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Inference in the Log-Linear Birnbaum-Saunders Model under type I Censoring

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Abstract

In this paper inference for a log-linear Birnbaum-Saunders model under Type I censoring is presented. Methods of inference based on maximum likelihood, including normal approximation, profile likelihood, signed deviance statistics, as well as parametric bootstrap are presented. Inference for both shape and regression parameters are studied, as well as quantiles and survival probabilities. Results of a simulation study to compare small sample accuracy of the various approaches are discussed and two examples with real data are shown.

Mathematics Subject Classification: 62H10
Keywords: Likelihood; fatigue life distribution; censored data; bootstrap; asymptotic inference; Birnbaum-Saunders distribution; log-linear model

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1 Introduction

The Birnbaum-Saunders density is an attractive probability model derived by [6] for modeling the number of cycles necessary to force a fatigue crack to grow to a critical size that would produce fracture. An advantage of this model, compared to alternative ones such as the log-normal or Weibull models, is that it can be derived from basic characteristics of the fatigue process instead of just being used on ad-hoc basis.

The same model has been studied by [13], who derived the density from assumptions related to a different model and has been studied, among other authors, by [21], [1], [15], [39], [29]. Recent work, which relates the Bisa distribution to the physical crack propagation model, includes [41] and [2], the latter including a new parameterization with direct physical interpretations. [3] also apply a truncated version of the Bisa model to a problem in financial risk modelling.

In this paper we consider a Birnbaum-Saunders log-linear regression model under Type I censoring, and study the coverage of different confidence intervals for combinations of sample size, percent of censoring and values of a shape parameter.

The organization of the paper is as follows. In section 2 a summary of related work is presented; then in section 3 the sinh normal distribution and the log-linear model, under Type I censoring with the presence of covariates, are introduced as well as some details regarding likelihood based inference. Details about methods of constructing confidence intervals are given in sections 5 and 6 and results of a simulation study to investigate the coverage of the different confidence intervals are reported in section 7. In section 8 two real life data sets are analyzed using the methods discussed.

2 Related Work

[18] consider the problem of prediction under the $BS(\alpha, \beta)$ distribution, as do [15] from both frequentist and Bayesian perspectives. [1] discusses Bayesian inference under the model $BISA(\alpha, \gamma)$ and, by assuming complete samples and non-informative priors, he applies the Laplace approximation to the integrals

appearing in the Fisher Information Matrix and obtains analytical expressions of the approximations to the entries of such a matrix. A repeated application of the Laplace Approximation enables the same author to obtain analytical approximations to the marginal posterior of each component in the joint posterior, as well as of the predictive distribution of a future observation. [7] discuss confidence interval estimation of quantiles of the Birnbaum Saunders distribution, as well as the construction of tolerance limits. [39] presented a Bayesian analysis of model (1) using a non-informative prior for α and γ, which can be derived from the expected Fisher Information matrix, and analyze the marginal posteriors of some functions of interest, assuming an informative prior. To simulate from a conditional density, [39] implements Monte Carlo Gibbs sampling and, to simulate from the conditional involved, the Metropolis algorithm is applied to one component; the author addresses only the case of complete observations. More recently, [27] proposed reduced bias modified moment estimators of the two parameters of the Birnbaum-Saunders distribution and reported results from a Monte Carlo simulation study, comparing the probability coverage of the discussed methods.

Much of the earlier work on inference for Bisa distributed data dealt with complete samples. In relatively early work, [12] developed maximum likelihood equations for censored BISA data for both Type 1 and 2 censored samples, and pointed out the intimate connection between the Bisa distribution and the inverse Gaussian. More recently, [8] has studied inferential techniques for censored Bisa data via both frequentist and Bayesian approaches; he also outlined residual quantities for graphical assessment of fit of Bisa regression model and studied for the first time a mixed effects log-linear Bisa regression model allowing for censored data; see, e.g., [17] which is based on [8]. There has been much related work on censored Bisa data recently. Representative papers include [34], [42], [28], [24], [16] and [31].

3 The Log-linear Model

Let $T > 0$ be an absolutely continuous random variable representing life time. If $\alpha > 0$ and $\beta > 0$ are two constants, then the random variable T follows the Birnbaum-Saunders density with parameters α and β, written

$T \sim BISA(\alpha, \beta)$, if $Z = \alpha^{-1}(\sqrt{T/\beta} - \sqrt{\beta/T}) \sim N(0,1)$, where $N(0,1)$ stands for the standard normal density.

If $T \sim BISA(\alpha, \beta)$ then the density of T is given by

$$f(t) = (2\alpha t)^{-1}\big(\sqrt{a} + \sqrt{b}\big)\frac{1}{\sqrt{2\pi}} \exp\left(\frac{1}{\alpha}\Big(\sqrt{a} - \sqrt{b}\Big)\right)$$

where $a = t/\beta$ and $b = \beta/t$, and it can be seen that α and β are shape and scale parameters respectively.

Let $T \sim BISA(\alpha, \beta)$ and consider the transformation $Y = \log T$, then Y follows the density

$$f(y) = \frac{1}{\sqrt{2\pi}\alpha} \cosh\left(\frac{y-\gamma}{2}\right) \exp\left(-\frac{2}{\alpha^2}\sinh^2\left(\frac{y-\gamma}{2}\right)\right), \quad \alpha > 0,\ -\infty < \gamma < \infty, \tag{1}$$

which is known as the Sinh Normal distribution with parameters α and γ, and we write $Y \sim SN(\alpha, \gamma)$.

