Theoretical Mathematics & Applications, vol.3, no.2, 2013, 129-141
ISSN: 1792- 9687 (print), 1792-9709 (online)
Scienpress Ltd, 2013

Analysis of MHD flow of a dusty viscoelastic fluid through a horizontal circular channel

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Abstract

An analysis was carried out to study the motion of an incompressible viscoelastic dusty fluid through a horizontal circular pipe placed under a transverse magnetic field such that the pressure gradient is varying in magnitude. The partial differential equations governing the flow are converted into ordinary differential equations which are then solved numerically by finite element method. The effects of the viscoelastic and magnetic field parameter on the velocity profile of both the fluid and particles are thoroughly examined.

Mathematics Subject Classification: 76R50

Keywords: Viscoelastic fluid, Magnetohydrodynamics, Maxwell model, Finite element model, Oscillating pressure gradient

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Article Info: *Received* : March 12, 2013. *Revised* : May 29, 2013
Published online : June 25, 2013

1 Introduction

The flow of a dusty viscoeleastic fluid through the circular pipes has many applications in various fields of science and engineering. For example, in the field of bio-engineering, this investigation is of particular interest since blood in veins is forced by a periodic pressure gradient. Some problems that involve the dynamic response of the fluid to the frequency of the periodic pressure gradient are normally occurring in chemical and petroleum industries. Fluids such as honey, oil, blood and some polymer solutions exhibit both viscous and elastic properties. These types of fluid are often refers to as viscoelastic fluids. It is very difficult to choose models that will exhibit all the properties of viscolelastic fluids. There exist many viscoelastic models and constitution equations among those the Maxwell Model which consists of a series arrangement of a purely viscous element assigned as dashpot and a perfectly elastic body is assigned as a spring.
However, the Maxwell Model is a linear viscoelastic rheological model and it is proper only under the condition that non-linear effects are negligible, such as very low strain and stress.

Andrienko et. Al [1] studied the unidirectional oscillating flow of viscoelastic fluid with the Maxwell Model in a tube; they found that the instantaneous velocities drastically increase at certain frequencies of the oscillating pressure gradient and called this phenomenon as the resonance-like behaviour of viscoeleastic fluids. Attia and Ewis [2] investigate the unsteady MHD couette flow of a viscoelastic fluid under exponential decaying pressure gradient. They found that the viscoeleastic parameter has a marked effect on the velocity and their steady state times for all values of magnetic field and the suction velocity. Mohyuddin [3] discussed the oscillating poisueille flow of a linear viscoelastic fluid for small radii of the pipe. Approximate resonance frequency and possible velocity enhancement are obtained and observed graphically.

Tan et al [4] studied unsteady flow of a viscoeleastic fluid with the traditional Maxwell model between two infinite parallel plates. The oscillating flow of a viscoelastic fluid in a pipe with the traditional Maxwell model was studied by Yin and Zhu [5]. They discussed amplitudes of resonance peaks which decay rapidly with frequency.

The development in the study of dusty viscoelastic fluid flow in a circular channel has been continuously going on because of its unavoidable applications in various fields. Therefore in this paper we would try to evolve a mathematical model for a dusty viscoelastic fluid flow in a circular channel. As the magnetic field (both natural and artificial) plays an important role in the motion of fluids, we have considered flows in horizontal circular channel subjected to applied magnetic field so that the results obtained may be applied in different branches of science and technology, industries and problems of medical sciences.

2 Formulation of the Problem

Consider the flow of a dusty viscoelastic incompressible fluid through a circular channel placed under transverse applied magnetic field taking along the axis of the channel.

