Financial Risk Assessment with Cauchy Distribution under a Simple Transformation of dividing with a Constant

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Abstract

Financial risk is the risk that value of the investment will change due to the moves in the market risk factors. Typical market risk factors are the stock price returns which are commonly assumed to be log normally distributed. In this paper we investigate using the Cauchy distribution under a simple transformation of dividing with a constant in financial risk assessment. We characterize this distribution by the first four moments: mean, variance, skewness and kurtosis, since these moments are use in many risk management applications. We use the simulated data to show the performance of model.

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1   Introduction

In probability theory, the "standard" Cauchy distribution is the probability distribution whose probability density function is

$$f(x) = \frac{1}{\pi(1 + x^2)}$$

for x real. This has median 0, and first and third quartiles respectively −1 and +1. Generally, a *Cauchy distribution* is any probability distribution belonging to the same location-scale family as this one. Thus, if X has a standard Cauchy distribution and μ is any real number and σ > 0, then

$$Y = \mu + \sigma X$$

has a Cauchy distribution whose median is μ and whose first and third quartiles are respectively $\mu - \sigma$ and $\mu + \sigma$.

McCullagh's parameterization, introduced by [1], used the two parameters of the non-standardized distribution to form a single complex-valued parameter, specifically, the complex number

$$\theta = \mu + i\sigma,$$

where i is the imaginary unit. It also extends the usual range of scale parameter to include $\sigma < 0$.

Although the parameter is notionally expressed using a complex number, the density is still a density over the real line. In particular the density can be written using the real-valued parameters μ and σ, which can each take positive or negative values, as

$$f(x) = \frac{1}{\pi|\sigma|\left(1 + \frac{(x - \mu)^2}{\sigma^2}\right)}$$

where the distribution is regarded as degenerate if $\sigma = 0$. An alternative form for the density can be written using the complex parameter $\theta = \mu + i\sigma$ as

$$f(x) = \frac{\Im\theta}{\pi|x - \theta|^2}$$

where $\Im\theta = \sigma$.

To the question "Why introduce complex numbers when only real-valued random variables are involved?", McCullagh wrote: To this question I can give no better answer than to present the curious result that

$$Y^* = \frac{\alpha Y + b}{cY + d} \sim C\left(\frac{\alpha\theta + b}{c\theta + d}\right)$$

for all real numbers α, b, c and d. … the induced transformation on the parameter space has the same fractional linear form as the transformation on the sample space only if the parameter space is taken to be the complex plane.

In other words, if the random variable Y has a Cauchy distribution with complex parameter θ, then the random variable Y^* defined above has a Cauchy distribution with parameter

$$(\alpha\theta + b)/(c\theta + d).$$

The distribution of the first exit point from the upper half-plane of a Brownian particle starting at θ is the Cauchy density on the real line with parameter θ." In addition, the complex-valued parameterization allows a simple relationship to be made between the Cauchy and the "circular Cauchy distribution".

Conventionally, Cauchy distribution is regarded as undefined. [2] made an attempt to solve the problem of non-definition of the Cauchy distribution through truncation. A renormalized pdf of the form

$$f(x) = \frac{1}{2\tan x}\left(\frac{1}{1 + x^2}\right)$$

where the first and the third moments are zero was obtained. Since interest is limited to the first four moments because of the basic statistical measures namely; mean, variance, skewness and kurtosis, [3] obtained a Cauchy distribution whose

first four moments are non-zero.

The purpose of this article is to investigate this model as a tool to evaluate the uncertainty (risk) in future stock price returns.

2   Mathematical Model

Let L_t be a time–continuous stochastic process with independent and stationary increments; for every $t > s \geq 0$, $L_t - L_s$ is independent of L_s and its distribution depends only on the time increment $t - s$ and not on t or s. Examples are the Brownian motion which has continuous paths and the Poisson which process is a non–decreasing process and thus has paths of bounded variation over finite time horizons. A Brownian motion does not have monotone paths and in fact its paths are unbounded variation over finite time horizons are unbounded variation over finite time horizons.

Except for Brownian motion, the paths of a levy process have jumps at random time points, and that is precisely what happens in a market of securities if it is of the view that the value is the price agreed last [4]. By the levy model imposition, the sizes of the jumps (whether positive or negative), have no relation to the price level reached, and the continuous time viewpoint forces the distribution of the increments $L_t - L_s$, to belongs to the infinitely divisible ones, example is the Cauchy distribution.

