t, r)$ as

$$\frac{1}{B(t,r)}\frac{\partial B(t,r)}{\partial r} = \frac{1}{P^{t+D^*(t.r)}(t,r)}\frac{\partial P^{t+D^*(t.r)}(t,r)}{\partial r}, \tag{3.18}$$

where $P^T(t, r_t)$ denotes the price at time t of a zero-coupon bond with maturity T.

3.2.2 Stochastic duration and convexity for affine models of the TSIR

The zero-coupon bond prices in affine models of the TSIR, e.g. the Vasicek and CIR models, are of the form

$$P^T(t,r) = e^{a(T-t)-b(T-t)r}, \tag{3.19}$$

for some functions $a(\cdot), b(\cdot)$. Now, the current price at time $t = 0$ of a vector $c = (0, C_1, ..., C_n)$ of non-negative future cash-flows, which is independent of any interest rate movements (under the assumptions made at the beginning of Section 3), is given by

$$P^c(0,r) = \sum_{k=1}^{n} P^{k\delta}(0,r)\cdot C_k\ . \tag{3.20}$$

Therefore, the current stochastic duration of the future cash-flows in affine models equals

$$D^c(0,r) = \frac{-1}{P^c(0,r)}\frac{\partial P^c(0,r)}{\partial r} = \sum_{k=1}^{n} w_k \cdot b(k\cdot\delta), \quad w_k = \frac{P^{k\delta}(0,r)\cdot C_k}{P^c(0,r)}. \tag{3.21}$$

The corresponding current stochastic convexity is similarly given by

$$C^c(0,r) = \frac{1}{P^c(0,r)}\frac{\partial^2 P^c(0,r)}{\partial r^2} = \sum_{k=1}^{n} w_k \cdot b(k\cdot\delta)^2\ . \tag{3.22}$$

A comparison of the traditional deterministic and stochastic risk measures for the CIR model is provided in [25], Section 12.3.3 (see also Section 5).

In the spirit of Section 3.1.2, and for later use, let us reinterpret these measures in probabilistic terms. For an arbitrary portfolio of future cash-flows $c = c^{+} - c^{-}$, let $\alpha_k^{aff} = P^{k\delta}(0,r) \cdot C_k^{+}$ and $\ell_k^{aff} = P^{k\delta}(0,r) \cdot C_k^{-}$ be the current prices of the future cash-flows $C_k^{\pm}$, $k = 1,...,n$, in an affine model of the TSIR.

Definitions 3.2. The random variable $S_c^{aff,+}$ with support $\{b(\delta),...,b(n\delta)\}$ and probabilities $\{q_1^{aff},...,q_n^{aff}\}$, with $q_k^{aff} = \alpha_k^{aff} \cdot \left(\sum_{i=1}^{n} \alpha_i^{aff}\right)^{-1}$, is called *affine positive cash-flow risk*. Similarly, the random variable $S_c^{aff,-}$ with support $\{b(\delta),...,b(n\delta)\}$ and probabilities $\{p_1^{aff},...,p_n^{aff}\}$, with $p_k^{aff} = \ell_k^{aff} \cdot \left(\sum_{i=1}^{n} \ell_i^{aff}\right)^{-1}$, is called *affine negative cash-flow risk.*

Then, the (stochastic) *affine duration*, *convexity* and *M-square index* of the future cash-flow vectors $c^{\pm}$ are defined similarly to (3.4) and (3.6) as

$$D_{c^{\pm}}^{aff} = E\left[S_c^{aff,\pm}\right], \quad C_{c^{\pm}}^{aff} = E\left[(S_c^{aff,\pm})^2\right], \quad M_{c^{\pm}}^{2,aff} = Var\left[S_c^{aff,\pm}\right]. \tag{3.23}$$

Examples 3.1: Vasicek and CIR duration and convexity

In the Vasicek model, the short rate follows an Ornstein-Uhlenbeck process of the form $dr_t = \kappa(\theta - r_t)dt + \sigma \cdot dW_t$ and the functions $a(\cdot), b(\cdot)$ in (3.19) with the time to maturity $\tau = T - t$ as argument are given by

$$b(\tau) = \frac{1 - e^{-\kappa\tau}}{\kappa}, \quad a(\tau) = \left(\theta - \frac{1}{2}\left(\frac{\sigma}{\kappa}\right)^2\right)(b(\tau) - \tau) - \frac{1}{4}\frac{[\sigma b(\tau)]^2}{\kappa}. \tag{3.24}$$

In the CIR model, the short rate follows the square-root process $dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t} \cdot dW_t$ and one has

$$b(\tau)=\frac{2\left(e^{\gamma\tau}-1\right)}{(\gamma+\kappa)\left(e^{\gamma\tau}-1\right)+2\gamma},\ a(\tau)=\frac{2\kappa\theta}{\sigma^2}\ln\left\{\frac{\gamma e^{\frac{1}{2}(\gamma+\kappa)\tau}b(\tau)}{e^{\gamma\tau}-1}\right\},\ \gamma=\sqrt{\kappa^2+2\sigma^2}\ . \quad (3.25)$$

Inserting these functions into the defining equations (3.19)-(3.22) yield by definition (3.23) concrete affine measures called *Vasicek* (respectively *CIR*) *duration*, *convexity* and *M-square index*. They are denoted $D_{c^\pm}^{V}, C_{c^\pm}^{V}, M_{c^\pm}^{2,V}$ (respectively $D_{c^\pm}^{CIR}, C_{c^\pm}^{CIR}, M_{c^\pm}^{2,CIR}$).