If we considered that the flow is to be fully developed and symmetric and the velocity of fluid and particle phase are function of radial distance r and time t only. The governing equation of motion of a dusty viscoelastic incompressible fluid are given by

$$\rho \frac{\partial u}{\partial t} = P(t) + \frac{1}{r}\frac{\partial (r\tau)}{\partial r} - KN\left(u_p - u\right) - \sigma B_0^2 u \tag{1}$$

and for particle phase, we have

$$\frac{\partial u_p}{\partial t} = \frac{K}{m}\left(u - u_p\right) \tag{2}$$

where

$$P(t) = -\frac{dp}{dt}$$

with initial and boundary condition

$$r\frac{du}{dr} = r\frac{du_p}{dr} = 0 \quad at \quad r = 0$$

$$u(1) = u_p(1) = 0 \quad at \;\; r = 1$$

where

$u(r,t)$: Axial velocity of fluid

$u_p(r,t)$: Axial velocity of dust particles

$P(t)$: Pressure

K: Stokes resistance coefficient

m: Mass of each particle

N: Number density of particles assumed to be constant

ρ: Density of fluid

σ: Electrical Conductivity

B_0: Magnetic Inductor

$\tau = \tau_{r,z}(r,t)$ is the shear stress.

We assume a linear viscoelastic fluid represented by the Maxwell Model. This model is obtained by thought of the Maxwell element which is subjected to a sudden elongation and the force is then calculated as a function of time. As seen from the mechanical assembly in Figure 1, the Maxwell element has no unique reference length and it will determine indefinitely when a force (per unit area) τ is applied. This behavour is analogous to the liquid-like behaviour of a melt of an uncross-linked polymeric material over its glass transition.

Assume that the stress τ (the force per unit area) in the spring $G\gamma_1$ and the stress τ_2 in the dashpot is

$$\mu\left(\frac{\partial\gamma_2}{\partial t}\right) = \mu\gamma_2, \qquad (3)$$

Hence, we have a relationship between τ_1 and τ_2 where

$$\tau = \tau_1 = \tau_2 \tag{4}$$

Since these are connected is series. In the system where the total strain of the system is written as

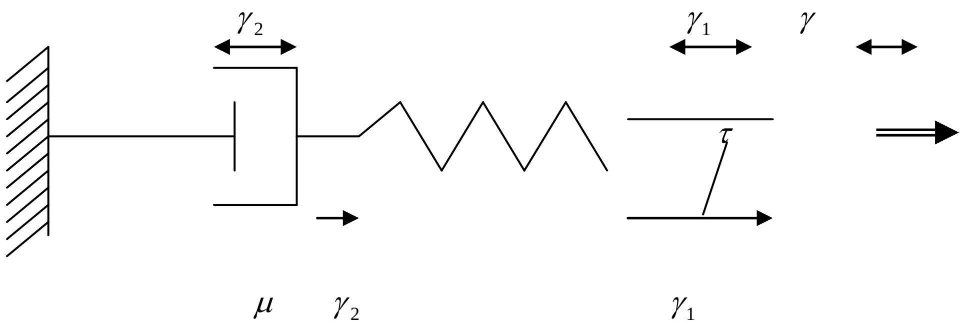

Figure 1: Maxwell element

$$\gamma = \gamma_1 + \gamma_2 \tag{5}$$

By differentiating equation (5) with represent to time t, we have

$$\frac{\partial \gamma}{\partial t} = \frac{\partial \gamma_1}{\partial t} + \frac{\partial \gamma_2}{\partial t} \tag{6}$$

So that with equation (4), we are able to write an equation relating the stress and strain rate

$$\begin{aligned}\frac{\partial \gamma}{\partial t} &= G^{-1}\left(\frac{\partial \gamma_1}{\partial t}\right) + (\tau_2)\,\mu^{-1} \\ &= G^{-1}\left(\frac{\partial \tau}{\partial t}\right) + (\tau)\,\mu^{-1}\end{aligned} \tag{7}$$

Equation (7) can be further reduced as an expression for τ where

$$\tau + \lambda \frac{\partial \tau}{\partial t} = \mu \frac{\partial \gamma}{\partial t} \tag{8}$$

$\lambda = \frac{\mu}{G}$ is called the relaxation time and G represents young modulus.

Equation (8) is the Maxwell model of a viscoelastic fluid such is a first order ordinary differential equation of τ. Since τ is only a function of time, equation (8) can be solved for τ to give

$$\tau = \ell^{-\frac{1}{\lambda}}\left[\int \frac{\mu}{\lambda}\left(\frac{\partial \gamma}{\partial t}\right)\ell^{\frac{t'}{\lambda}}dt' \ + c\right] \tag{9}$$

and using $\dot{\gamma}$

$$\tau = \int_{-\alpha}^{t}\left[\frac{\mu}{\lambda}\ell^{-\frac{t-t'}{\lambda}}\right]\gamma(t')dt' \tag{10}$$

Equation (8) is the differential equation for Maxwell Model and equation (10) is the integral equation for the Maxwell model, both of which are equivalent.