If $(X_1 + X_2 + \cdots + X_n)$ are independent and identically distributed random variable, each with a standard Cauchy distribution, then the sample mean $(X_1 + X_2 + \cdots + X_n)/n$ has the same standard Cauchy distribution. This can be seen if one computes the characteristics function of the sample mean,

$$f(x) = \frac{1}{2 \tan x}\left(\frac{1}{1 + x^2}\right) \tag{1}$$

Where $\tilde{x}$ is the sample mean. This example serves as a more generalized version of the central limit theorem that is the characteristic of all levy skew alpha –stable distribution, of which the Cauchy distribution is a special case. The location –scale family to which the Cauchy distribution belongs is closed under simple transformation of dividing with a constant [3]. The classical model proposed by [5] relates the stock price S_t to the levy process L_t through an exponential linear Brownian motion with drift

$$S_t = S_0 e^{L_t} = S_0 e^{\sigma B_t + \mu_t}, \quad t \geq 0 \tag{2}$$

Where $S_0 > 0$ is the initial price of stock, $B = \{B_t : t > 0\}$ is a standard Brownian motion, $\sigma > 0$ and $\mu \in \mathbb{R}$. This choice of model offers the feature the stock prices have multiplicative stationarity and independence in the sense that any $0 \leq u < t < \infty$,

$$S_t = S_u \times \tilde{S}_{t-u} \tag{3}$$

Where $\tilde{S}_{t-u}$ is of S_u and has same distribution as S_{t-u} [6].

The ratio of prices at t and $t - \tau$ becomes

$$R_t(\tau) = \ln(S_{t+\tau}) - \ln(S_t) = L_{t+\tau} - L_t, \tag{4}$$

and we pass in a discrete time an ordinary random walk base on independent increments $X_t = L_{t+\tau} - L_t$, the distribution of which being our modeling tool.

In [7] the risk neutral moments of τ - period of the ratio $R_t(\tau)$ in (4) of the price of stocks at time t on variance $VAR(t, \tau)$, skewness $SKEW(t, \tau)$ and kurtosis $KURT(t, \tau)$ was obtained as:

$$VAR(t,\tau) = e^{rt}V(t,\tau) - \mu(t,\tau)^2 \tag{5}$$

$$SKEW(t,\tau) = \frac{e^{rt}W(t,\tau) - 3\mu(t,\tau)e^{rt}V(t,\tau) + 2\mu(t,\tau)^3}{[e^{rt}V(t,\tau) - \mu(t,\tau)^2]^{3/2}} \tag{6}$$

$$KURT(t,\tau) = \frac{e^{rt}X(t,\tau) - 4\mu(t,\tau)e^{rt}W(t,\tau) + 6e^{rt}\mu(t,\tau)^2V(t,\tau) - 3\mu(t,\tau)^4}{[e^{rt}V(t,\tau) - \mu(t,\tau)^2]^2} \tag{7}$$

where $V(t,\tau)$, $W(t,\tau)$, $X(t,\tau)$ and $\mu(t,\tau)$ are given by

$$V(t,\tau) = \int_{S_t}^{\infty} \frac{2(1-\ln(K/S_t))}{K^2}C(t,\tau,K)dK + \int_0^{S_t} \frac{2(1-\ln(K/S_t))}{K^2}P(t,\tau,K)dK, \tag{8}$$

$$W(t,\tau) = \int_{S_t}^{\infty} \frac{6\ln(K/S_t) - 3(\ln(K/S_t))^2}{K^2}C(t,\tau,K)dK$$

$$+ \int_0^{S_t} \frac{6\ln(K/S_t) - 3(\ln(K/S_t))^2}{K^2}P(t,\tau,K)dK \tag{9}$$

$$X(t,\tau) = \int_{S_t}^{\infty} \frac{12(\ln(K/S_t))^2 - 4(\ln(K/S_t))^3}{K^2}C(t,\tau,K)dK$$

$$+ \int_0^{S_t} \frac{12(\ln(K/S_t))^2 - 4(\ln(K/S_t))^3}{K^2}P(t,\tau,K)dK \tag{10}$$