The density in (1) has been studied by [33], who derived some properties of it. In particular, if $Y \sim SN(\alpha, \gamma)$ then $\mathbb{E}(Y) = \gamma$, $\mathbb{V}ar(Y)$ can not be obtained as a closed form, α is a shape parameter, γ is a location parameter and, for a being a known constant, $Z = a + Y \sim SN(\alpha, a + \gamma)$. If γ is known, then the density belongs to the exponential family but if both parameters are unknown this property does not hold. The transformation $Y = \log(T)$ is useful because γ is a location parameter and has practical interpretation in applications.

If one wishes to study the effect of covariates, say $X_1, \cdots, X_p$, on the response Y, where Y is assumed to be such that $Y \sim SN(\alpha, \gamma)$, then the ith observation Y_i can be modeled as

$$Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_p X_{ip} + \varepsilon_i, \quad i = 1, \cdots, n \tag{2}$$

where the disturbances ε_i are such that $\varepsilon_i \sim SN(\alpha, 0)$, $i = 1, \cdots, n$. The model in (2) is referred to as the log-linear Birnbaum-Saunders regression model and was discussed by [33]. This model is particulary useful since it enables the applied researcher to relate the stress level to the the observed fatigue and, by using standard asymptotic likelihood results, approximate tests and confidence intervals on parameters of it can readily be obtained.

4 Maximum Likelihood Estimation

Assuming Type 1 censoring, denote by y_i the log observed failure times and L_j the log observed censoring times. Then, assuming independent censoring, the likelihood function corresponding to (1) can be written as

$$\begin{aligned}\mathcal{L}(\theta) = \prod_{i \in O} \frac{1}{\alpha\sqrt{2\pi}} \cosh\left(\frac{y_i - \gamma}{2}\right) \exp\left[-\frac{2}{\alpha^2}\sinh^2\left(\frac{y_i-\gamma}{2}\right)\right] \\ \times \prod_{j \in C}\left[1 - \Phi\left(\frac{2}{\alpha}\sinh\left(\frac{L_j - \gamma}{2}\right)\right)\right],\end{aligned} \tag{3}$$

where O and C denote the sets of observed and censored observations respectively, L_j represents the logarithm of the jth censoring time, Φ represents the cumulative distribution function of the standard normal density and $\theta = (\alpha\ \gamma)'$ is a vector of parameters.

If one writes $O_i = (Y_i - \gamma)/2$, $C_j = (L_j - \gamma)/2$ and $D_j = \sqrt{2}\sinh(C_j)/\alpha$, the log-likelihood corresponding to (3) is

$$\begin{aligned}l(\theta) = \sum_{i \in O}\left\{\log\left[\frac{1}{\alpha\sqrt{2\pi}}\cosh\left(\frac{y_i-\gamma}{2}\right)\right] - \frac{2}{\alpha^2}\sinh^2\left(\frac{y_i-\gamma}{2}\right)\right\} + \\ \sum_{j \in C}\log\left[1 - \Phi\left(\frac{2}{\alpha}\sinh\left(\frac{L_j-\gamma}{2}\right)\right)\right],\end{aligned} \tag{4}$$

and instead of solving the score equations

$$\begin{aligned}S_\alpha(\theta) = \sum_{i\in O}\left[\frac{4(2+\sinh(O_i))}{\alpha^3} - \frac{1}{\alpha}\right] + \sum_{j \in C}\left[\frac{\exp\big(-(2/\alpha^2)\sinh^2(C_j)\big)\sqrt{2/\pi}\,\sinh(C_j)}{\alpha^2\big[1 + (1/2)\,(\mathrm{erf}(D_j) - 1)\big]}\right] \\ = 0\end{aligned}$$

and

$$\begin{aligned}S_\gamma(\theta) = \sum_{j\in C}\left[\frac{\exp\big(-(2/\alpha^2)\ \sinh^2(C_j)\big)\cosh(C_j)}{\alpha\sqrt{2\,\pi}\ \big[1 + (1/2)\big(\mathrm{erf}(D_j) - 1\big)\big]}\right] - \sum_{i \in O}\left[\frac{\cosh(O_i)}{\alpha^2} - \frac{1}{2}\tanh(O_i)\right] \\ = 0,\end{aligned}$$

which are clearly non-linear in θ, the maximum likelihood (ML) estimates of θ can be obtained by optimizing $l(\theta)$ using a Newton or Quasi-Newton method.