By apply integration by part of equation (10) it can be reduced to

$$\tau(t) = -\int_{\alpha}^{t}\left[\mu\lambda^{-2}\ell^{-\frac{t-t'}{\lambda}}\right]\gamma(t,t')dt \tag{11}$$

In this study, we assume a linear viscoelastic fluid represented by the Maxwell Model from equation (11), setting $\tau = \tau_{x,y}$ as follows

$$\tau + \lambda\frac{\partial \tau}{\partial t} = \mu_0\frac{\partial u}{\partial r} \tag{12}$$

Substitute equation (12) into (1) gives

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{d\rho}{dz} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial r^2} + \frac{\mu}{\rho}\frac{\partial u}{r\,\partial r} - \frac{\lambda\mu}{\rho}\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial r^2}\right) - \frac{\lambda}{r\rho}\frac{\partial}{\partial t}\left(\mu\frac{\partial u}{\partial r}\right) - \frac{KN}{\rho}\left(u-u_p\right) + \frac{\sigma B_0^2 u}{\rho} \tag{13}$$

But $$\frac{-1}{\rho}\frac{d\rho}{dz} = \lambda_1\ell^{iwt} \tag{14}$$

Since the flow is purely oscillatory, where λ_1 is a constant and w is the frequency of oscillations.

Due to the selected form of pressure gradient we assume the solution of the form

$$u(r,t) = f(r)\ell^{iwt} \tag{15}$$

$$u_p(r,t) = g(r)\ell^{iwt} \tag{16}$$

Introducing the following dimensionless variable

$$\bar{r} = \frac{r}{r_o}, \quad \bar{\tau} = \frac{\tau}{r_o} \quad \bar{u} = \frac{u}{u_o}, \quad \bar{u}_p = \frac{u_p}{u_o}, \quad t = \frac{tu_o}{r_o}, \quad \bar{\rho} = \frac{\rho}{\rho u_o^2} \tag{17}$$

Substituting (17) into equation (13) and (2), then remove the caps, gives

$$\frac{\partial u}{\partial t} = \frac{-d\rho}{dt} + \frac{1}{\mathrm{Re}}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r\,\mathrm{Re}}\frac{\partial u}{\partial r} - \frac{Q}{r}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial r}\right) - \frac{R}{\mathrm{Re}}\left(u_p - u\right) - Q\frac{\partial}{\partial t}\left(\frac{\partial^2 u}{\partial r^2}\right) + \frac{H_a}{\mathrm{Re}}u \tag{18}$$

for fluid phase and

$$\frac{\partial u_p}{\partial t} = \frac{1}{R_p}\left(u - u_p\right) \tag{19}$$

for dust particle phase, where

Re = $\frac{\rho r_o u_o}{\mu}$ (Renold number)

R= $\frac{KNr_o^2}{\mu}$ (particle concentration parameter)

$H_a = \frac{\sigma \beta_o^2 r_o^2}{\mu}$ (Hartman Number)

$Q = \frac{\mu^2}{\rho k_1 r_o^2}$ (Viscoelastic Parameter)

$k_1 = \frac{\lambda_1 \mu_0}{\rho r_0}$ (Kinematic Coefficient of viscoelasticity)

$R_p = \frac{mu_0}{KNr_0}$ (Relaxation time parameters of dust particles)

Substituting equation (15) and (16) into (18) and (19) to obtain

$$\frac{1}{r}\frac{\partial}{\partial r}\left(A_1 r\frac{\partial f}{\partial r}\right) - A_2 f - \lambda_1 = 0 \tag{20}$$

$$g = \left(1 + iwR_p\right)^{-1} f \tag{21}$$

where $A_1 = Qiw - \frac{1}{\text{Re}}$, $A_2 = \frac{R}{\text{Re}} - iw + \frac{H_a}{\text{Re}} - \frac{R}{\text{Re}(1 + iwR_p)}$.