3   The Cauchy Distribution Under a Simple Transformation

A random variable Y follows a Cauchy distribution under a simple transformation of dividing through by a constant with parameter vector (a,b), in symbolic notation $X \sim Cauchy\ (a,b)$, if its probability density function is;

$$f(S_t;a,b) = \frac{b\theta}{\pi}\left(\frac{1}{b^2(s-a)^2}\right), \quad -1 < S_t < 1 \tag{11}$$

where θ is the stabilization term given as [3];

$$\theta = \frac{\pi}{2b \tan^{-1}(\frac{1-a}{b})} \tag{12}$$

The pdf (11) can therefore be written as

$$f(S_t;a,b) = \frac{1}{2 \tan^{-1}\left(\frac{1-a}{b}\right)}\left(\frac{1}{b^2 + (s-a)^2}\right) , \quad -1 < S_t < 1, \quad a < 1, b > 0 \tag{13}$$

Note that a and b are ordinary parameters of location and scales.

Under this transformation, the variance s is obtained by

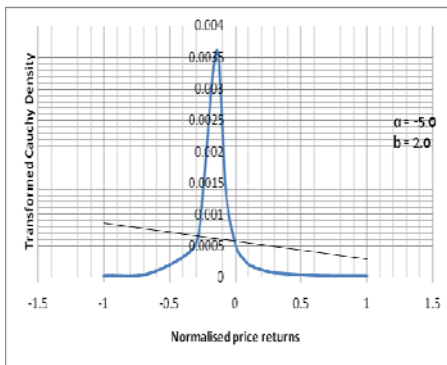$$S_t = \frac{x}{|x_m|} = \frac{(x_{-r},...,x_{-1},x_0,x_1,...x_q)}{|x_m|} \tag{14}$$

by letting

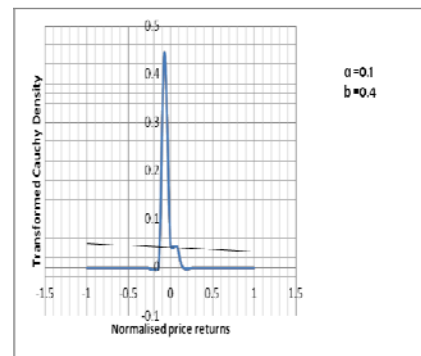$$|x_m| = \max|x_{-r},...,x_0,...,x_q|, \quad x \in R, \quad -r < x < q: \quad r,q \in R.$$

R is the real space without the points, $-\infty$ and ∞ [8, Ch. 3 and Ch.5.], X is a Cauchy random variable.

The characteristic function of the pdf is given by

$$\psi(y) = \frac{\theta}{\pi} \int_{-\infty}^{\infty} e^{ibs} \frac{1}{b^2 + (s-a)^2} = \frac{\theta}{\pi} e^{ias-bs} = \frac{1}{2 \tan^{-1}\left(\frac{1-a}{b}\right)} e^{ias-bs} \tag{15}$$

Figure 1: (a,b) shows Cauchy densities under transformation for different parameter sets.

Suppose y_1 and y_2 are independent and identically distributed as transformed Cauchy variables as

$$f(S_1;0,1) = k\left(\frac{1}{1+s_1^2}\right), \quad -1 < S_1 < 1$$

and

$$f(S_2;0,1) = k\left(\frac{1}{1+s_2^2}\right), \quad -1 < S_2 < 1$$

where $k = \dfrac{1}{\tan^{-1}1}$. Let $z = s_1 + s_2 \Rightarrow s_2 = z - s_1$ then by the convolution theorem in [9], we have

$$f(z;0,1) = \int_{-\infty}^{\infty} \{f(s_1)f(z-s_1)\} ds_1$$

$$= k^2 \int_{-\infty}^{\infty} \frac{dy_1}{(1+s_1^2)(1+(z-s_1)^2)}$$

$$= \frac{2k^2}{z}\int_{-\infty}^{\infty} \frac{dy_1}{y_1(4+z^2+s_1^2+2zs_1)} = \frac{2k}{4+z^2}$$

Hence the probability density function of Z is $f(z;0,1) = \dfrac{2k}{4+z^2}$. This suggests the general form for sum of n Cauchy variables under transformation

$$f(Z;A,B) = \frac{n}{2\tan^{-1}\left(\dfrac{1-A}{B}\right)}\left(\frac{1}{(nB)^2+(Z-A)^2}\right), \quad -\infty < Z < \infty \tag{16}$$

with $A = a_1 + a_2 + \cdots + a_n$, $B = b_1 + b_2 + \cdots + b_n$, $Z = s_1 + s_2 + \cdots + s_n$ and $n = 1,2,\ldots$ The sample path of the independent sum of the normalized stock price returns is shown in Figure 2 using (16).