On defining $x_i = (1\ x_{i1} \cdots x_{ip})'$, the likelihood and log-likelihood corresponding to the log-linear model given in (2), are

$$\begin{aligned}\mathcal{L}(\theta) = \prod_{i\in O} \frac{1}{\alpha\sqrt{2\pi}} \cosh\left(\frac{y_i-\gamma_i}{2}\right) \exp\left[-\frac{2}{\alpha^2}\sinh^2\left(\frac{y_i-\gamma_i}{2}\right)\right] \\ \times \prod_{j\in C}\left[1-\Phi\left(\frac{2}{\alpha}\sinh\left(\frac{L_j-\gamma_j}{2}\right)\right)\right],\end{aligned} \tag{5}$$

and

$$\begin{aligned} l(\theta) = \sum_{i\in O}\left\{\log\left[\frac{1}{\alpha\sqrt{2\pi}}\cosh\left(\frac{y_i-\gamma_i}{2}\right)\right] - \frac{2}{\alpha^2}\sinh^2\left(\frac{y_i-\gamma_i}{2}\right)\right\} + \\ \sum_{j\in C}\log\left[1-\Phi\left(\frac{2}{\alpha}\sinh\left(\frac{L_j-\gamma_j}{2}\right)\right)\right],\end{aligned} \tag{6}$$

respectively, where O, C, Y_i and L_j are defined as before, $\beta = (\beta_0\ \ \beta_1\ \ \cdots\ \beta_p)'$ and $\gamma_i = x_i'\beta$.

Again, the ML estimates must be obtained by using numerical methods, but since (2) is a linear model in the β_j, least squares point estimates of β_j could be easily obtained and used as starting values in the iterative procedure.

Since the ML estimators of α and γ can not be obtained in a closed form and their sampling distributions are impossible to obtain, approximate methods of inference must be considered, see [9], [37] or [30] for example.

The Fisher information matrix, required to perform likelihood based approximate inference, can be estimated by the expected, $I(\theta)$, or the observed, $J(\theta)$, information matrices. The observed and expected Fisher information matrices are

$$\begin{aligned} J(\theta) &= \left[-\frac{\partial^2 l(\theta)}{\partial\theta_k\,\partial\theta_r}\right] \\ &= \left[j_{kr}\right], \quad k, r = 1, \cdots, p \end{aligned}$$

and

$$\begin{aligned} I(\theta) &= -\mathbb{E}\left[\frac{\partial^2 l(\theta)}{\partial\theta_k\,\partial\theta_r}\right] \\ &= [\mathbb{E}(j_{kr})], \quad k, r = 1, \cdots, p \end{aligned}$$

where p is the dimension of the vector θ.

It can be seen that, under the model in (2),the expectation yielding the entry (k,r) of $I(\theta)$ can not be obtained as a closed form. For example, given a single observation, no covariates and setting $\gamma = \beta_0$, the first entry of $J(\theta)$ is

$$j_{11} = \alpha^{-2} - \frac{2\sqrt{\dfrac{2}{\pi}}\,\sinh(C_j)}{\alpha^3\,\exp\left[\dfrac{2\sinh^2(C_j)}{\alpha^2}\right]} + \frac{4\sqrt{\dfrac{2}{\pi}}\,\sinh^3(C_j)}{\alpha^5\,\exp\left[\dfrac{2\sinh^2(C_j)}{\alpha^2}\right]} - \frac{12\ (2+\sinh(O_i))}{\alpha^4},$$

where O_i, C_j and D_j are defined as above; clearly, the expected value $\mathbb{E}(j_{kr})$ can only be estimated numerically using, for example, a quadrature rule, Monte Carlo simulation or the Laplace approximation. On the contrary, $J(\theta)$ can be readily obtained and it is usually favored as compared to $I(\theta)$, see [19]. In this paper all the inference based on asymptotic likelihood results is performed via the observed information.

5 Confidence Intervals for Parameters

In this paper we consider approximate confidence intervals for α and β based on the normal approximation, the profile likelihood, the parametric bootstrap and the signed profiled likelihood.

Confidence intervals based on the normal approximation rely on the asymptotic normality of the ML estimates $\widehat{\theta}_j$ and are defined as

$$CI(\theta_j) = \widehat{\theta}_j \pm z_{1-(\delta/2)}\widehat{\sigma}_j, \tag{7}$$

where $\widehat{\sigma}_j^2$ represents the asymptotic estimated variance of $\widehat{\theta}_j$ and it can be obtained from the observed or expected Fisher's information matrix, $J(\theta)$ and $I(\theta)$.

Let $\theta = (\theta_1, \theta_2')'$, where θ_1 is univariate, and define the scaled profile likelihood function

$$\mathcal{LR}_p(\theta_1) = \underset{\theta_2}{\text{arg-max}}\ \frac{\mathcal{L}(\theta_1, \theta_2(\theta_1))}{\mathcal{L}(\widehat{\theta})},$$

and

$$G(\theta_1) = -2\log\mathcal{LR}_p(\theta_1), \tag{8}$$

where $\widehat{\theta}$ is the unrestricted ML estimate of θ. Since the limiting distribution of $G(\theta_1)$ is χ_1^2, an approximate confidence region for θ_1 can be obtained by solving the nonlinear equation

$$G(\theta_1) - \chi_{1,1-a}^2 = 0, \tag{9}$$

where $\chi_{1,1-a}^2$ is the $(1-a)$ quantile of the chi square distribution with one degree of freedom.

Generally, the left hand side of (9) is concave and so it has two different roots bracketing $\widehat{\theta}$. Let θ_1^L and θ_1^U be two such roots, where $\theta_1^L < \theta_1^U$, then an approximate confidence interval for θ_1, with confidence coefficient $100(1-\alpha)\%$, is

$$(\theta_1^L, \theta_1^U).$$

Again, since the equation in (8) is nonlinear in θ_1, a numerical method must be used to solve (9). Facilities to implement this task are available in popular software such as R [32], SAS' IML [35], and MATHEMATICA [43], among others.