The initial and boundary condition becomes

$$r\frac{\partial f(0)}{\partial r} = r\frac{\partial g(0)}{\partial r} = 0 \quad \text{at} \quad r = 0, \quad f(1) = g(1) = 0 \quad \text{at} \quad r = 1. \tag{22}$$

Weak formulation

In developing the weak formulation for equation (20), we multiply with weight function $v(r)$ and integrate over the volume of the cylindrical channel of the unit element. This gives

$$0 = 2\pi\int_{r_a}^{r_b}\left[A_1 r\frac{\partial f}{\partial r}\frac{\partial v}{\partial r} + A_2 rvf\right]dr + 2\pi\int_{r_q}^{r_b}\lambda_1 rvdr - v(r_a)Q_1^e - v(r_b)Q_2^e \tag{23}$$

where (r_a, r_b) is the domain of a typical element along the radial direction

$$Q_1^e = -2\pi\left(A_1 r\frac{\partial f}{\partial r}\right) \text{ at } r = r_a, \quad Q_1^e = -2\pi\left(A_1 r\frac{\partial f}{\partial r}\right) \text{ at } r = r_b. \tag{24}$$

Finite Element Model

The finite element model is obtained by substituting the approximation

$$f(r) \approx \sum_{j=1}^{n}\psi_i^e\psi_i^e(r) \tag{25}$$

and $v = \psi_1, \psi_2, \ldots\ldots\ldots\ldots, \psi_n$ into equation(23).

The finite element model is given by

$$[K^e][F^e] = [\lambda^e] + [Q] \tag{26}$$

where

$$K_j^e = 2\pi\int_{r_a}^{r_b}\left[A_1 r\frac{\partial\psi_i^e}{\partial r}\frac{\partial\psi_j^e}{\partial r} + A_2 r\psi_i^e\,\psi_j^e\right]dr \tag{27}$$

$$\lambda_j^e = 2\pi\int_{r_a}^{r_b}\psi_i^e\ \lambda_1 r\ \ dr \tag{28}$$

ψ_i^e are the interpolation functions expressed in term of the radial coordinate r and the linear interpolation function used in the studies are of the form

$$\psi_1^e(r) = \frac{r_b - r}{h_e} \tag{29}$$

$$\psi_1^e(r) = \frac{r - r_a}{h_e}. \tag{30}$$

Finite Element Computation

We consider a mean of four linear elements to solve equation (26) and obtained the following results

$$f_1 = \frac{2\lambda_1 - A_4 f_2}{A_3} \tag{31}$$

$$f_2 = \frac{24\lambda_1 - A_6 F_3 - 5A_4 f_4}{3A_4} \tag{32}$$

$$f_3 = \frac{\left(A_3 A_5 - A_4^2\right)\left(72A_5\lambda_1 - 180A_4\lambda_1\right) + 18A_4^2 A_5 \lambda_1}{\left(A_3 A_5 - A_4^2\right)\left(3A_5 A_6 - 25A_4^2\right) - 27A_4^2 A_5 A_3} \tag{33}$$

$$f_4 = \frac{36\lambda_1 - 5A_4 f_3}{3A_5} \tag{34}$$

where

$$A_3 = 96A_1 + A_2$$
$$A_4 = A_2 - 96A_1$$
$$A_5 = 384A_1 + 8A_2$$
$$A_6 = 768A_1 + 21A_2$$

3 Results and Discussion

The aim of this study is to investigate the effect of magnetic field parameter H_a (magnetic Hartmann number) and viscoelastic parameter (Q) at different constant values on the velocity of both the fluid and particles. This effect will be examined at different values of radius r in the medium (cylindrical channel). Figures 2-5 show the graphical representation of the velocity profile (f) with radius r for various values of magnetic and viscoelastic parameter.