The first four moments of this pdf are obtained [3] as

$$E(s) = \frac{1}{2\tan^{-1}\left(\frac{1-a}{b}\right)} \left[\begin{array}{c} \frac{1}{2}\left[\ln(1+a^2+b^2-2a)-\ln(1+a^2+b^2+2a)\right] \\ +\frac{a}{b}\left(\tan^{-1}\left(\frac{1-a}{b}\right)-\tan^{-1}\left(\frac{-1-a}{b}\right)\right) \end{array} \right] \tag{17}$$

$$E(s^2) = \frac{1}{\tan^{-1}\left(\frac{1-a}{b}\right)} \left[\begin{array}{c} 1+a\left[\ln(1+a^2+b^2-2a)-\ln(a^2+b^2)\right] \\ +\frac{a^2-b^2}{b}\left(\tan^{-1}\left(\frac{1-a}{b}\right)-\tan^{-1}\left(\frac{-a}{b}\right)\right) \end{array} \right] \tag{18}$$

$$E(s^3) = \frac{1}{2\tan^{-1}\left(\frac{1-a}{b}\right)} \left[\begin{array}{c} 4a+\frac{3a^2-b^2}{2}\left[\ln(1+a^2+b^2-2a)-\ln(1+a^2+b^2+2a)\right] \\ +\frac{a^2-3ab^2}{b}\left(\tan^{-1}\left(\frac{1-a}{b}\right)-\tan^{-1}\left(\frac{-1-a}{b}\right)\right) \end{array} \right] \tag{19}$$

and

$$E(s^4) = \frac{1}{2\tan^{-1}\left(\frac{1-a}{b}\right)} \left[\begin{array}{c} \frac{1}{3}+a-b^2+3a^2+2a(a^2-b^2)\ln(1+a^2+b^2-2a) \\ +b(b^2-6)\left(\tan^{-1}\left(\frac{1-a}{b}\right)\right)-2a(a^2-b^2)\ln(a^2-b^2) \\ +b(b^2-6)\left(\tan^{-1}\left(\frac{-a}{b}\right)\right) \end{array} \right] \tag{20}$$

From which we obtain the expression for the skewness and kurtosis as:

$$skew(a,b) = \frac{\varphi^2(a,b)+\varnothing^2(a,b)}{\left[\varphi^{(1)}(a,b)+\varnothing^{(1)}(a,b)\right]^{\frac{3}{2}}} \tag{21}$$

where

$$\varphi^{(1)}(a,b) = 1+\ln\left[\frac{(1+a^2+b^2-2a)^a}{a^2+b^2}\right]$$

$$\varnothing^{(1)}(a,b) = \frac{a^2-b^2}{b}\left[\tan^{-1}\left(\frac{1-a}{b}\right)+\tan^{-1}\left(\frac{1-a}{b}\right)\right]$$

$$\varphi^2(a,b) = 4a + \left[\ln \frac{(1+a^2+b^2-2a)^{\frac{3a^2-b^2}{2}}}{1+a^2+b^2-2a} \right]$$

$$\varnothing^2(a,b) = \frac{a^3-3ab^2}{b}\left[\tan^{-1}\left(\frac{1-a}{b}\right) + \tan^{-1}\left(\frac{1-a}{b}\right) \right]$$

and

$$Kurt(a,b) = \frac{\varphi^3(a,b)+\varnothing^3(a,b)}{\left[\varphi^{(1)}(a,b)+\varnothing^{(1)}(a,b)\right]^2} \qquad (22)$$

with

$$\varphi^3(a,b) = \frac{1}{3}+a-b^2+3a^2+\left[\ln \frac{(1+a^2+b^2-2a)^{\frac{2a(a^2-b^2)}{2}}}{a^2+b^2} \right]$$

$$\varnothing^3(a,b) = b(b^2-6)\left[\tan^{-1}\left(\frac{1-a}{b}\right) + \tan^{-1}\left(\frac{a}{b}\right) \right]$$

