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Convergence theorems for a finite family of nonspreading and nonexpansive multivalued mappings and equilibrium problems with application

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Abstract

We introduce a finite family of nonspreading multivalued mappings and a finite family of nonexpansive multivalued mappings in Hilbert space. We establish some weak and strong convergence theorems of the sequences generated by our iterative process. Some new iterative sequences for finding a common element of the set of solutions of an equilibrium problem and the set of common fixed points it was introduced. The results improve and extend the corresponding results of Mohammad Eslamian (Optim. Lett., 7(3):547-557, 2012).

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1 Introduction

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively. Let E be a nonempty closed subset of a real Hilbert space H. Let $N(E)$ and $CB(E)$ denote the family of nonempty subsets and nonempty closed bounded subsets of E, respectively. *The Hausdorff metric* on $CB(E)$ is defined by

$$H(A_1, A_2) = \max\{\sup_{x\in A_1} d(x, A_2), \sup_{y\in A_2} d(y, A_1)\}$$

for $A_1, A_2 \in CB(E)$, where $d(x, A_1) = \inf\{||x - y||, y \in A_1\}$. An element $p \in E$ is called *a fixed point* of $T : E \to N(E)$ if $p \in T(p)$. The set of fixed points of T is represented by $F(T)$.

A subset $E \subset H$ is called proximal if for each $x \in H$, there exists an element $y \in H$ such that

$$\|x - y\| = d(x.E) = \inf\{\|x - z\| : z \in E\}.$$

We denote by $K(E)$ and $P(E)$ nonempty compact subsets and nonempty proximal bounded subsets of E, respectively.

The multi-valued mapping $T : E \to CB(E)$ is called nonexpansive if

$$H(Tx, Ty) \le ||x - y||, \quad for \quad \forall x, y \in E.$$

The multi-valued mapping $T : E \to CB(E)$ is called quasi-nonexpansive if $F(T) \ne \emptyset$ and

$$H(Tx, Tp) \le ||x - p||, \quad for \quad \forall x \in E \quad p \in F(T).$$

Iterative process for approximating fixed points (and common fixed points) of nonexpansive multivalued mappings have been investigated by various authors (see[1], [2], [3]).

Recently, Kohsaka and Takahashi (see[4], [5])introduced an important class of mappings which they called the class of nonspreading mappings. Let E be a subset of Hilbert space H, then they called a mapping $T : E \to E$ nonspreading if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in E.$$

Lemoto and Takahashi [6] proved that $T : E \to E$ is nonspreading if and only if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Ty, y - Ty\rangle, \quad \forall x, y \in E.$$

Now, inspired by [4] and [5], we propose a definition as follows,

Definition 1.1. *The multivalued mapping $T : E \to CB(E)$ is called nonspreading if*

$$2\|u_x - u_y\|^2 \leq \|u_x - y\|^2 + \|u_y - x\|^2, \tag{1}$$

for $u_x \in Tx,\ u_y \in Ty, \forall x, y \in E$.

By Takahashi [6], We get also the multivalued mapping $T : E \to CB(E)$ is nonspreading if and only if

$$\|u_x - u_y\|^2 \leq \|x - y\|^2 + 2\langle x - u_x, y - u_y\rangle, \tag{2}$$

for $u_x \in Tx, u_y \in Ty, \forall x, y \in E$. In fact,

$$\begin{aligned}
&2\|u_x - u_y\|^2 \leq \|u_x - y\|^2 + \|u_y - x\|^2 \\
&\Leftrightarrow 2\|u_x - u_y\|^2 \leq \|u_x - x\|^2 + 2\langle u_x - x, x - y\rangle + \|x - y\|^2 \\
&\qquad\qquad + \|u_x - x\|^2 + 2\langle x - u_x, u_x - u_y\rangle + \|u_x - u_y\|^2 \\
&\Leftrightarrow 2\|u_x - u_y\|^2 \leq 2\|u_x - x\|^2 + \|x - y\|^2 + \|u_x - u_y\|^2 \\
&\qquad\qquad + 2\langle u_x - x, x - u_x - (y - u_y)\rangle \\
&\Leftrightarrow \|u_x - u_y\|^2 \leq \|x - y\|^2 + 2\langle x - u_x, y - u_y\rangle.
\end{aligned}$$