It is noticed that at low value of magnetic field and viscoelastic parameter, the velocity of both the fluid and particles is not affected. But as the magnetic field and viscoelastic parameter increases, the horizontal velocity of fluid and particle decrease rapidly. When the radius is one (i.e at boundary of the channel), the velocity is zero at each value of magnetic field and viscoelastic parameter. This shows that increasing in magnetic field decrease the velocity of fluid and particle because the dust particle experiences an additional force while moving in a magnetic field

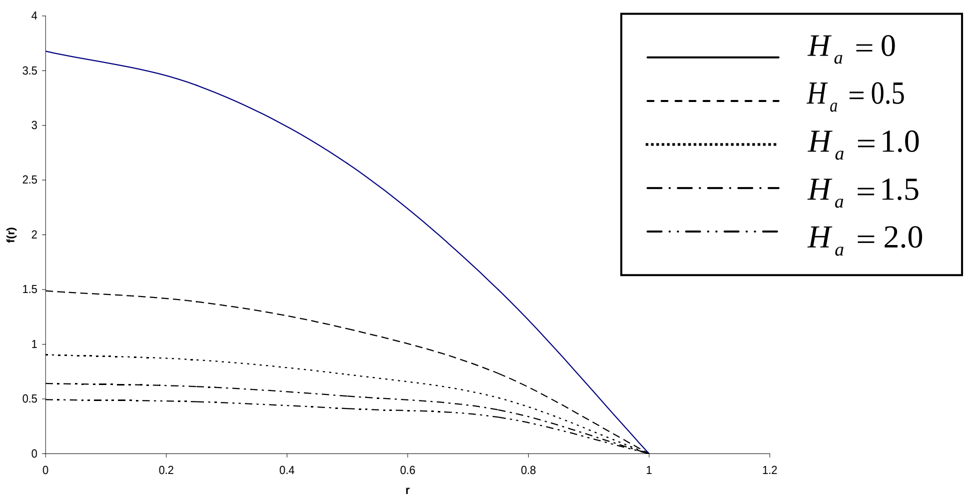

Figure 2: Velocity profile of fluid for different values of H_a,
when $\lambda = 1$, $\omega = 0.1$, $R_e = 100$, $R = 1$, $Q = 1$, and $R_p = 1$

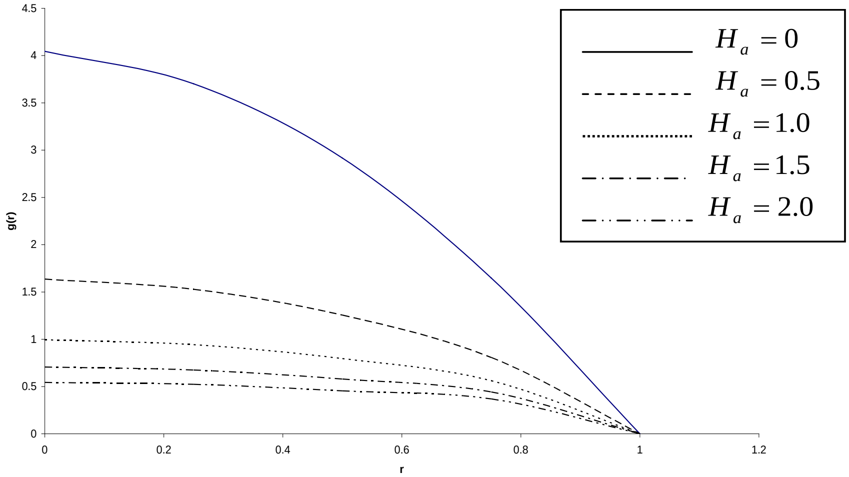

Figure 3: Velocity profile of particle for different values of H_a,
when $\lambda = 1$, $\omega = 0.1$, $R_e = 100$, $R = 1$, $Q = 1$, and $R_p = 1$