Theorem 1. Suppose that random variable y is Cauchy (a,b) distributed with mean and variance a and b respectively. Then the parameters a and b are given by;

$$b = \sum_{i=1}^{n} \sqrt{(1-a)(s_i-a)} \qquad (23)$$

from which we have

$$a = \sum_{i=1}^{n} (s_i+1) \pm \sqrt{(s_i+1)^2 - 4(s_i+b^2)} \qquad (24)$$

Proof: We apply the maximum likelihood (MLE) estimation on the pdf (13) to maximize log-likelihood function

$$b = -n\log\tan^{-1}\left(\frac{1-a}{b}\right) - \frac{1}{2}\sum_{i=1}^{n}\log\left[b^2+(s_i-a)^2\right] \qquad (25)$$
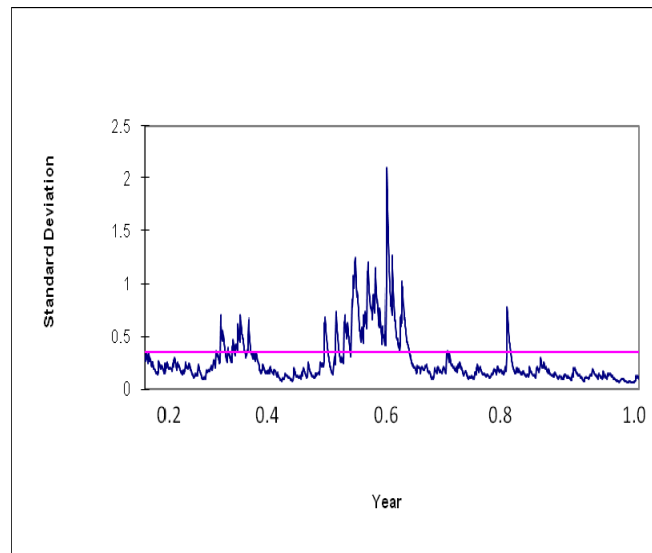
and we get the MLE for b and then a as in (23) and (24).

Figure 2: A sample path of the independent sum of the normalized stock price returns;

$$f(Z; A, B) = \frac{n}{2 \tan^{-1}\left(\dfrac{1-A}{B}\right)} \cdot \frac{1}{(nB)^2 + (Z-A)^2}$$

The Cauchy distribution appears as the opposite limit as $a \to 0$ for the inverse Gaussian distribution;

$$NIG(x, a, \beta, \mu, \delta) \sim k |x|^{-1.5} \exp(-\frac{a}{\delta}|x| + \frac{\beta}{\delta}x)$$

when $|x| \to 0$. In practice therefore, given sample data, we have the central moments of a transformed Cauchy distributed random variable $X \sim Cauchy\ (a,b)$ as [10];

$$m_1 = \frac{a}{b} \ , \ m_2 = \frac{a}{b^2} \ , \ \Im = \frac{3}{\sqrt{a}} \ , \ \Re = 3 + \frac{15}{a}$$

It is easy to see that

$$b = \frac{m_1}{m_2} \ , \ a = b^2 m_2$$

where

$$m_1 = \frac{1}{n}\sum_{i=1}^{n} s_i \ , \qquad\qquad m_2 = \frac{1}{n}\sum_{i=1}^{n}(s_i - m_1)^2$$

$$m_3 = \frac{1}{n}\sum_{i=1}^{n}(s_i - m_1)^3 \ , \qquad m_4 = \frac{1}{n}\sum_{i=1}^{n}(s_i - m_1)^4$$

$$\Im = \frac{m_1}{(\sqrt{m_2})^3} \ , \quad \Re = \frac{m_4}{(m_2)^2}$$

One can also attempt to estimate a by solving for $\tan^{-1}\left(\dfrac{1-a}{b}\right)$ in (17) and (19)

and equating the results using $b^2 = (1-a)(s-a)$.