Under the parametric bootstrap approach, see [20] for example, given the ML estimates $\widehat{\alpha}$ and $\widehat{\gamma}$ obtained from a random sample of size n, B bootstrap random samples of size n are simulated from the density $SN(\widehat{\alpha}, \widehat{\gamma})$ and from each the corresponding bootstrap estimates $\widehat{\alpha}_k^*$ and $\widehat{\gamma}_k^*$ are obtained, $k = 1, \cdots, B$. By using the empirical distribution of $\widehat{\alpha}_k^*$ and $\widehat{\gamma}_k^*$, approximate confidence intervals for α and γ, or functions of them, can be constructed. Specifically, if an approximate $100(1-\delta)\%$ confidence interval for θ_j is desired, $0 < \delta < 1$, such an interval is formed by obtaining the $\delta/2$ and $1-(\delta/2)$ sample quantiles corresponding to the bootstrap estimates $\widehat{\theta}_{jk}^*$, $k = 1, \cdots, B$. For large B, the empirical distribution should resemble the theoretical one.

As suggested by [4], if a more accurate inference is desired, the signed log likelihood ratio statistic can be used. Let the be vector of parameters θ, of size d, be partitioned as $\theta = (\theta_1\ \theta_2')'$, where θ_1 is scalar and θ_2 of size $d-1$. Then the signed log-likelihood ratio statistic, [4], $r(\theta_1)$ say, is defined as

$$r(\theta_1) = \mathrm{sgn}(\widehat{\theta}_1 - \theta_1)\{2[l(\widehat{\theta}_1, \widehat{\theta}_2) - l(\theta_1, \widehat{\theta}_2(\theta_1))]\}^{1/2}$$

where $\widehat{\theta}_1$ and $\widehat{\theta}_2$ are the unrestricted ML estimates of θ_1 and θ_2 respectively, l stands for the log-likelihood and $\widehat{\theta}_2(\theta_1)$ is the ML estimate of θ_2 conditional on θ_1. The statistic $r(\theta_1)$ is such that

$$r(\theta_1) \xrightarrow{\text{D}} N(0,1). \tag{10}$$

From (10), approximate confidence intervals and tests on components of θ are easily obtained.

6 Confidence Intervals for Quantiles

It is frequently of interest to study quantiles of the distribution and several approaches are available to do inference about these quantities. As to the density (1), the p-quantile, $0 < p < 1$, is given by

$$Y_p = 2\sinh^{-1}\left(\frac{\alpha}{2}\Phi^{-1}(p)\right) + \gamma. \tag{11}$$

Let $\widehat{\alpha}$ and $\widehat{\gamma}$ be the ML estimators of α and γ, then, by the invariance property of the ML estimators, the ML estimate of Y_p is simply

$$\widehat{Y}_p = 2\sinh^{-1}\left(\frac{\widehat{\alpha}}{2}\Phi^{-1}(p)\right) + \widehat{\gamma}, \tag{12}$$

and by the delta method, an estimate of the approximate variance of $\widehat{Y}_p$ is

$$\widehat{\sigma}^2_{y_p} = g'(\widehat{\theta})J^{-1}(\widehat{\theta})g(\widehat{\theta}), \tag{13}$$

where $g(\theta) = 2\sinh^{-1}\left((\alpha/2)\,\Phi^{-1}(p)\right) + \gamma$ and $g'(\theta) = \partial g(\theta)/\partial\theta$ is the gradient vector of g.

Therefore an approximate $100(1-\delta)\%$ confidence interval for Y_p is

$$CI(Y_p) = \widehat{Y}_p \pm z_{1-(\delta/2)}\widehat{\sigma}_{y_p}. \tag{14}$$

An approximate confidence interval for Y_p can also be obtained by using the profile likelihood of Y_p. Under this approach, the scaled profile likelihood of the p-quantile, Y_p, is

$$\mathcal{PL}(Y_p) = \sup_{\alpha} \frac{\mathcal{L}(\alpha, c_p)}{\mathcal{L}(\widehat{\theta})}, \tag{15}$$

where $c_p = Y_p - 2\,\sinh^{-1}\left((\alpha/2)\Phi^{-1}(p)\right)$. By the same arguments discussed above, an approximate $100(1-\delta)\%$ confidence interval for Y_p, based on the likelihood ratio, is obtained by solving the nonlinear equation

$$h(Y_p) - \chi^2_{1,1-\delta} = 0 \tag{16}$$

where $h(Y_p) = -2\log\mathcal{PL}(Y_p)$ and $\chi^2_{1,1-\delta}$ is defined as above.

If the parametric bootstrap is to be used, then the approximate $100(1-\delta)\%$ confidence interval for Y_p is constructed by obtaining the $\delta/2$ and $1-(\delta/2)$ sample quantiles of the empirical distribution of $Y^*_{p_k}$, $k = 1, \cdots, B$, where $Y^*_{p_k}$ is defined as

$$Y^*_{p_k} = 2\sinh^{-1}\left(\frac{\widehat{\alpha}^*_k}{2}\Phi^{-1}(p)\right) + \gamma^*_k, \tag{17}$$

$\widehat{\alpha}^*_k$ and $\widehat{\gamma}^*_k$ are the bootstrap ML estimates based on the k-th bootstrap replicate drawn from $SN(\widehat{\alpha}, \widehat{\gamma})$.