3.3 Duration and convexity measures for swaps

Based on the preliminaries of Section 2.4, we show the equivalence of a payer swap $SW^p(T,H,K,r,\delta)$ with a portfolio consisting of a long initial cash position of amount H and a short position in a bond $B(T,H,K,\delta)$. This equivalent characterization of a swap follows immediately by considering the cash-flows associated to the fixed and floating legs.

Lemma 3.1 (*Cash-flows of a payer swap*). The cash-flows c^{fix}, c^{fl} associated to the fixed and floating legs of a payer swap $SW^p(T,H,K,r,\delta)$ are given by

$$c^{fix}=(0,HK\delta,...,HK\delta),\quad c^{fl}=(H,0,...,0,-H)\,. \quad (3.26)$$

Proof. For the fixed leg this follows immediately by noting that the fixed payments of the payer swap are those of a straight-coupon bond $B(T,H,K,\delta)$ omitting the final face value payment. For the floating leg we use the representation (2.15) for the value of the floating leg. At the start date $t=0$ the market value (in the notations of Section 2) can be rewritten as

$$V_0^{fl}=H\cdot\delta\cdot\sum_{i=1}^{n}R(t,(i-1)\delta,i\delta)\cdot P(0,i\delta)=H\cdot\{1-P(0,n\delta)\}, \quad (3.27)$$

which implies the desired result. ◊

Subtracting the floating leg and fixed leg cash-flow vectors in (3.26), one obtains the cash-flow vector of a payer swap, namely $c_{sw^p} = c^{fl} - c^{fix} = (H, -HK\delta, ..., -HK\delta, -H(1+K\delta))$, which implies the stated equivalence. Obviously, the duration and convexity of the *future* cash-flows of a payer swap, up to the sign, are identical to those of a bond $B(T, H, K, \delta)$, whatever notion is used for the duration and convexity measures (Macaulay, Fisher-Weil, Vasicek, CIR).

4 Hedging strategies for portfolios of fixed income securities

Though interest rate risk management can and should be formulated for arbitrary portfolios of fixed income securities, our single illustration will focus for clearness on hedging portfolios of swaps through portfolios of bonds. Several motivations support this emphasis:

(i) The first one is directly related to the practical use of swaps (e.g. [25], Section 6.5.1). An investor can transform a floating rate loan into a fixed rate loan by entering into an appropriate swap, where the investor receives floating rate payments (netting out the payments on the original loan) and pays fixed rate payments. This process is called a liability transformation. Conversely, an investor who has lent money at a floating rate, that is owns a floating rate bond, can transform this to a fixed rate bond by entering into a swap, where he pays floating rate payments and receives fixed rate payments, a so-called asset transformation. Hence, interest rate swaps can be used for hedging interest rate risk on both (certain) assets and liabilities. On the other hand, interest rate swaps can also be used for taking advantage of specific expectations of future interest rates, that is for speculation.

(ii) A second explanation is related to the nature of the “first order interest rate risk”, also called delta vector (e.g. [22], Chap. 8). For any given set of cash-flows the delta vector represents the sensitivity of the portfolio in units of account per basis point to a shift in any input rate (or price, in the case of futures). It is obtained through calculation of the price or *present value of a basis point*, abbreviated PVBP (e.g. [22], Section 9.3) or PV01 (e.g. [1], Section III.1.8). To cover the delta exposure of a long position into a swap, a trader has several options (e.g. [22], Section 9.1). A perfect hedge can only be obtained by paying fix to another market participant. Another possibility is to achieve a net zero delta across the whole yield curve by paying fixed in a different maturity date. In this situation, the size of the new deal will be different from that of the original (paying into a longer maturity requires a smaller nominal while a shorter maturity requires a larger nominal). The calculation of the exact required amount, i.e. the hedge ratio of swaps of different maturities, is considered in [22], Section 9.3. While achieving a total zero delta this strategy is coupled with a “yield curve position”, which may be quite risky in case rates do not move favourably. Since a perfect hedge is seldom achieved without any extra cost, the only effective way remains the possibility to hedge portfolios of swaps using other financial instruments than swaps. The only instruments that reduce the delta and do not introduce non-interest rate exposures are forward rate agreements (or FRAs), bonds and interest rate or bond futures. Of these the most common method of reducing absolute interest rate exposure is by hedging with government bonds, which have the advantage of liquidity. When there is a liquid bond market, the bond-swap spread (or simply the spread) is less volatile than the corresponding absolute swap rate, making the bond a natural hedging instrument (e.g. [22], Section 9.2, p.169).