The equilibrium problem for $\phi : E \times E \to R$ is to find $x \in E$ such that $\phi(x, y) \geq 0, \forall y \in E$. The set of such solution is denoted by $EP(\phi)$. Given a mapping $T : E \to CB(E)$, let $\phi(x, y) = \langle x, y\rangle$ for all $y \in E$. The $x \in EP(\phi)$ if and only if $x \in E$ is a solution of the variational inequality $\langle Tx, y\rangle \geq 0$ for all $y \in E$.

Numerous problems in physics, optimization, and economics can be reduced to find a solution of the equilibrium problem. Some methods have been proposed to solve the equilibrium problem. For instance, see Blum and Oettli [7], Combettes and Hirstoaga [8], Li and Li [9], Giannessi, Maugeri, and Pardalos [10], Moudafi and Thera [11] and Pardalos,Rassias and Khan [12]. In the recent years, the problem of finding a common element of the set of solutions of equilibrium problems and the set of fixed points of singlevalued nonexpansive mappings in the framework of Hilbert spaces has been intensively studied by many authors.

In this paper, inspired by [13], we propose an iterative process for finding a common element of the set of solutions of equilibrium problem and the set of common fixed points of a countable family of nonexpansive multivalued mappings and a countable family of nonspreading multivalued mappings in the setting of real Hilbert spaces. We also prove the strong and weak convergence of the sequences generated by our iterative process. The results presented in the paper improve and extend the corresponding results in [15].

2 Preliminaries

In the sequel, we begin by recalling some preliminaries and lemmas which will be used in the proof. Let H be a real Hilbert spaces with the norm $\|\cdot\|$. It is also known that H satisfies Opial's condition, i.e., for any sequence x_n with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $y \neq x$.

Lemma 2.1. *(see[16]) Let H be a real Hilbert space, then the equality*

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$$

holds for all $x, y \in H$

Lemma 2.2. *(see[17]) Let H be a real Hilbert space. Then for all $x, y, z \in H$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\begin{aligned}\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 \\ -\beta\gamma\|y-z\|^2 - \gamma\alpha\|z-x\|^2.\end{aligned}$$

For solving the equilibrium problem, we assume that the bifunction $\phi : E \times E \to R$ satisfies the following conditions for all $x, y, z \in E$:

(A_1) $\phi(x, x) = 0$,

(A_2) ϕ is monotone, *i.e.*, $\phi(x, y) + \phi(y, x) \leq 0$,

(A_3) ϕ is upper-hemicontinuous, i.e.$\limsup_{t\to 0^+} \phi(tz + (1-t)x, y) \leq \phi(x, y)$,

(A_4) $\phi(x, .) = 0$ is convex and lower semicontinuous.

Lemma 2.3. *(see[7]) Let E be a nonempty closed convex subset of H and let ϕ be a bifunction satisfying $(A_1) - (A_4)$. Let $r > 0$ and $x \in H$. Then, there existsz $\in E$ such that*

$$\phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in E.$$

The following Lemma is established in [8].

Lemma 2.4. *Let ϕ be a bifunction satisfying $(A_1) - (A_4)$. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to E$ as follows*

$$T_r x = \{z \in E : \phi(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in E\}.$$

Then, the following results hold

(a) T_r *is single valued;*

(b) T_r *is firmly nonexpansive,i.e.,for any $x, y \in H$,$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;*

(c) $F(T_r) = EP(\phi)$;

(d) $EP(\phi)$ *is closed and convex.*

The following Lemma is established in [3].