7 Simulation Results

To study the coverage of the confidence intervals, for α and γ, based on the normal approximation (NA), the parametric bootstrap (PB), the signed root of the log-likelihood ratio statistic (SS) and the profile likelihood ratio (PLR), a small simulation experiment was implemented.

The factors included in the experiment were: Method of construction, value of the parameter α, sample size n, and percent of censoring P. The factors and levels are shown in Table 1. The above values of α cover a range of interest in practical applications and, since it is a location parameter, without loss of generality γ was set at 0. Since γ is of most interest, and usually α can be considered as a nuisance parameter, the SS and PLR methods were implemented only for γ.

The number of simulations was 10,000 and to implement the parametric bootstrap, 2000 samples were obtained for each estimate $(\widehat{\alpha}, \widehat{\gamma})$. The code for the simulations was written in FORTRAN 90 and the IMSL libraries, [40],

Table 1: Factors and levels in the Monte Carlo simulation study.

Factor	Levels				
α	1.0	0.5	0.25		
P	20	40	60		
n	20	30	50	70	100
Method	NA	PB	SS	PLR	

were used. The results from the simulation experiment are shown in Tables 2, 3 and 4.

The results showed that, for all the methods, the greater the percent of censored data the greater the departures of the observed coverages from the nominal value. Overall, the coverage of the profile likelihood intervals is close to the nominal value, as long as the percent of censoring did not exceed 60. As to the confidence intervals for α, based on the normal approximation, they performed satisfactorily for sample sizes no less than 100 and percent of censoring of 40 or less; this kind of interval seems to perform better when the parameter is γ and the percent of censoring is about 40 or less.

When the parametric bootstrap was used, overall, the observed coverages were closest to the nominal value if the percent of censoring was about 40 or less and the sample size was at least 50. As to the signed root of the log-likelihood ratio statistic, overall, its coverage is close to the nominal value only if the percent of censoring is about 40 or less; however, overall, its coverage was less than the observed for the remaining methods.

Table 2: Estimated coverage for α and γ for different methods.

			Method					
			NA		PB		PLR	SS
α	n	P	α	γ	α	γ	γ	γ
0.25	20	60	0.8846	0.9334	0.9168	0.9182	0.9194	0.9288
		40	0.8997	0.9570	0.9104	0.9384	0.9308	0.9312
		20	0.9018	0.9300	0.9048	0.9204	0.9376	0.9414
	30	60	0.8988	0.9438	0.9369	0.9328	0.9264	0.9128
		40	0.9166	0.9500	0.9108	0.9380	0.9357	0.9432
		20	0.9204	0.9352	0.9428	0.9240	0.9368	0.9412
	50	60	0.9212	0.9408	0.9388	0.9228	0.9046	0.9098
		40	0.9334	0.9512	0.9484	0.9527	0.9302	0.9410
		20	0.9414	0.9500	0.9501	0.9482	0.9376	0.9412
	70	60	0.9367	0.9484	0.9268	0.9262	0.8868	0.8910
		40	0.9372	0.9536	0.9468	0.9382	0.9421	0.9390
		20	0.9244	0.9484	0.9440	0.9500	0.9424	0.9526
	100	60	0.9251	0.9393	0.9268	0.9404	0.8588	0.8722
		40	0.9443	0.9258	0.9242	0.9424	0.9505	0.9418
		20	0.9660	0.9700	0.9503	0.9463	0.9426	0.9484

8 Examples of Application

8.1 Example 1

The data in Table 5 above, taken from [23], were discussed by [26] and show the number of thousand of miles at which different locomotive controls failed in a life test involving 96 items. The test was terminated at 135000 miles and there were 59 units not failing before this time, yielding Type I censored data. The observed data, t_i, are shown in Table 5 and the common censoring time was 135 thousand of miles.

[23] analyzed these data under a Log-Normal model, so a reasonable model

Table 3: Estimated coverage for α and γ for different methods.

			Method					
			NA		PB		PLR	SS
α	n	P	α	γ	α	γ	γ	γ
0.50	20	60	0.8895	0.9253	0.8968	0.9228	0.9246	0.9264
		40	0.8905	0.9523	0.9226	0.9288	0.9414	0.9348
		20	0.9225	0.9360	0.9188	0.9327	0.9402	0.9488
	30	60	0.9055	0.9335	0.9084	0.9486	0.9414	0.9227
		40	0.9125	0.9568	0.9288	0.9466	0.9462	0.9473
		20	0.9220	0.9515	0.9283	0.9166	0.9424	0.9462
	50	60	0.9195	0.9445	0.9304	0.9182	0.9424	0.9083
		40	0.9371	0.9625	0.9406	0.9457	0.9410	0.9426
		20	0.9490	0.9402	0.9468	0.9425	0.9424	0.9444
	70	60	0.9282	0.9484	0.9382	0.9308	0.9408	0.9012
		40	0.9180	0.9485	0.9407	0.9344	0.9488	0.9401
		20	0.9385	0.9455	0.9326	0.9448	0.9486	0.9505
	100	60	0.9445	0.9412	0.9425	0.9343	0.9490	0.8839
		40	0.9455	0.9415	0.9432	0.9642	0.9504	0.9433
		20	0.9437	0.9531	0.9548	0.9324	0.9498	0.9494

could be the $SN(\alpha, \gamma)$ density.