4.1 Classical static immunization theory and convex order

In classical immunization theory ([26], Section 3), one assumes that cash-flows are independent of interest rate movements. In the situation of Section

3.1.2, consider a portfolio of future cash-flows $c = c^+ - c^-$ with vanishing present value of future cash-flows at time $t = 0$, i.e. such that

$$V_c = \sum_{k=1}^{n} \alpha_k - \sum_{j=1}^{n} \ell_j = 0 . \quad (4.1)$$

This corresponds to the setting studied in [14]. One is interested in the possible changes of the value of a portfolio at a time immediately following the current time $t = 0$ under a change of the TSIR's from $P(s)$ to $P^*(s)$ such that $f(s) = \frac{P^*(s)}{P(s)}$ is the *shift factor*. Immediately following the initial time, the post-shift change in portfolio value is given by

$$\Delta V_c = V_c^* - V_c = \sum_{k=1}^{n} \alpha_k f(k\delta) - \sum_{j=1}^{n} \ell_j f(j\delta) . \quad (4.2)$$

The classical immunization problem consists of finding conditions under which (4.2) is non-negative, and give precise bounds on this change of value in case it is negative. In the probabilistic setting of Section 3.1.2, the positive and negative cash-flow risks S_c^+ and S_c^- associated to an arbitrary portfolio of future cash-flows $c = c^+ - c^-$ have been introduced. In view of (4.1) the normalization assumption $\sum_{k=1}^{n} \alpha_k = \sum_{j=1}^{n} \ell_j = 1$ will be made from now on. In this setting, the change in portfolio value (4.2) identifies with the mean difference

$$\Delta V_c = E\left[f(S_c^+)\right] - E\left[f(S_c^-)\right]. \quad (4.3)$$

Under duration matching, that is $D_c := D_{c^+} - D_{c^-} = 0$, assumed in immunization theory, the non-negativity of the difference (4.3) is best analyzed within the context of stochastic orders. The notion of convex order, as first propagated by [28]-[30] ((see also [18] and [32]), yields the simplest and most useful results.

Definition 4.1. A random variable X precedes Y in *convex order* or *stop-loss order* by equal means, written $X \le_{cx} Y$ or $X \le_{sl,=} Y$, if

$E[X] = E[Y]$ and one of the following equivalent properties is fulfilled :

(CX1) $E[f(X)] \le E[f(Y)]$ for all convex real functions $f(x)$ for which the expectations exist

(CX2) $E[(X-d)_+] \le E[(Y-d)_+]$ for all real numbers d

(CX3) $E[|X-d|] \le E[|Y-d|]$ for all real numbers d

(CX4) There exists a random variable $Y' =_d Y$ (equality in distribution) such that $E[Y'|X] = X$ with probability one

The equivalence of (CX1), (CX2) and (CX4) is well-known from the literature ([18], [32]). The equivalence of (CX2) and (CX3) follows immediately from the identity $(X-d)_+ = \frac{1}{2}\{|X-d| + X - d\}$ using that the means are equal. The partial order induced by (CX3) has also been called *dilation order* ([30], [31]).

Deterministic hedging strategies for portfolios of fixed income securities are based on the following three main results.

Theorem 4.1. Let S_c^+ and S_c^- be the positive and negative cash-flow risks of a portfolio of future cash-flows $c = c^+ - c^-$ with vanishing duration $D_c = 0$, and let $f(s)$ be a convex shift factor of the TSIR. If $V_c = 0$ the portfolio of future cash-flows is *immunized*, that is $\Delta V_c = E[f(S_c^+)] - E[f(S_c^-)] \ge 0$ if, and only if, one has $S_c^- \le_{cx} S_c^+$.

Proof. This is immediate by the property (CX1). ◊

Theorem 4.2. Under the assumptions of Theorem 4.1, a portfolio of future cash-flows is immunized if, and only if, the difference between the mean absolute deviation indices of the positive and negative cash-flows is non-negative, that is

$$E\left[\left|S_c^+ - k\delta\right|\right] \geq E\left[\left|S_c^- - k\delta\right|\right], \quad k = 1,\dots,n. \tag{4.4}$$

Proof. This generalization of earlier results by [9], [10] (sufficient condition under constant shift factors) and [33] (necessary condition under convex shift factors) has been derived in [14]. ◊

In case the shift factor of the TSIR is not convex, immunization results can be obtained through generalization of the notion of convex function (see [15] for another extension).

Definition 4.2. Given are real numbers α and β, and an interval $I \subseteq R$. A real function $f(x)$ is called α*-convex* on I if $f(x) - \frac{1}{2}\alpha x^2$ is convex on I. It is called *convex-*β on I if $\frac{1}{2}\beta x^2 - f(x)$ is convex on I.

Note that a twice differentiable shift factor $f(s)$ on the support $[\delta, n\delta]$ is automatically α-convex with $\alpha = \inf_{s\in[\delta,n\delta]}\{f''(s)\}$, and convex-$\beta$ with $\beta = \sup_{s\in[\delta,n\delta]}\{f''(s)\}$.