Lemma 2.5. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T : E \to K(E)$ be a nonexpansive multivalued mapping. If $x_n \rightharpoonup p$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, then $p \in Tp$.*

Lemma 2.6. *Let E be a nonempty closed convex subset of a real Hilbert space H. Let $T : E \to E$ be a nonspreading multivalued mapping such that $F(T) \neq \emptyset$. Then S is demiclosed, i.e., $x_n \rightharpoonup p$ and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$, we have $p \in Tp$.*

Proof. Let $\{x_n\} \subset E$ be a sequence such that $x_n \rightharpoonup p$ and

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Then the sequences $\{x_n\}$ and Tx_n are bounded. Suppose that $p\bar{\in}Tp$. From Definition 1.1 and Opial's condition, we have

$$\begin{aligned}
&\liminf_{n\to\infty} \|x_n - p\|^2 \\
&< \liminf_{n\to\infty} \|x_n - u_p\|^2 \\
&= \liminf_{n\to\infty} (\|x_n - u_{x_n}\|^2 + \|u_{x_n} - u_p\|^2 + 2\langle x_n - u_{x_n}, u_{x_n} - u_p\rangle) \\
&\leq \liminf_{n\to\infty} (\|x_n - u_{x_n}\|^2 + \|x_n - p\|^2 + 2\langle x_n - u_{x_n}, p - u_p\rangle \\
&\qquad\qquad +2\langle x_n - u_{x_n}, u_{x_n} - u_p\rangle) \\
&= \liminf_{n\to\infty} \|x_n - p\|^2,
\end{aligned}$$

where $u_p \in Tp$ and $u_{x_n} \in Tx_n$. This is a contradiction. Hence we get the conclusion. □

3 Main results

Theorem 3.1. *Let E be a nonempty closed convex subset of a real Hilbert space H, ϕ be a bifunction of $E \times E$ into R satisfying $(A_1) - (A_2)$. Let for $i = 1, 2, 3, \cdots N$, $f_i : E \to E$ be a finite family of nonspreading multivalued mappings and $T_i : E \to K(E)$ be a finite family of nonexpansive multivalued mappings. Assume that $\mathfrak{F} = \bigcap_{i=1}^{N} F(T_i) \bigcap F(f_i) \bigcap EP(\phi) \neq \emptyset$ and $T_i p = \{p\}$ for each $p \in \mathfrak{F}$. Let $\{x_n\}$ and $\{w_n\}$ be sequences generated initially by an arbitrary element $x_1 \in E$ and*

$$\begin{cases} \phi(w_n, y) + \frac{1}{r_n}\langle y - w_n, w_n - x_n\rangle \geq 0, & \forall y \in E, \\ x_{n+1} = \alpha_n w_n + \beta_n u_n^n + \gamma_n z_n, & \forall n \geq 1, \end{cases}$$

where $u_n^i \in f_i w_n$,$z_n \in T_n w_n$, $T_n = T_{N(mod(N))}$ and *(i)* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\} \subset [a, 1] \subset (0, 1)$,

(ii) $\alpha_n + \beta_n + \gamma_n = 1, \forall n \geq 1$,

(iii) $r_n > 0$, *and* $\liminf_{n\to\infty} r_n > 0$.

Then, the sequences $\{x_n\}$ *and* $\{w_n\}$ *converge weakly to an element of* $\mathfrak{F}$.

Proof. **(I)** First, We claim that $\lim_{n\to\infty} \|x_n - q\|$ exist for all $q \in \mathfrak{F}$. Indeed, let $q \in \mathfrak{F}$, then from the definition of T_r in lemma 2.4,we have $w_n = T_{r_n}x_n$ and therefore

$$\|w_n - q\| = \|T_{r_n}x_n - T_{r_n}q\| \leq \|x_n - q\| \quad \forall n \geq 1.$$