On defining $y_i = \log(t_i)$, the optimization of the log-likelihood (4) yields the ML estimates $\widehat{\alpha} = 0.771$ and $\widehat{\gamma} = 5.137$. The approximate estimated variances, based on the observed information matrix $J(\widehat{\theta})$, are $\widehat{\sigma}^2_{\widehat{\alpha}} = 0.012443$ and $\widehat{\sigma}^2_{\widehat{\gamma}} = 0.01390$ and the approximate 95% confidence interval for α is $(0.571\,, 1.015)$, whereas for γ such an interval is $(4.905\,, 5.368)$.

A plot of the normalized likelihood is shown in Figure 1 and it reveals that the normal approximation could not be useful, since the likelihood is skewed. A plot of the function $G(\gamma) - \chi^2_{1,0.95}$, where G is defined in (8), is shown in Figure 2 and from an inspection of it is clear that $l(\gamma, \alpha(\gamma)) = \log \mathcal{LR}_p(\gamma)$ is quite asymmetric. The approximate confidence interval for γ, based on the

Table 4: Estimated coverage for α and γ for different methods.

			Method					
			NA		PB		PLR	SS
α	n	P	α	γ	α	γ	γ	γ
1.00	20	60	0.8784	0.9222	0.9146	0.9288	0.9372	0.9260
		40	0.8997	0.9495	0.9235	0.9301	0.9448	0.9370
		20	0.9048	0.9384	0.9030	0.9340	0.9482	0.9386
	30	60	0.8878	0.9310	0.8912	0.9413	0.9455	0.9313
		40	0.9178	0.9568	0.9203	0.9398	0.9430	0.9400
		20	0.9226	0.9477	0.9227	0.9441	0.9474	0.9380
	50	60	0.9128	0.9398	0.9207	0.9373	0.9340	0.9324
		40	0.9203	0.9492	0.9412	0.9428	0.9434	0.9330
		20	0.9382	0.9449	0.9483	0.9418	0.9408	0.9460
	70	60	0.9344	0.9502	0.9302	0.9302	0.9288	0.9218
		40	0.9420	0.9554	0.9421	0.9364	0.9324	0.9438
		20	0.9214	0.9505	0.9402	0.9406	0.9434	0.9446
	100	60	0.9260	0.9476	0.9412	0.9426	0.9159	0.9164
		40	0.9354	0.9531	0.9467	0.9601	0.9414	0.9332
		20	0.9485	0.9578	0.9508	0.9412	0.9472	0.9460

likelihood ratio, is (4.940,5.427).

If the parametric bootstrap is used, the 95% approximate confidence intervals for α and γ, based on 5000 simulations from the $SN(\widehat{\alpha},\widehat{\gamma})$ density, are (0.573 , 1.036) and (4.936 , 5.415) respectively. Histograms of the empirical distributions of $\widehat{\alpha}$ and $\widehat{\gamma}$ are shown in Figures 3 and 4. Both figures confirm that the sample distributions of $\widehat{\alpha}$ and $\widehat{\gamma}$ are slightly skewed, despite the relatively large sample size, and suggest an alternative method to construct the confidence intervals.

As to the survivor function, Lawless (1982) constructs an approximate 95% likelihood ratio based confidence interval for the survivor function at $t = 80$, $S(\log(80))$, under the a log-normal model. The confidence intervals

Table 5: Number of thousand of miles at which different locomotive controls failed in a life test.

22.5	37.5	46.0	48.6	51.5	53.0	54.5	57.5	66.5	68.0
69.5	76.5	77.0	78.5	80.0	81.5	82.0	83.0	84.0	91.5
93.5	102.5	107.0	108.5	112.5	113.5	116.0	117.0	118.5	119.0
120.0	122.5	123.0	127.5	131.0	132.5	134.0			

for $S(\log(80))$, obtained via the normal approximation, profile likelihood ratio and parametric bootstrap are (0.758768, 0.925195), (0.7623,0.9192) and (0.780 , 0.901) respectively. The bootstrap distribution of the ML estimate of $S(\log(80))$ is shown in Figure 5.

As regards quantiles, the .1 quantile is often of interest in reliability problems, see [25]; approximate 95% confidence intervals for the 0.1 quantile, $Y_{0.1}$, obtained using the expression in (14) and the profile likelihood are (3.99,4.38) and (3.961,4.362) respectively.

8.2 Example 2

Here we consider the log-linear model including covariates and the motorette data found in [36] and analyzed by [38]. Ten motorettes were tested at each of four different temperatures and the time to failure was recorded.

[38] analyzed these data assuming a linear regression model and disturbances normally distributed. Here we re-analyze such data but under the log-linear Birnbaum-Saunders regression model

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i, \quad i = 1, \cdots, n, \tag{18}$$

where $Y_i = \log(T_i)$, $x_i = 1000/(273.2 + C^\circ)$, $\varepsilon_i \sim SN(\alpha, 0)$, and $i = 1, \cdots, 40$.