Theorem 4.3. Let S_c^+ and S_c^- be the positive and negative cash-flow risks of a portfolio of future cash-flows $c = c^+ - c^-$ with vanishing duration $D_c = 0$, and let the shift factor $f(s)$ be α-convex and convex-β on the support $[\delta, n\delta]$. If $V_c = 0$ and $S_c^- \leq_{cx} S_c^+$ the change in portfolio under the shift factor satisfies the upper and lower bounds

$$\frac{1}{2}\alpha \cdot (M_{c^+}^2 - M_{c^-}^2) \leq \Delta V_c = E\left[f(S_c^+)\right] - E\left[f(S_c^-)\right] \leq \frac{1}{2}\beta \cdot (M_{c^+}^2 - M_{c^-}^2). \tag{4.5}$$

Proof. This result by [34], which expands on ideas by [23], is a simple consequence of the characterizing property (CX1) in Definition 4.1. By assumption, one has the inequalities

$$E\left[f(S_c^-)\right]-\frac{1}{2}\alpha\cdot E\left[(S_c^-)^2\right]\le E\left[f(S_c^+)\right]-\frac{1}{2}\alpha\cdot E\left[(S_c^+)^2\right],$$
$$\frac{1}{2}\beta\cdot E\left[(S_c^-)^2\right]-E\left[f(S_c^-)\right]\le\frac{1}{2}\beta\cdot E\left[(S_c^+)^2\right]-E\left[f(S_c^+)\right] \qquad (4.6)$$

Since $D_c=0$ one has $E\left[(S_c^+)^2\right]-E\left[(S_c^-)^2\right]=Var\left[S_c^+\right]-Var\left[S_c^-\right]=M_{c^+}^2-M_{c^-}^2$, hence (4.5). ◊

Remarks 4.1.

(i) In the terminology of [34] the condition $S_c^-\le_{cx}S_c^+$ means that the portfolio of future cash-flows $c=c^+-c^-$ is *Shiu decomposable*. In this situation, one has necessarily $D_c=0$ and $M_{c^+}^2-M_{c^-}^2\ge 0$ (Proposition 1 in [34]). An algorithm to generate Shiu decomposable portfolios by given negative cash-flow risk S_c^- is found in [13], Corollary A.1. Half of the difference in M-square indices, that is $\frac{1}{2}(M_{c^+}^2-M_{c^-}^2)$, has been called *Shiu risk measure*. For further details consult [14].

(ii) For the interested reader we mention that it is possible to extend some of the above results to the immunization of economic cash-flow products as considered first in [3] (for this consult [16], Sections 7 and 8).

(iii) An extension to directional immunization along the line of [17] can also be formulated.

4.2 Static immunization bounds with stochastic affine measures of duration and convexity

The three main results of Section 4.1 are also valid mutatis mutandis for the Macaulay and stochastic affine measures of duration and convexity. In particular, Theorem 4.3 extends to the stochastic affine risk measurement context as follows.

Theorem 4.4. Let $S_c^{aff,+}$ and $S_c^{aff,-}$ be the affine positive and negative cash-flow risks of a portfolio of future cash-flows $c = c^+ - c^-$ with vanishing affine duration $D_c^{aff} := D_{c^+}^{aff} - D_{c^-}^{aff} = 0$, and let the shift factor $f(s)$ be α-convex and *convex-* β on the support $[b(\delta), b(n\delta)]$. If $V_c^{aff} = 0$ and $S_c^{aff,-} \leq_{cx} S_c^{aff,+}$ the change in portfolio under the shift factor satisfies the upper and lower bounds

$$\begin{aligned} &\frac{1}{2}\alpha \cdot (M_{c^+}^{2,aff} - M_{c^-}^{2,aff}) \\ &\leq \Delta V_c^{aff} = E\left[f(S_c^{aff,+})\right] - E\left[f(S_c^{aff,-})\right]. \\ &\leq \frac{1}{2}\beta \cdot (M_{c^+}^{2,aff} - M_{c^-}^{2,aff}) \end{aligned} \tag{4.7}$$

Similarly to the remark preceding Theorem 4.3 one notes that a twice differentiable shift factor $f(s)$ on the support $[b(\delta), b(n\delta)]$ is automatically α -convex with $\alpha = \inf_{s \in [b(\delta), b(n\delta)]} \{f''(s)\}$, and convex- β with $\beta = \sup_{s \in [b(\delta), b(n\delta)]} \{f''(s)\}$. As observed at the beginning of Section 3, the Fisher-Weil measures are only consistent with Merton's model. Therefore, Theorem 4.4 has a more realistic and wider range of application than Theorem 4.3, which has been initially derived in [14], Theorem 2.3. The significance of the new formulation for hedge optimization is illustrated in Section 5.

Remark 4.2. The topic of dynamic immunization strategies, which is not touched upon within the present work, can be treated as in [25], Section 12.4.

5 Static immunization bounds for a single swap liability

As motivated at the beginning of Section 4, we illustrate the (static) hedging of portfolios of swaps (as liabilities) through portfolios of bonds (as assets). To illustrate the main features we focus solely on hedging a single swap liability with two bonds (as assets), and observe that the general case can be treated similarly. For simplicity, we assume an affine TSIR such that $P_k = P(k\delta) = P^{k\delta}(0,r),\ k = 1,...,n$, where $T = n\delta$ is the maximum maturity of the considered swaps and bonds. We restrict ourselves to the Vasicek and CIR models described in the Examples 3.1. By the results of Section 3.3 the durations and convexities of a portfolio of bonds and swaps reduce to the durations and convexities of two bond portfolios corresponding to the asset respectively liability side. For simplicity, we fix the tenor $\delta = 1$ and suppose the asset side is represented by a bond portfolio $B^+ = \{B_1^+, B_2^+\}$ with $B_i^+ = B(n_i, H_i, R_i, 1),\ i = 1,2$. Without loss of generality we assume that $n_1 < n_2$.