Figure 2 shows the Cauchy densities under transformation of dividing with a constant for different parameter sets, obtained by using simulated data. Figure 3 below is the log stock price returns with the Cauchy Density Function under Transformation, with simulated data.
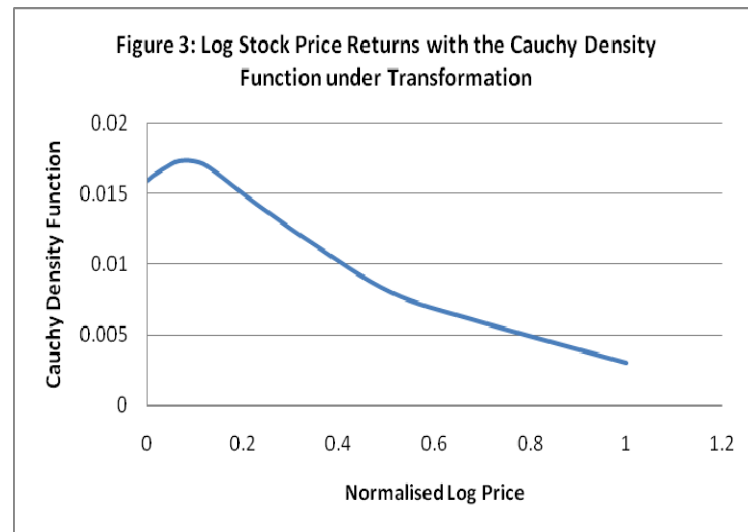

Figure 3 : Log Stock Price Returns with the Cauchy Density Function
        under Transformation

4   Risk Measure with Cauchy Distribution Under a Simple Transformation

We consider herein a kind of distortion risk measure for a random variable $S(y)$ with distortion function given by

$$\rho(y) = \int_0^\infty g(S(y))dy \qquad (26)$$

where $g(S(y))$ is the Cauchy distribution under transformation. That is

$$g[S(y)] = Cauchy(a, b, S(y)) = \frac{1}{2\tan^{-1}\left(\dfrac{1-a}{b}\right)} \int_0^{S(y)} \frac{1}{b^2 + (y-a)^2} dy = F_c(S(y)) \qquad (27)$$

Given $0 < a < 1$, the $S_a(y)$, determined by $\overline{F}(S_a) = 1 - F(S_a) = a$ and denote by $V_a R_S(1-a)$ is called the value at risk $V_a R$ with a degree of confidence $1-a$. The conditional expectation of $S(y)$ given by $S > S_a$, denoted by $TCE_S(S_a) = E((S \mid S > S_a))$ is called the tail conditional expectation TCE of S at $V_a RS_a$.

Using distorted probabilities, it is possible to define a distortion $g_v(\cdot)$ that will produce the traditional $V_a R$ measure, V_a as the risk measure

$$g[v(y)] = \begin{cases} 1, & \text{if } 1-a < y < 1 \\ 0, & \text{if } 0 < y < 1-a \end{cases} \qquad (28)$$

So that the risk measure is

$$\rho[v(y)] = \int_0^\infty g[v(y)]dy = \int_0^{V_a} dy = V_a \qquad (29)$$

where V_a is $F_S^{-1}(y)$. The tail conditional expectation (TCE) or tail-$V_a R$ is defined for smooth distortion functions, given the parameter a, $0 < a < 1$, as:

$$TCE_a = E((S \mid S > F_S^{-1}(a))) \qquad (30)$$

Where F_S^{-1} is the inverse distortion function of the variable S. It is well known that the *TCE* can also be expressed in terms of a distortion risk measure as follows:

$$g_c(y) = \begin{cases} 1, & \text{if } 1-a < y \leq 1 \\ \dfrac{y}{1-a}, & \text{if } 0 < y < 1-a \end{cases} \tag{31}$$

Definition 1.

Let $(\Omega, F, (F_t)_{t \geq 0}, P)$ be a filtered probability space. An adapted cadlag R-valued process. $X = \{X(t)\}_{t \geq 0}$ with $x(0) = 0$ is a Cauchy –levy process under transformation if x(t) has independent and stationary increment distributed as Cauchy $(; a, b)$.

Now we choose to θ such that the discounted price process

$$\{\exp(-(r-q)t)\, s_t,\ t \geq 0\}$$

is marginal ie

$$s_0 = \exp(-(r-q)t)E^\theta[s_t] \tag{32}$$

where expectation is taken with respect to the law with density $f_t^{(\theta)}(x)$, q the rate of yield of compound dividends per annum and r the interest rate.