By Lemma 2.2 and Since for each i,f_i is nonspearding multivalued mappings, we have

$$\begin{aligned}
&\|x_{n+1} - q\|^2 \\
&= \|\alpha_n w_n + \beta_n u_n^n + \gamma_n z_n - q\|^2 \\
&\leq \alpha_n\|w_n - q\|^2 + \beta_n\|u_n^n - q\|^2 + \gamma_n\|z_n - q\|^2 - \alpha_n\beta_n\|w_n - u_n^n\|^2 \\
&\qquad\qquad -\alpha_n\gamma_n\|w_n - z_n\|^2 \\
&= \alpha_n\|w_n - q\|^2 + \beta_n\|u_n^n - q\|^2 + \gamma_n d^2(z_n, T_n q) - \alpha_n\beta_n\|w_n - u_n^n\|^2 \\
&\qquad\qquad -\alpha_n\gamma_n\|w_n - z_n\|^2 \\
&\leq \alpha_n\|w_n - q\|^2 + \beta_n\|u_n^n - q\|^2 + \gamma_n H^2(T_n w_n, T_n q) - \alpha_n\beta_n\|w_n - u_n^n\|^2 \\
&\qquad\qquad -\alpha_n\gamma_n\|w_n - z_n\|^2 \\
&\leq \alpha_n\|w_n - q\|^2 + \beta_n\|w_n - q\|^2 + \gamma_n\|w_n - q\|^2 - \alpha_n\beta_n\|w_n - u_n^n\|^2 \\
&\qquad\qquad -\alpha_n\gamma_n\|w_n - z_n\|^2 \\
&= \|w_n - q\|^2 - \alpha_n\beta_n\|w_n - u_n^n\|^2 - \alpha_n\gamma_n\|w_n - z_n\|^2 \\
&\leq \|x_n - q\|^2 - \alpha_n\beta_n\|w_n - u_n^n\|^2 - \alpha_n\gamma_n\|w_n - z_n\|^2. \qquad (3)
\end{aligned}$$

This implies that $\|x_{n+1} - q\| \leq \|x_n - q\|$,hence we get $\lim_{n\to\infty} \|x_n - q\|$ exist for all $q \in \mathfrak{F}$.

(II) Next, we prove that $\lim_{n\to\infty} \|w_n - w_{n+i}\| = 0$, for all $i \geq 1$.

From (3), we have

$$\alpha_n\beta_n\|w_n - u_n^n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

It follows from our assumption that

$$a^2\|w_n - u_n^n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

Taking the limit as $n \to \infty$, we have $\lim_{n\to\infty} \|w_n - u_n^n\| = 0$. Observing (3) again and our assumption, we have

$$a^2\|w_n - z_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

Taking the limit as $n \to \infty$, we get $\lim_{n\to\infty} \|w_n - z_n\| = 0$. Thus

$$\lim_{n\to\infty} d(w_n, T_n w_n) \leq \lim_{n\to\infty} \|w_n - z_n\| = 0. \tag{4}$$

Also we have

$$\lim_{n\to\infty} \|x_{n+1} - w_n\| \leq \lim_{n\to\infty} \beta_n \|w_n - u_n^n\| + \lim_{n\to\infty} \gamma_n \|w_n - z_n\| = 0. \tag{5}$$

Since $w_n = T_{r_n} x_n$, we have

$$\begin{aligned}
\|w_n - q\|^2 &= \|T_{r_n} x_n - T_{r_n} q\|^2 \\
&\leq \langle T_{r_n} x_n - T_{r_n} q, x_n - q\rangle \\
&= \langle w_n - q, x_n - q\rangle \\
&= \frac{1}{2}(\|w_n - q\|^2 + \|x_n - q\|^2 - \|x_n - w_n\|^2),
\end{aligned}$$

and hence

$$\|w_n - q\|^2 \leq \|x_n - q\|^2 - \|x_n - w_n\|^2. \tag{6}$$

Using (3) and the last inequality (6), we have

$$\|x_{n+1} - q\|^2 \leq \|x_n - q\|^2 - \|x_n - w_n\|^2$$

and hence

$$\|x_n - w_n\|^2 \leq \|x_n - q\|^2 - \|x_{n+1} - q\|^2.$$

This imply that

$$\lim_{n\to\infty} \|x_n - w_n\| = 0. \tag{7}$$

Using (5) and (7), we get

$$\lim_{n\to\infty} \|w_n - w_{n+1}\| \leq \lim_{n\to\infty} (\|x_{n+1} - w_{n+1}\| + \|x_{n+1} - w_n\|) = 0. \tag{8}$$