Moreover, one usually has $R_1 \le R_2$ (higher interest reward for longer bond maturities). Similarly, the liability side is represented by a bond $B^- = B(m, H, K, 1)$. Recall that the fixed interest rate of a swap, or *swap rate* (e.g. Munk (2011), Section 6.5.1, equation (6.32)), is set such that the swap has zero net present value at contract agreement, i.e.

$$K = (1 - P(m)) / \sum_{j=1}^{m} P(j). \tag{5.1}$$

For duration matching one needs the assumption $n_1 < m < n_2$. Therefore, the maximum maturity $T = n$ is described by the integer $n = n_2$. The cash-flow vector $c = c^+ - c^-$ of this portfolio is given by

$$\begin{aligned} c^+ &= (C_0^+, C_1^+, ..., C_n^+),\ C_0^+ = 0,\ C_j^+ = \sum_{i=1}^{2} H_i \left[1\{j = n_i\} + R_i 1\{j \le n_i\} \right], \quad j = 1,...,n, \\ c^- &= (C_0^-, C_1^-, ..., C_m^-),\ C_0^- = -H,\ C_j^- = H\left[1\{j = m\} + K 1\{j \le m\} \right], \quad j = 1,...,m. \end{aligned} \tag{5.2}$$

To be able to apply the static immunization bounds, the following assumptions are made (normalization and duration matching assumptions):

$$\sum_{k=1}^{n} \alpha_k = \sum_{j=1}^{n} \ell_j = 1, \quad \sum_{k=1}^{n} \alpha_k^{aff} = \sum_{j=1}^{n} \ell_j^{aff} = 1, \tag{5.3}$$

$$D_{c^+} = D_{c^-} \quad \text{(Fisher-Weil)}, \qquad D_{c^+}^{aff} = D_{c^-}^{aff} \quad \text{(affine risk measures)}. \tag{5.4}$$

For a given TSIR the parameters of the swap liability are fixed, but those of the two bond assets may vary in order to obtain an "optimal" or best possible hedge. In the notations of Section 3, and with the above simplifying assumption on the TSIR, the current prices of the future liability cash-flows satisfy the following relationships

$$\ell_j = P(j)C_j^- = \ell_j^{aff}, \quad j = 1,...,m. \tag{5.5}$$

Since $\ell_j = P(j)HK,\ j = 1,...,m-1,\ \ell_m = P(m)H(1+K)$, and in virtue of the relations (5.1) and (5.5), the normalization assumptions $\sum_{j=1}^{m} \ell_j = \sum_{j=1}^{m} \ell_j^{aff} = 1$ are fulfilled if, and only if, one has $H = 1$. The Fisher-Weil and affine durations of the future liability cash-flows are given by

$$D_{c^-} = mP(m) + K \cdot \sum_{j=1}^{m} jP(j), \quad D_{c^-}^{aff} = b(m)P(m) + K \cdot \sum_{j=1}^{m} b(j)P(j). \tag{5.6}$$

For the bonds indexed $i = 1,2$ on the asset side consider the quantities defined and denoted by

$V_i = P(n_i) + R_i \cdot \sum_{k=1}^{n_i} P(k)$: present value of B_i^+ per unit of principal

$D_i = n_i P(n_i) + R_i \cdot \sum_{k=1}^{n_i} kP(k)$: Fisher-Weil duration of B_i^+ per unit of principal

$D_i^{aff} = b(n_i)P(n_i) + R_i \cdot \sum_{k=1}^{n_i} b(k)P(k)$: affine duration of B_i^+ per unit of principal

It is clear that $D_i \neq D_i^{aff},\ i = 1,2$. Therefore, by fixed interest rates and maturities of the bonds, the duration matching assumptions can only be fulfilled if one assumes different bond principals in the Fisher-Weil and affine cases, that is $H_i \neq H_i^{aff},\ i = 1,2$. With these definitions the present value of the future asset cash-flows, denoted V, is by no-arbitrage uniquely given by

$$V = \sum_{k=1}^{n} \alpha_k = H_1 V_1 + H_2 V_2 = \sum_{k=1}^{n} \alpha_k^{aff} = H_1^{aff} V_1 + H_2^{aff} V_2, \tag{5.7}$$

and the corresponding Fisher-Weil and affine durations are given by

$$D_{c^+} = H_1 D_1 + H_2 D_2, \quad D_{c^+}^{aff} = H_1^{aff} D_1^{aff} + H_2^{aff} D_2^{aff}. \tag{5.8}$$

It follows that the normalization and duration matching assumptions are equivalent to the following systems of linear equations. For the Fisher-Weil duration one has the linear system

$$H_1 V_1 + H_2 V_2 = 1, \quad H_1 D_1 + H_2 D_2 = D_{c^-}, \tag{5.9}$$

and for the affine duration one has

$$H_1^{aff} V_1 + H_2^{aff} V_2 = 1, \quad H_1^{aff} D_1^{aff} + H_2^{aff} D_2^{aff} = D_{c^-}^{aff}. \tag{5.10}$$