Optimizing the log-likelihood function in (6) which corresponds to model (18) gives the following ML estimates: $\widehat{\beta}_0 = -14.137$, $\widehat{\beta}_1 = 10.050$ and $\widehat{\alpha} =$

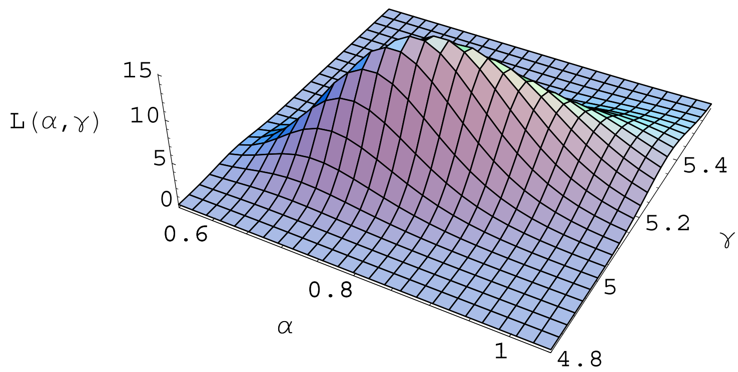

Figure 1: Normalized likelihood; data of example 1.

0.642. Since the inverse of the observed information matrix is

$$J^{-1}(\widehat{\theta}) = \begin{bmatrix} 0.01549701 & -0.07909928 & 0.0406174 \\ -0.07909928 & 5.80148913 & -2.6773590 \\ 0.04061740 & -2.67735904 & 1.2395089 \end{bmatrix}$$

then the asymptotic standard errors of the estimates are $\widehat{\sigma}_{\widehat{\alpha}} = 0.124487$, $\widehat{\sigma}_{\widehat{\beta}_0} = 2.408628$ and $\widehat{\sigma}_{\widehat{\beta}_1} = 1.113332$; hence, the approximate 95% confidence intervals for α, β_0 and β_1 are (0.398, 0.885), (-18.857, -9.416) and (7.867, 12.232) respectively. Using the ML point estimates and the observed information matrix, the normal approximation provides a 95% confidence interval for the expected value of Y given that $x = 130$, i.e. $\mathbb{E}(Y \mid X = 130) = \mu_{Y|x=130}$, given by (1008.61,1576.11).

As to β_1, a plot of $G(\beta_1) - \chi^2_{1,0.95}$ is shown in Figure 6. From this, it can be concluded that the form of the profile log likelihood of β_1, $\log \mathcal{LR}_p(\beta_1)$, is not markedly skewed. The confidence interval for β_1, obtained by solving the equation in (9), is (7.99,12.593). Plots of the profile log-likelihood and

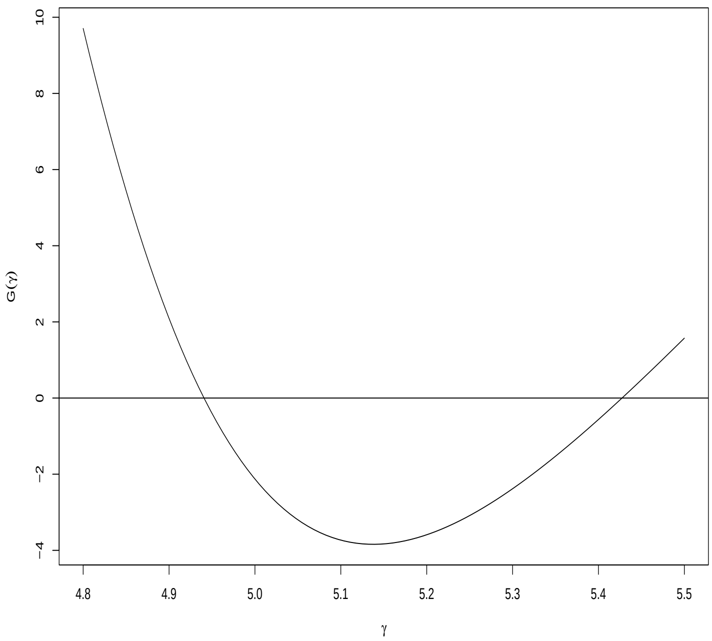

Figure 2: Plot of the function $G(\gamma) - \chi^2_{1,0.95}$, data of example 1.

bootstrap distributions of α and β_0 can also be readily obtained, but they are omitted because of space reasons.