It follows that

$$\lim_{n\to\infty} \|w_n - w_{n+i}\| = 0, \qquad \forall i \in \{1, 2, 3, \cdots N\}. \tag{9}$$

Applying (7)-(9), we obtain that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{10}$$

This also implies that

$$\lim_{n\to\infty} \|x_{n+i} - x_n\| = 0, \qquad \forall i \in \{1, 2, 3, \cdots N\}. \tag{11}$$

(III) We claim that for $i \in \{1, 2, \cdots N\}$,

$$\begin{aligned}\lim_{n\to\infty} d(w_n, T_i w_n) &= \lim_{n\to\infty} d(w_n, f_i w_n) = 0\\ \lim_{n\to\infty} d(x_n, T_i x_n) &= \lim_{n\to\infty} d(x_n, f_i x_n) = 0.\end{aligned}$$

Observing that for $i \in N$,

$$\begin{aligned}&\lim_{n\to\infty} d(w_n, T_{n+i} w_n)\\ &\quad\leq \lim_{n\to\infty} [\|w_n - w_{n+i}\| + d(w_{n+i}, T_{n+i} w_{n+i}) + H(T_{n+i} w_{n+i}, T_{n+i} w_n)]\\ &\quad\leq \lim_{n\to\infty} [2\|w_n - w_{n+i}\| + d(w_{n+i}, T_{n+i} w_{n+i})] = 0,\end{aligned}$$

which implies that the sequence

$$\bigcup_{i=1}^{N} \{d(w_n, T_{n+i} w_n)\} \to 0,$$

we have

$$\lim_{n\to\infty} d(w_n, T_i w_n) = \lim_{n\to\infty} d(w_n, T_{n+(i-n)} w_n) = 0.$$

Also we have

$$\begin{aligned}&\lim_{n\to\infty} d(w_n, f_{n+i} w_n)\\ &\leq \lim_{n\to\infty} [\|w_n - w_{n+i}\| + d(w_{n+i}, f_{n+i} w_{n+i}) + H(f_{n+i} w_{n+i}, f_{n+i} w_n)]\\ &\leq \lim_{n\to\infty} [\|w_n - w_{n+i}\| + \|w_{n+i} - u_{n+i}^{n+i}\| + \|u_{n+i}^{n+i} - u_n^{n+i}\|]\\ &\leq \lim_{n\to\infty} [\|w_n - w_{n+i}\| + \|w_{n+i} - u_{n+i}^{n+i}\|\\ &\qquad + (\|w_n - w_{n+i}\|^2 + 2|\langle w_{n+i} - u_{n+i}^{n+i}, w_n - u_{n+i}^{n}\rangle|)^{\frac{1}{2}} = 0\end{aligned}$$

which imply that

$$\lim_{n\to\infty} d(w_n, f_i w_n) = 0, \forall i \in \{1, 2, \cdots N\}.$$

Hence for $\forall i \in \{1, 2, \cdots N\}$, we have

$$\begin{aligned}\lim_{n\to\infty} d(x_n, T_i x_n) &\leq \lim_{n\to\infty} [\|x_n - w_n\| + d(w_n, T_i w_n) + H(T_i w_n, T_i x_n)] \\ &\leq \lim_{n\to\infty} [2\|x_n - w_n\| + d(w_n, T_i x_n)] = 0,\end{aligned}$$

and

$$\begin{aligned}\lim_{n\to\infty} d(x_n, f_i x_n) &\leq \lim_{n\to\infty} [\|x_n - w_n\| + d(w_n, f_i w_n) + H(f_i w_n, f_i x_n)] \\ &\leq \lim_{n\to\infty} [\|x_n - w_n\| + d(w_n, f_i w_n) \\ &\quad +(\|w_n - x_n\|^2 + 2|\langle w_n - u_n^i, x_n - u_n^i\rangle|)^{\frac{1}{2}} = 0.\end{aligned}$$