In the following the determinants of the linear systems (5.9)-(5.10) do not vanish, an assumption which holds in practical applications. Solving these equations one sees that under the normalization and duration matching assumptions the principals of the asset bonds are uniquely determined. For the Fisher-Weil duration one obtains

$$H_1 = \frac{D_2 - D_{c^-} V_2}{D_2 V_1 - D_1 V_2}, \quad H_2 = \frac{D_{c^-} V_1 - D_1}{D_2 V_1 - D_1 V_2}, \tag{5.11}$$

and for the affine duration one has

$$H_1^{aff} = \frac{D_2^{aff} - D_{c^-}^{aff} V_2}{D_2^{aff} V_1 - D_1^{aff} V_2}, \quad H_2^{aff} = \frac{D_{c^-}^{aff} V_1 - D_1^{aff}}{D_2^{aff} V_1 - D_1^{aff} V_2}. \tag{5.12}$$

If *short* bond positions are allowed for hedging, i.e. $H_i < 0$ for some $i = 1,2$, then there exists a unique bond portfolio satisfying the normalization and duration matching assumptions for all maturity choices $n_1 < m < n_2$. A bond portfolio is *strictly feasible* if only long bond positions are allowed for hedging, i.e. for $i = 1,2$ one has $H_i V_i \in (0,1)$ respectively $H_i^{aff} V_i \in (0,1)$. The conditions under which (5.11) and (5.12) yield strictly feasible bond portfolios are not simple. Counterexamples to strict feasibility are found in the Tables below.

For hedge optimization it is further most important to find feasible bond portfolios such that the corresponding (affine) negative and positive cash-flow risks are stochastically ordered in the convex sense, that is such that $S_c^- \leq_{cx} S_c^+$ respectively $S_c^{aff,-} \leq_{cx} S_c^{aff,+}$. Indeed, according to Theorem 4.1 and its stochastic affine pendant, these stochastic inequalities are the necessary and sufficient conditions under which the swap liability will be immunized under arbitrary convex shift factors. A feasible bond portfolio satisfying the convex ordering will be called a *convex hedge*. With the Theorems 4.3 and 4.4 it is then possible to construct lower and upper static immunization bounds for the change in portfolio value.

The following numerical examples are based on a Vasicek model with parameters $\kappa = 0.15, \theta = 0.05, \sigma = 0.015$ and a CIR model with $\kappa = 0.15, \theta = 0.05, \sigma = 0.065$, both with an initial short rate $r = 0.055$ (see the Examples 3.1). The shift factor reflects a change in the short rate of amount $\Delta r = -0.01$ (increase of 1% in the short rate) and takes the form $f(s) = \exp\{b(s)\Delta r\}$. To evaluate the immunization bounds in the Theorems 4.3 and 4.4 we need the second derivative $f''(s) = \Delta r \cdot \{b''(s) + \Delta r \cdot b'(s)^2\} \cdot f(s)$ with

$$b''(s) = -\kappa \cdot \exp\{-\kappa s\}, \text{ Vasicek model,} \tag{5.13}$$

$$\begin{aligned} &b''(s) = h(s)^{-1} \cdot \{g''(s) - h''(s) \cdot b(s) - 2h'(s) \cdot b'(s)\}, \\ &b'(s) = h(s)^{-1} \cdot \{g'(s) - h'(s) \cdot b(s)\}, \quad b(s) = h(s)^{-1} \cdot g(s), \quad \text{CIR model,} \\ &g(s) = 2\left(e^{\gamma s} - 1\right), \quad h(s) = (\gamma + \kappa)\left(e^{\gamma s} - 1\right) + 2\gamma \end{aligned} \tag{5.14}$$

The tenor of the bonds and swaps is fixed at $\delta = 1$ and the interest rates of the bonds are set equal to $R_1 = 0.05, R_2 = 0.06$. The Tables 1 to 4 list our results for some triples $n_1 < m < n_2$ with varying swap maturity $m \in \{2,...,15\}$.

Let us comment on the obtained results. If short bond positions are allowed for hedging, then the narrowest triples $(n_1 = m-1, m, n_2 = m+1)$ in our numerical examples almost always lead to convex hedges. An exception is the triple (3, 4, 5) for the Fisher-Weil duration in both the Vasicek and CIR models, for which the convex ordering $S_c^- \le_{cx} S_c^+$ only slightly fails. But, even in this case, the "formal" immunization bounds (4.5), marked bold in the Tables, seem to work well (though by Theorem 4.1 there will be some convex shift factor for which this does not hold). Another exception is the triple (14, 15, 16) for the affine CIR duration (exploding amounts of principals due to an almost vanishing determinant). Strictly feasible narrow triples $(n_1 = m-1, m, n_2 = m+1)$ seem to yield the best possible convex hedges with the smallest range of variation for the immunization bounds by fixed maturity of the swap liability. However, this is not true if short bond positions are allowed as counterexamples in the Tables suggest (e.g. the triples (12, 13, 14) for the affine Vasicek and CIR models).

