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Simple Approximations for the Distribution of the Range of a Brownian Motion

Erhard Reschenhofer[1]

Abstract

This paper proposes two simple approximations of the asymptotic density function of the range of a standard Brownian motion. The approximations are obtained from a flexible asymmetric density by imposing moment restrictions and by minimizing the Kullback–Leibler divergence, respectively.

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1 Introduction

The precision of estimators for the daily volatility of stock returns can be improved by using the intraday range rather than the difference between the opening and the closing price [4, 5]. The distribution of the intraday range is usually approximated by that of a Brownian motion. The asymptotic density function of the range

$$R = \max_{0 \le s \le 1}(W(s)) - \min_{0 \le s \le 1}(W(s))$$

of a standard Brownian motion W(s), $s \in [0,1]$, is given by

$$\delta(r) = 8\sum_{k=1}^{\infty}(-1)^{k-1}k^2\phi(kr)$$

[2], where ϕ denotes the density function of the standard normal distribution. In this form, the asymptotic density is difficult to handle. It is not even obvious that it is positive [2, p.

[1]Department of Statistics and Operations Research, University of Vienna.

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430]. Moreover, its numerical instability for small values of r is also a serious issue. Thus, only simple measures such as $E(R^2)$ are used in quantitative finance. To enable more complex applications, the next section proposes two approximations of $\delta(r)$ which are both easier to handle and numerically stable. Section 3 concludes.

2 Approximations

The density δ is approximated by densities of the form

$$h(x) = I_{(-\infty,\mu]}(x)\underbrace{\frac{\lambda p}{\Gamma(\frac{1}{p})\sigma}e^{-(\frac{\mu-x}{\sigma})^p}}_{h_1(x)} + I_{(\mu,\infty)}(x)\underbrace{\frac{(1-\lambda)q}{\Gamma(\frac{1}{q})\tau}e^{-(\frac{x-\mu}{\tau})^q}}_{h_2(x)} ,$$

where $\mu \approx 1.34577$ is the mode of δ and $p,q,\sigma,\tau>0$, $0<\lambda<1$ are unknown parameters. Using for $m>0$

$$\int_0^\infty x^m e^{-x^p} dx = \frac{\Gamma(\frac{m+1}{p})}{p}$$

[3, p. 337], we obtain

$$E(X) = \frac{\lambda p}{\Gamma(\frac{1}{p})\sigma}\int_{-\infty}^{\mu} x e^{-(\frac{\mu-x}{\sigma})^p} dx + \frac{(1-\lambda)q}{\Gamma(\frac{1}{q})\tau}\int_{\mu}^{\infty} x e^{-(\frac{x-\mu}{\tau})^q} dx$$

$$= \frac{\lambda p}{\Gamma(\frac{1}{p})\sigma}\int_{\infty}^{0}(\mu-\sigma z)\, e^{-z^p}(-\sigma)dz + \frac{(1-\lambda)q}{\Gamma(\frac{1}{q})\tau}\int_0^\infty(\mu+\tau z)\, e^{-z^q}\tau dz$$

$$= \mu - \lambda\frac{\sigma\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})} + (1-\lambda)\frac{\tau\Gamma(\frac{2}{q})}{\Gamma(\frac{1}{q})} ,$$

$$E|X-\mu|^m = \frac{\lambda p}{\Gamma(\frac{1}{p})\sigma}\int_{-\infty}^{\mu}(\mu-x)^m e^{-(\frac{\mu-x}{\sigma})^p} dx + \frac{(1-\lambda)q}{\Gamma(\frac{1}{q})\tau}\int_{\mu}^{\infty}(x-\mu)^m e^{-(\frac{x-\mu}{\tau})^q} dx$$

$$= \frac{\lambda p}{\Gamma(\frac{1}{p})\sigma}\int_{\infty}^{0}(z\sigma)^m e^{-z^p}(-\sigma)dz + \frac{(1-\lambda)q}{\Gamma(\frac{1}{q})\tau}\int_0^\infty(z\tau)^m e^{-z^q}\tau dz$$

$$= \frac{\lambda\Gamma(\frac{m+1}{p})\sigma^m}{\Gamma(\frac{1}{p})} + \frac{(1-\lambda)\Gamma(\frac{m+1}{q})\tau^m}{\Gamma(\frac{1}{q})} ,$$

and

$$Var(X) = E(X-\mu)^2 - (E(X)-\mu)^2$$

$$= \frac{\lambda\Gamma(\frac{3}{p})\sigma^2}{\Gamma(\frac{1}{p})} + \frac{(1-\lambda)\Gamma(\frac{3}{q})\tau^2}{\Gamma(\frac{1}{q})} - \left(-\lambda \frac{\sigma\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})} + (1-\lambda) \frac{\tau\Gamma(\frac{2}{q})}{\Gamma(\frac{1}{q})} \right)^2 .$$