Let $\varnothing(u) = E[\exp(u\,i\,X_1)]$ denote the characteristics functions of X_1. Then from (32) it can be shown that in order to let the discounted price process be a Martingale, we need to have [11];

$$\exp(r-q) = \frac{\varnothing(-i(\theta+1))}{\varnothing(-\theta)} \tag{33}$$

First we note that in the Black-Scholes world the historical measure of the log returns over a period of length 1 follows a normal $\left(\mu - \dfrac{1}{2}r^2,\ \sigma^2\right)$ law and this in this case

$$\varnothing(x) = \exp\left[iu(\mu - \frac{1}{2}\sigma^2) - \sigma^2 \frac{u^2}{2} \right].$$

So that (33) becomes

$$r - q = \mu - \frac{1}{2}r^2 + \frac{1}{2}\sigma^2(2\theta + 1)$$

or

$$\theta^* = \frac{r - q - u}{\sigma^2}.$$

The solution of this equation θ say, gives us the Esscher transform martingale measure through the density time $f_t^{\theta^*}(x)$.

Now we choose θ such that the discounted price present

$$\{\exp(-(r-q)t)s_t, \ t \geq 0\}$$

is a martingale, i.e;

$$s_0 = \exp(-(r-q)t)E^{\theta}[s_t] \tag{34}$$

where expectation is taken with respect to the law with density $f_t^{\theta^*}(x)$.

Let $\varnothing(r) = E[\exp(u\,i\,X_i)]$ denote the characteristic formation of X_i. Then from (34) it can be shown that in order to let the discounted price process be a martingale, we need to (33).

where expectation is taken with respect to the law with density $f_t^{\theta^*}(x)$.

Pricing through the characteristic function using (15) and applying (33), we have

$$r - q = a + b(\theta + 1)$$

or

$$\theta^* = \frac{r - q - a - b}{b}.$$

5   Conclusion

Let S_t be the exponential Cauchy-Levy (under transformation) price process defined in (2) with parameter $[a,b]$. One possible arbitrage-free price of a European –type contingent pay-off $f(S_t)$ at time t is

$$C(k,t) = \exp(-qt)\, S_0\, Cauchy(d_1) - K \exp(-rt)\, Cauchy(d_2)$$

Where

$$d_1 = \log\left(\frac{S_0}{K}\right) + \frac{r-q-a-b}{b}\tau,$$

$$d_2 = \log\left(\frac{S_0}{K}\right) + \frac{r-q+a+b}{b}\tau.$$

References

[1]   P. McCullagh, Conditional inference and Cauchy models, *Biometrika*, **79**, (1992), 247–259.

[2]   C. Wakk, *Hand-book on Statistical Distributions for Experimentalists*, Particle physics Group, Fasikum, University of Stockholm, 2000.

[3]   J. Ohakwe and B.O. Osu, The Existence of the moments of the Cauchy Distribution under a Simple Transformation of Dividing with a Constant,   to appear *Theoretical Mathematics and Application*, (2011).

[4]   E. Bolviken and F.E. Benth, Quantification of risk in Norwegian stocks via the Normal inverse Gaussian distribution, Papers, (2000), ssrn.com/sol3/Delivery.cfm.nbj006275.309.

[5]   P. Samuleson, Rational theory of Warrant pricing, *Ind. Management Review*, **6**, (1965), 13-32.

[6]   A.E. Kyprianou, An Introduction to the theory of levy processes. *Lecture*

notes. Satellite summerschool on levy processes: *Theory and applications*, Sonderborg Denmark, (2007).

[7]   G. Bakshi, N. Kapadia and D. Madan, Stock Return characteristics, Skew laws, and Differential pricing of individual equity options, *Review of Financial studies*, **16**(1), (2003), 101-143.

[8]   A.S. Kapadia, W. Chan and L.A. Moye, *Mathematical statistics with Applications*, CRC press, Taylor and Francis Group, 2005.

[9]   J.F.C. Kingman and S.J. Taylor, *Introduction to measure and probability*, Cambirdge University press, 1973.

[10] N. Johnson, S. Kotz and N. Balakrishan, *Continuous Univariate distributions*, vol. **1**, Wiley-Interscience Publication, 1994.

[11] W. Schoutens, *Levy processes in Finance: Pricing Financial Derivatives*, John Wiley and Sons, 2003.