9 Discussion and conclusion

The objective of this paper was to study the coverage of confidence intervals constructed using the normal approximation, parametric bootstrap, signed deviance statistic and profile likelihood methods for different combinations of sample size, percent of censoring and values of the shape parameter of the $SN(\alpha, \gamma)$ model. The underlying model considered was a log-linear Birnbaum-Saunders model, corresponding to the $SN(\alpha, \gamma)$ density, under Type I censoring, and the cases of a single location parameter and the location depending

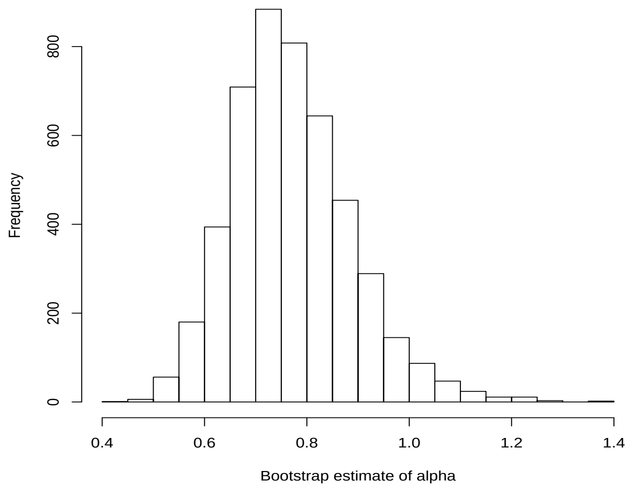

Figure 3: Bootstrap distribution of $\widehat{\alpha}$, data of example 1.

on covariates were addressed. Methods to construct approximate confidence intervals, both for parameters and quantiles, were presented.

Monte Carlo simulation results showed that, overall and in terms of coverage, the confidence intervals based on the profile likelihood function perform satisfactorily for most of the cases. The performance of the normal approximation, as dictated by theory, is affected by small sample sizes and, in general, a heavy percent of censored observations, 60 or more, decreased the observed coverages of the methods presented.

A simple application of the methods was shown on two real data sets.

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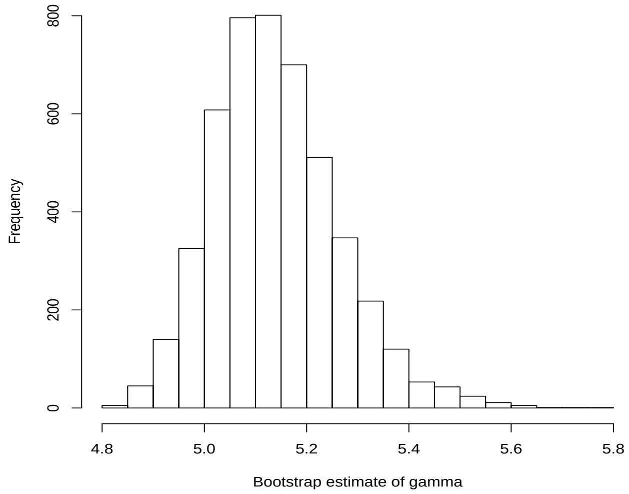

Figure 4: Bootstrap distribution of $\widehat{\gamma}$, data of example 1.

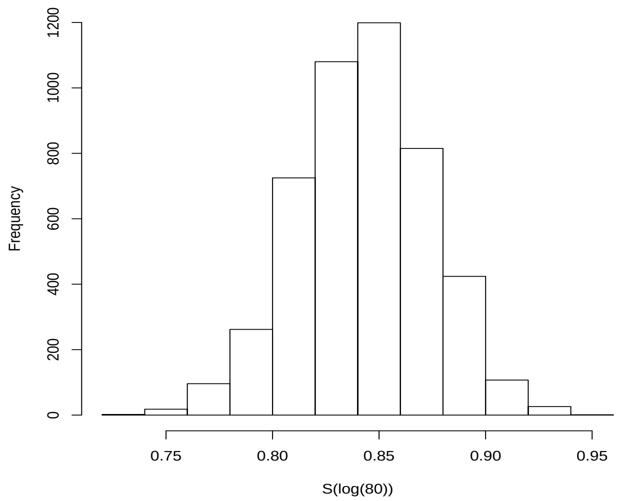

Figure 5: Bootstrap distribution of $\widehat{S}(\log(80))$, data of example 1.

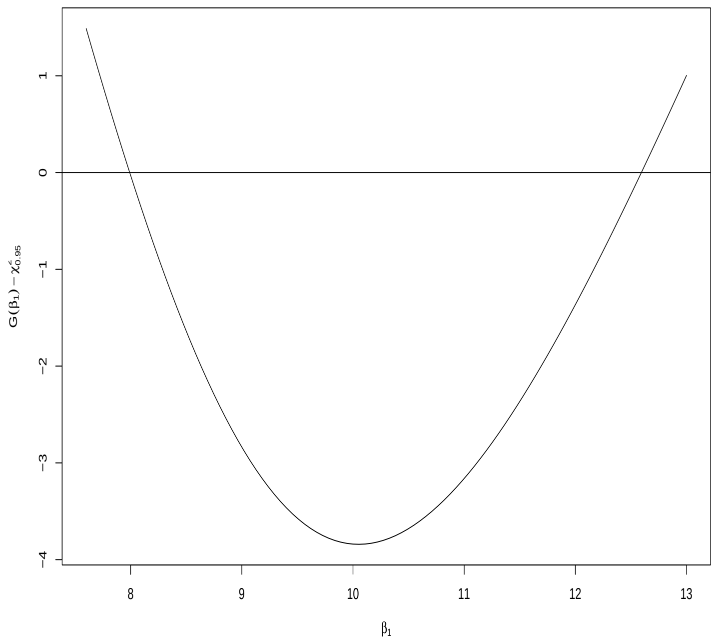

Figure 6: Plot of the function $G(\beta_1) - \chi^2_{1,0.95}$ for the data of example 2.

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