The continuity restriction

$$h_1(\mu) = \frac{\lambda p}{\Gamma(\frac{1}{p})\sigma} = \frac{(1-\lambda)q}{\Gamma(\frac{1}{q})\tau} = h_2(\mu)$$

implies that

$$\tau = \frac{(1-\lambda)q\Gamma(\frac{1}{p})}{\lambda p\Gamma(\frac{1}{q})}\,\sigma$$

and the unbiasedness restriction

$$E(X) = \mu - \lambda \frac{\sigma\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})} + (1-\lambda) \frac{\tau\Gamma(\frac{2}{q})}{\Gamma(\frac{1}{q})} = 2\sqrt{\tfrac{2}{\pi}} = E(R)$$

that

$$\tau = \left(2\sqrt{\tfrac{2}{\pi}} - \mu + \lambda \frac{\sigma\Gamma(\frac{2}{p})}{\Gamma(\frac{1}{p})} \right) \frac{\Gamma(\frac{1}{q})}{(1-\lambda)\Gamma(\frac{2}{q})} .$$

Thus,

$$\sigma = \frac{\Gamma(\frac{1}{q})\,(2\sqrt{\frac{2}{\pi}} - \mu)}{(1-\lambda)\Gamma(\frac{2}{q})} \left(\frac{(1-\lambda)q\Gamma(\frac{1}{p})}{\lambda p\Gamma(\frac{1}{q})} - \frac{\lambda\Gamma(\frac{1}{q})\Gamma(\frac{2}{p})}{(1-\lambda)\Gamma(\frac{1}{p})\Gamma(\frac{2}{q})} \right)^{-1}$$

$$= \frac{\lambda p\Gamma(\frac{1}{p})\Gamma^2(\frac{1}{q})\,(2\sqrt{\frac{2}{\pi}} - \mu)}{(1-\lambda)^2 q\Gamma^2(\frac{1}{p})\Gamma(\frac{2}{q}) - \lambda^2 p\Gamma^2(\frac{1}{q})\Gamma(\frac{2}{p})} .$$

Noting that

$$E(R) = 2\sqrt{\tfrac{2}{\pi}}\,, \quad E(R^2) = 4\log(2)\,, \quad E(R^3) = \tfrac{2}{3}\sqrt{2\pi^3}\,, \quad E(R^4) = 9\zeta(3)$$

[5], where

$$\zeta(3) = \sum_{j=1}^{\infty} j^{-3} \approx 1.2020569$$

(for tables of values of the Riemann zeta function ζ see [1]), and imposing in addition the higher order restrictions

$$E(X-\mu)^2 = \frac{\lambda\Gamma(\frac{3}{p})\sigma^2}{\Gamma(\frac{1}{p})} + \frac{(1-\lambda)\Gamma(\frac{3}{q})\tau^2}{\Gamma(\frac{1}{q})} = 4\log(2) - 4\mu\sqrt{\tfrac{2}{\pi}} + \mu^2 = E(R-\mu)^2 ,$$

$$E(X^3) = \frac{\lambda p}{\Gamma(\frac{1}{p})}\int_0^{\infty}(\mu - \sigma z)^3\, e^{-z^p}\, dz + \frac{(1-\lambda)q}{\Gamma(\frac{1}{q})}\int_0^{\infty}(\mu + \tau z)^3\, e^{-z^q}\, dz$$

$$= \mu^3 + \frac{\lambda}{\Gamma(\frac{1}{p})}\left(-3\mu^2\sigma\Gamma(\tfrac{2}{p}) + 3\mu\sigma^2\Gamma(\tfrac{3}{p}) - \sigma^3\Gamma(\tfrac{4}{p})\right)$$

$$+ \frac{(1-\lambda)}{\Gamma(\frac{1}{q})}\left(3\mu^2\tau\Gamma(\tfrac{2}{q}) + 3\mu\tau^2\Gamma(\tfrac{3}{q}) + \tau^3\Gamma(\tfrac{4}{q})\right)$$

$$= \tfrac{2}{3}\sqrt{2\pi^3} = E(R^3),$$

and

$$E(X-\mu)^4 = \frac{\lambda\Gamma(\frac{5}{p})\sigma^4}{\Gamma(\frac{1}{p})} + \frac{(1-\lambda)\Gamma(\frac{5}{q})\tau^4}{\Gamma(\frac{1}{q})}$$

$$= 9\zeta(3) - \tfrac{8}{3}\sqrt{2\pi^3}\,\mu + 24\log(2)\,\mu^2 - 8\sqrt{\tfrac{2}{\pi}}\,\mu^3 + \mu^4$$

$$= E(R-\mu)^4$$

yields:

σ=0.4045, τ=0.7544, λ=0.3466, p=2.879, q=1.525. (1)

Alternatively, all unknown parameters (except τ which is determined by the continuity restriction) can be selected by minimizing the Kullback-Leibler divergence

$$D_{KL}(\delta \| h) = \int \delta(x)\frac{\delta(x)}{h(x)}dx$$

of h from δ. Using [0.385,7] as the interval of integration, we obtain

σ=0.4090, τ=0.7600, λ=0.3478, p=2.920, q=1.539. (2)

Figure 1 shows that both (1) and (2) approximate δ very well. Minor discrepancies occur only when δ is practically zero.