(IV) Now we prove that $\omega_w(x_n) \subset \mathfrak{F}$ where

$$\omega_w(x_n) = \{x \in H : x_{n_i} \rightharpoonup x, \{x_{n_i}\} \subset \{x_n\}\}.$$

Since $\{x_n\}$ is bounded and H is reflexive,$\omega_w(x_n)$ is nonempty. Let $\mu \in \omega_w(x_n)$ is be an arbitrary element. Then there exists a subsequence $\{x_{n_i} \subset \{x_n\}$ converging weakly to μ. From (3.5) we get that $w_{n_i} \rightharpoonup \mu$ as $i \to \infty$. We show that $\mu \in EP(\phi)$. Since $w_n = T_{r_n} x_n$, we have

$$\phi(w_n, y) + \frac{1}{r_n}\langle y - w_n, w_n - x_n\rangle \geq 0, \quad \forall y \in E.$$

By (A_4), we have

$$\frac{1}{r_n}\langle y - w_n, w_n - x_n\rangle \geq \phi(y, w_n),$$

and hence

$$\langle y - w_{n_i}, \frac{w_{n_i} - x_{n_i}}{r_{n_i}}\rangle \geq \phi(y, w_{n_i}).$$

Therefore, we have $\phi(y, \mu) \leq 0$ for all $y \in E$.

For $t \in (0, 1]$, let $y_t = ty + (1 - t)\mu$.Since $y, \mu \in E$ and E is convex, we have $y_t \in E$ and hence $\phi(y_t, \mu) \leq 0$. Form $(A_1), (A_4)$ we have

$$\phi(y_t, y_t) = \phi(y_t, ty_+(1 - t)\mu) \leq t\phi(y_t, y) + (1 - t)\phi(y_t, \mu) \leq t\phi(y_t, y),$$

which gives $\phi(y_t, y) \geq 0$. From (A_3) we have $\phi(\mu, y) \geq 0$, and hence $\mu \in EP(\phi)$. By Lemma 2.5 and 2.6, we have $\mu \in \bigcap_{i=1}^{N} F(T_i) \bigcap F(f_i)$. Therefore $\omega_w(x_n) \subset \mathfrak{F}$.

(V) We show that $\{x_n\}$ and $\{w_n\}$ converge weakly to an element of $\mathfrak{F}$.
To verify that the aberration is valid, it is sufficient to show that $\omega_w(x_n)$ is a single point set. Let $\mu_1, \mu_2 \in \omega_w(x_n)$, and $\{x_{n_k}\}, \{x_{n_m}\} \subset \{x_n\}$, such that $x_{n_k} \to \mu_1, x_{n_m} \to \mu_2$. Since $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in \mathfrak{F}$, and $\mu_1, \mu_2 \in \mathfrak{F}$, we have $\lim_{n\to\infty} \|x_n - \mu_1\|$ and $\lim_{n\to\infty} \|x_n - \mu_2\|$. Now let $\mu_1 \neq \mu_2$, then by Opial's property,

$$
\begin{aligned}
\lim_{n\to\infty} \|x_n - \mu_1\| &= \lim_{k\to\infty} \|x_{n_k} - \mu_1\| \\
&< \lim_{k\to\infty} \|x_{n_k} - \mu_2\| \\
&= \lim_{n\to\infty} \|x_n - \mu_2\| \\
&= \lim_{m\to\infty} \|x_{n_m} - \mu_2\| \\
&< \lim_{m\to\infty} \|x_{n_m} - \mu_1\| \\
&= \lim_{n\to\infty} \|x_n - \mu_1\|,
\end{aligned}
$$

which is a contradiction. Therefore $\mu_1 = \mu_2$, This show that $\omega_w(x_n)$ is a single point set. This complete the proof of Theorem 3.1. □

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