Table 1: Convex hedges and immunization bounds for a single swap (Fisher-Weil Vasicek)

swap		first bond				second bond				immunization bounds		
m	K	n1	V1	D1	H1	n2	V2	D2	H2	ΔV_min	ΔV	ΔV_max
											per mill	
2	0.05571	1	0.99420	0.99420	0.48651	3	1.01270	2.87065	0.50984	0.44762	0.52839	0.62203
3	0.05529	1	0.99420	0.99420	0.31225	4	1.01795	3.74263	0.67740	0.72587	0.90661	1.18562
4	0.05488	3	0.98575	2.81771	0.48637	5	1.02356	4.57847	0.50857	**0.29177**	0.34316	**0.55928**
4	0.05488	2	0.98948	1.93161	0.31035	5	1.02356	4.57847	0.67697	0.58834	0.73347	1.12775
5	0.05448	4	0.98292	3.65737	0.48424	6	1.02946	5.38119	0.50903	0.23849	0.28062	0.53575
6	0.05411	5	0.98086	4.45485	0.48017	7	1.03555	6.15332	0.51086	0.19846	0.23486	0.52184
7	0.05376	6	0.97946	5.21380	0.47376	8	1.04177	6.89702	0.51448	0.16954	0.20388	0.52130
8	0.05343	7	0.97862	5.93736	0.46463	9	1.04805	7.61407	0.52030	0.14980	0.18583	0.53810
9	0.05312	8	0.97823	6.62821	0.45240	10	1.05434	8.30603	0.52872	0.13744	0.17901	0.57635
10	0.05283	9	0.97822	7.28865	0.43662	11	1.06059	8.97418	0.54016	0.13090	0.18190	0.64039
11	0.05257	10	0.97852	7.92066	0.41681	12	1.06677	9.61966	0.55508	0.12883	0.19314	0.73486
12	0.05232	11	0.97905	8.52597	0.39235	13	1.07284	10.24345	0.57405	0.13012	0.21164	0.86495
13	0.05210	12	0.97978	9.10607	0.36249	14	1.07879	10.84642	0.59774	0.13388	0.23653	1.03671
13	0.05210	12	0.97978	9.10607	0.61839	15	1.08459	11.42934	0.36337	0.10604	0.16534	0.95618
14	0.05189	13	0.98066	9.66229	0.32625	15	1.08459	11.42934	0.62702	0.13945	0.26723	1.25748
14	0.05189	13	0.98066	9.66229	0.60570	16	1.09023	11.99293	0.37241	0.09901	0.16792	1.03932
15	0.05170	14	0.98164	10.19583	0.28235	16	1.09023	11.99293	0.66301	**0.14637**	0.30350	**1.53650**
15	0.05170	14	0.98164	10.19583	0.59074	17	1.09571	12.53783	0.38341	0.09416	0.17505	1.15031

Table 2: Convex hedges and immunization bounds for a single swap (Fisher-Weil CIR)

	swap			first bond				second bond			immunization bounds	
m	K	n1	V1	D1	H1	n2	V2	D2	H2	ΔV_min	ΔV	ΔV_max
											per mill	
2	0.05570	1	0.99420	0.99420	0.48651	3	1.01270	2.87069	0.50983	0.46870	0.54801	0.63607
3	0.05528	1	0.99420	0.99420	0.31225	4	1.01796	3.74268	0.67739	0.76268	0.94554	1.21237
4	0.05487	3	0.98576	2.81774	0.48636	5	1.02357	4.57849	0.50858	**0.30627**	0.36044	**0.57192**
4	0.05487	2	0.98948	1.93162	0.31034	5	1.02357	4.57849	0.67697	0.61756	0.76929	1.15321
5	0.05448	4	0.98293	3.65742	0.48423	6	1.02944	5.38106	0.50905	0.24915	0.29450	0.54786
6	0.05411	5	0.98086	4.45487	0.48018	7	1.03548	6.15284	0.51088	0.20563	0.24530	0.53352
7	0.05377	6	0.97944	5.21367	0.47386	8	1.04163	6.89588	0.51447	0.17367	0.21125	0.53261
8	0.05345	7	0.97855	5.93688	0.46491	9	1.04781	7.61191	0.52020	0.15124	0.19057	0.54896
9	0.05315	8	0.97809	6.62709	0.45297	10	1.05397	8.30235	0.52843	0.13636	0.18146	0.58653
10	0.05288	9	0.97798	7.28652	0.43767	11	1.06007	8.96846	0.53956	0.12728	0.18217	0.64940
11	0.05263	10	0.97815	7.91706	0.41854	12	1.06606	9.61129	0.55401	0.12249	0.19314	0.74196
12	0.05240	11	0.97854	8.52036	0.39500	13	1.07284	10.23179	0.57231	0.12077	0.20707	0.86912
13	0.05220	12	0.97909	9.09789	0.36637	14	1.07765	10.83081	0.59509	0.12118	0.22888	1.03660
13	0.05220	12	0.97909	9.09789	0.62102	15	1.08320	11.40911	0.36185	0.09377	0.15355	0.95397
14	0.05200	13	0.97977	9.65093	0.33169	15	1.08320	11.40911	0.62317	0.12300	0.25584	1.25143
14	0.05200	13	0.97977	9.65093	0.60936	16	1.08858	11.96739	0.37018	0.08498	0.15335	1.02868
15	0.05183	14	0.98054	10.18064	0.28976	16	1.08858	11.96739	0.65763	**0.12578**	0.28756	**1.52253**
15	0.05183	14	0.98054	10.18064	0.59565	17	1.09378	12.50633	0.38028	0.07837	0.15752	1.12915