3 Conclusion

The approximations of the asymptotic distribution of the range of a standard Brownian motion $W(s)$, $s\in[0,1]$, which have been obtained in the previous section, are probably good enough for all practical purposes. However, whether they are also simple enough must be determined in each individual case. A major challenge for future research is to find an application where only the approximations allow the derivation of an analytic, closed-form solution of a relevant problem.

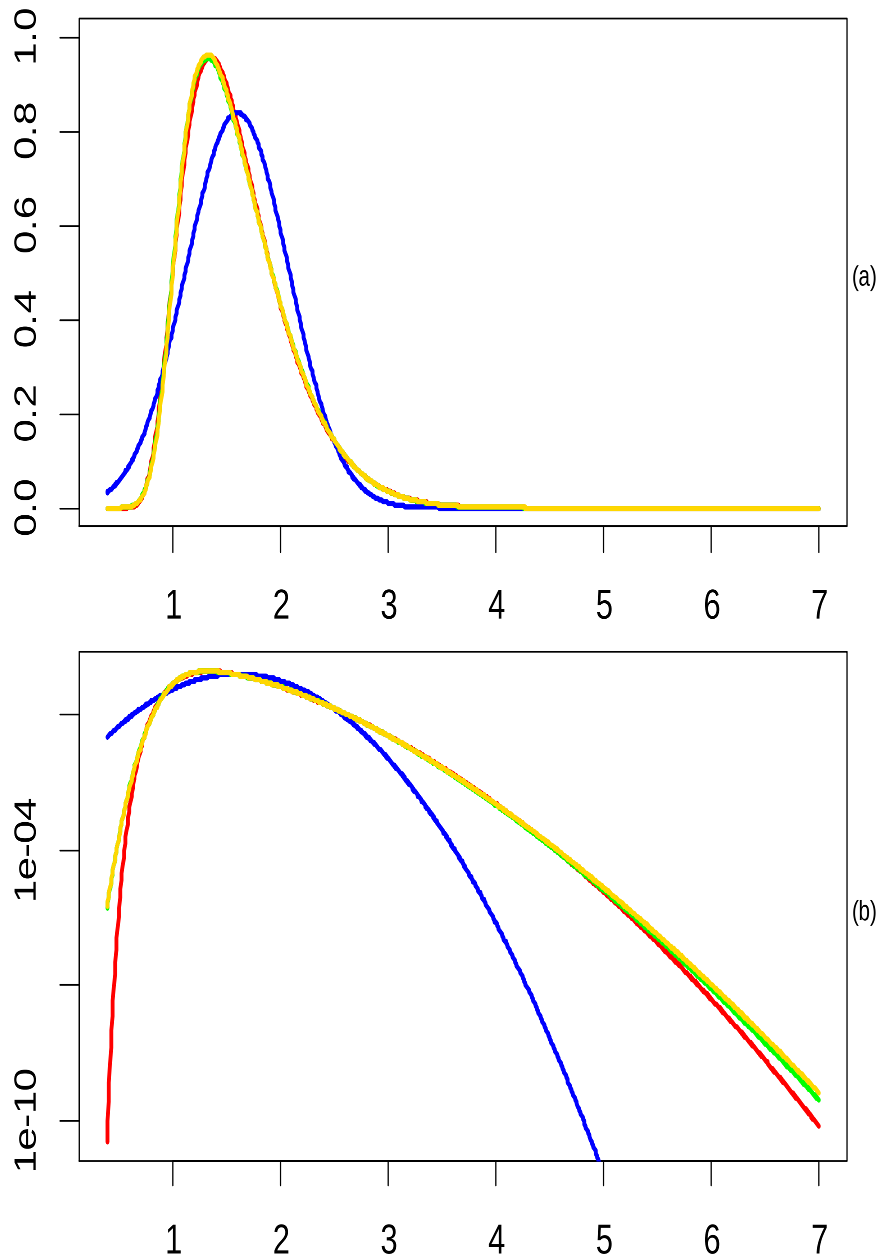

Figure 1: Normal approximation (blue) and new approximations (1: yellow, 2: green) of the asymptotic density function (red) of the range of a standard Brownian motion on [0,1]. (a): Linear scale, (b): Log scale

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