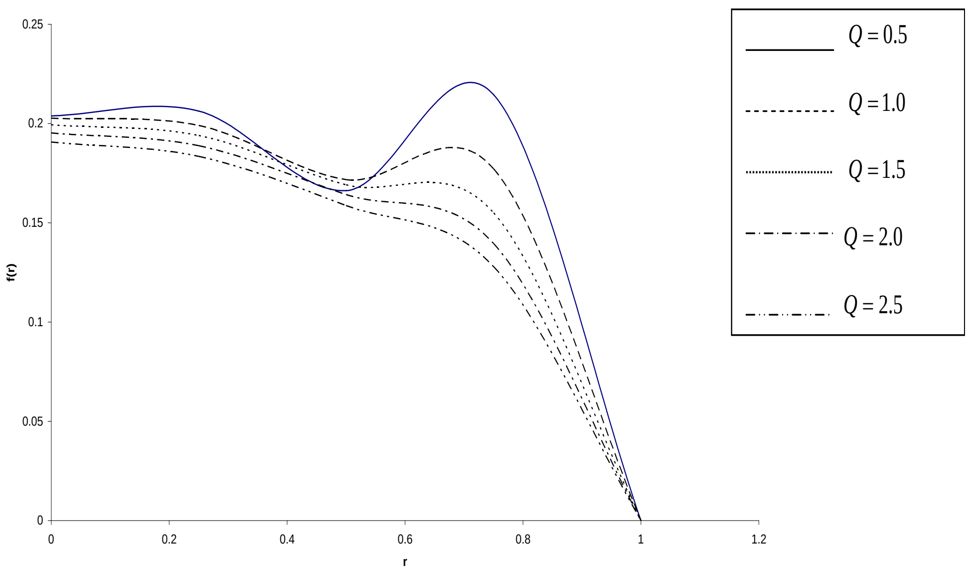

Figure 4: Velocity profile of fluid for different values of Q,
when $\lambda = 1, \omega = 0.1$, $R_e = 100$, $R = 1$, $H_a = 5$, and $R_p = 1$

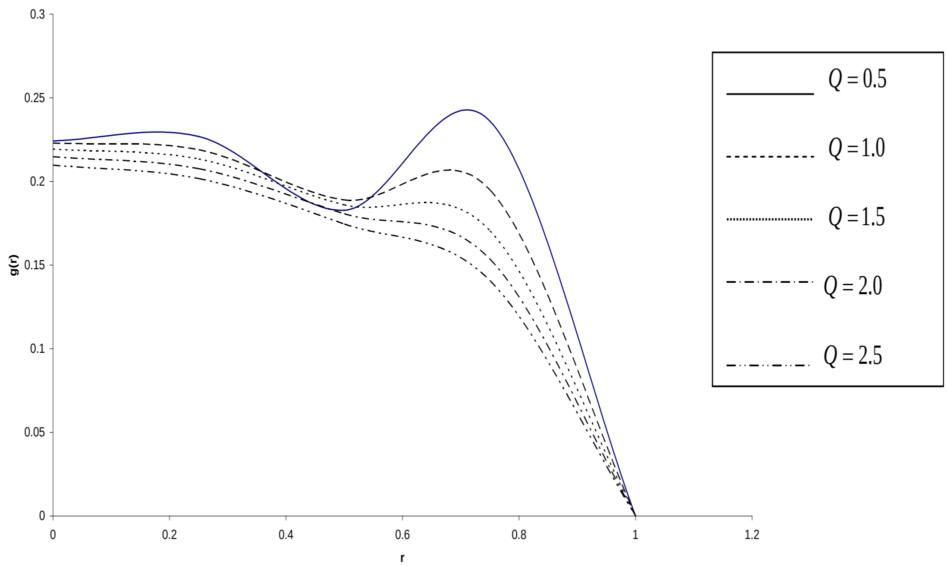

Figure 5: Velocity profile of fluid for different values of Q,
when $\lambda = 1,\ \omega = 0.1,\ R_e = 100,\ R = 1,\ H_a = 5,$ and $R_p = 1$

4 Conclusion

The flow of dusty viscoelastic fluid in a circular channel under the influence of an applied uniform magnetic field has been studied. The effect of magnetic field and viscoelastic parameter on both the fluid and particles has been investigated. It is interest to find that increase in the value of magnetic field and viscoelastic parameter reduce the horizontal velocity of the fluid and particles and thereby reducing the boundary layer thickness and hence induces an increase in the absolute value of the velocity gradient at the surface.

This shows that the thickness of the boundary layer is much larger for higher values of viscoelastic parameter than that of magnetic field parameter. This is because the rate of transport is considerably reduced with increase in magnetic field parameter which shows clearly that the transverse magnetic field opposes the transport phenomena.

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