Table 3: Convex hedges and immunization bounds for a single swap (affine Vasicek)

	swap			first bond				second bond			immunization bounds	
m	K	n1	V1	D1	H1	n2	V2	D2	H2	ΔV_min	ΔV	ΔV_max
											per mill	
2	0.05571	1	0.99420	0.92323	0.44884	3	1.01270	2.32498	0.54682	0.27013	0.30473	0.34537
3	0.05529	1	0.99420	0.92323	0.26449	4	1.01795	2.84464	0.72405	0.37381	0.43378	0.52636
4	0.05488	3	0.98575	2.28012	0.44740	5	1.02356	3.27605	0.54611	0.11205	0.12239	0.17135
4	0.05488	2	0.98948	1.67186	0.26181	5	1.02356	3.27605	0.72389	0.24326	0.27636	0.37202
5	0.05448	4	0.98292	2.77548	0.44378	6	1.02946	3.63597	0.54766	0.07527	0.08145	0.12354
6	0.05411	5	0.98086	3.17991	0.43695	7	1.03555	3.93779	0.55180	0.05335	0.05770	0.09304
7	0.05376	6	0.97946	3.51096	0.42553	8	1.04177	4.19226	0.55983	0.04110	0.04501	0.07550
8	0.05343	7	0.97862	3.78271	0.40758	9	1.04805	4.40803	0.57358	0.03533	0.03968	0.06786
9	0.05312	8	0.97823	4.00642	0.38009	10	1.05434	4.59204	0.59580	0.03412	0.03955	0.06811
10	0.05283	9	0.97822	4.19112	0.33810	11	1.06059	4.74991	0.63103	0.03649	0.04354	0.07526
11	0.05257	10	0.97852	4.34408	0.27251	12	1.06677	4.88617	0.68745	0.04225	0.05156	0.08965
12	0.05232	11	0.97905	4.47117	0.16467	13	1.07284	5.00453	0.78183	0.05234	0.06488	0.11381
13	0.05210	12	0.97978	4.57712	**-0.03095**	14	1.07879	5.10798	**0.95507**	0.07014	**0.08784**	0.15573
13	0.05210	12	0.97978	4.57712	0.47422	15	1.08459	5.19897	0.49361	0.02876	0.03531	0.06500
14	0.05189	13	0.98066	4.66576	**-0.46277**	15	1.08459	5.19897	**1.34043**	0.10766	**0.13574**	0.24336
14	0.05189	13	0.98066	4.66576	0.39526	16	1.09023	5.27949	0.56170	0.03527	0.04407	0.08098
15	0.05170	14	0.98164	4.74020	**-2.05163**	16	1.09023	5.27949	**2.76453**	0.24085	**0.30500**	0.55291
15	0.05170	14	0.98164	4.74020	0.25278	17	1.09571	5.35120	0.68619	0.04694	0.05930	0.10921

Table 4: Convex hedges and immunization bounds for a single swap (affine CIR)

swap		first bond				second bond				immunization bounds		
m	K	n1	V1	D1	H1	n2	V2	D2	H2	ΔV_min	ΔV	ΔV_max
											per mill	
2	0.05570	1	0.99420	0.92262	0.44704	3	1.01270	2.31377	0.54858	0.27758	0.31009	0.34703
3	0.05528	1	0.99420	0.92262	0.26172	4	1.01796	2.82246	0.72674	0.38142	0.43801	0.52178
4	0.05487	3	0.98576	2.26906	0.44413	5	1.02357	3.23973	0.54925	0.11199	0.12170	0.16554
4	0.05487	2	0.98948	1.66790	0.25818	5	1.02357	3.23973	0.72739	0.24434	0.27546	0.36118
5	0.05448	4	0.98293	2.75365	0.43980	6	1.02944	3.58310	0.55147	0.07370	0.07942	0.11642
6	0.05411	5	0.98086	3.14425	0.43215	7	1.03548	3.86676	0.55638	0.05119	0.05517	0.08556
7	0.05377	6	0.97944	3.45923	0.41964	8	1.04163	4.10217	0.56545	0.03885	0.04244	0.06814
8	0.05345	7	0.97855	3.71342	0.39994	9	1.04781	4.29855	0.58087	0.03323	0.03725	0.06070
9	0.05315	8	0.97809	3.91881	0.36932	10	1.05397	4.46333	0.60606	0.03228	0.03733	0.06103
10	0.05288	9	0.97798	4.08502	0.32110	11	1.06007	4.60247	0.64710	0.03502	0.04162	0.06817
11	0.05263	10	0.97815	4.21978	0.24196	12	1.06606	4.72076	0.71603	0.04152	0.05035	0.08285
12	0.05240	11	0.97854	4.32928	0.10065	13	1.07284	4.82204	0.84102	0.05363	0.06590	0.10925
13	0.05220	12	0.97909	4.41850	**-0.19770**	14	1.07765	4.90943	**1.10757**	0.07836	**0.09710**	0.16245
13	0.05220	12	0.97909	4.41850	0.43860	15	1.08320	4.98540	0.52674	0.02788	0.03409	0.05866
14	0.05200	13	0.97977	4.49140	**-1.14281**	15	1.08320	4.98540	**1.95688**	0.15373	**0.19147**	0.32343
14	0.05200	13	0.97977	4.49140	0.32181	16	1.08858	5.05196	0.62898	0.03681	0.04564	0.07841
15	0.05183	14	0.98054	4.55119	**46.744**	16	1.08858	5.05196	**-41.186**	**n.d.**	**-4.47898**	**n.d.**
15	0.05183	14	0.98054	4.55119	0.05617	17	1.09378	5.11074	0.86391	0.05667	0.07083	0.12198

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