Journal of Statistical and Econometric Methods, vol.8, no.1, 2019, 97-133
ISSN: 1792-6602 (print), 1792-6939 (online)
Scienpress Ltd, 2019

Conditional Dependence Modelling with Regular Vine Copulas

Cyprian Omari[1], Peter Mwita[2] and Anthony Waititu[3]

Abstract

Modelling sophisticated high-dimensional dependence structures for financial assets in a portfolio framework require flexible dependence models. In this paper, a regular vine-copula based model is employed to analyze financial dependencies and co-movements of a six-dimensional portfolio of currency exchange rates starting from January 2001 to April 2018. The regular-vine copula based model employs partial correlations to construct the regular vine structure and offer superior flexibility in the selection of the distributions to model financial dependence structure. The model also captures the asymmetry between multivariate variables using bivariate copulas with flexible tail dependence. Empirical evidence suggests that co-movements in currency markets are most likely to experience a crash and boom together thus, concluding that currency markets are integrated due to the nature of the global financial systems. The C-Vine copula specification is favoured over the other

[1] Department of Statistics and Actuarial Science, Dedan Kimathi University of Technology, Kenya. E-mail: cyomari@dkut.ac.ke
[2] Department of Mathematics and Statistics, Machakos University, Kenya. E-mail: petermwita@mks.ac.ke
[3] Department of Statistics and Actuarial Sciences, Jomo Kenyatta University of Agriculture and Technology, Kenya. E-mail: agwaititu@yahoo.com

Article Info: *Received* : September 25, 2018. *Revised* : October 23, 2018.
Published online : January 1, 2019

copula specifications in modeling the dependence dynamics between currency exchange rates.

Keywords: Copula; regular vines; C-Vine, D-Vine; currency exchange rates; tail dependence; pair-copula constructions.

1 Introduction

Many statistical problems and in particular econometrics applications require modelling sophisticated dependence structures between multivariate variables. Conventionally, modelling dependence between multivariate variables has mainly focused on using linear correlation as the standard measure of dependence. For this purpose multivariate elliptical distributions such as Gaussian, and Student-t distributions are widely used in modelling the distributions of multivariate variables. However, empirical evidence suggest that most univariate financial variables are non-normal and usually exhibit stylized characteristics such as volatility clustering, asymmetry, excess kurtosis and heavy tailed distributions. Therefore, using elliptical distributions especially multivariate Gaussian distribution is limited since it does not account for asymmetry and tail dependencies usually present between different pairs of financial variables.

Following the 2008 global financial crisis (GFC), regulators, practitioners, academic researchers and media acknowledged that the use of unreliable models for dependence as one of the main causes of the crisis [1]. In the aftermath of the financial crisis, modelling the dependence structure between financial securities has been a hot topic of research in econometrics, finance and statistics. This has motivated the application copula-based approaches in modelling financial assets, such as portfolio asset returns and currency returns. Copulas are functions that describe the dependence structure between random variables. The main result related to the copula function is the Sklar theorem [2], which states that every multivariate distribution can be decomposed into its marginal distribution and a copula which gives the dependence structure. Patton [3] extended and proved the validity of Sklar's theorem in the conditional case, thus extending the applications of copula in finance and econometrics.

For bivariate copulas, there exists a large class of copula families with salient statistical properties that are comprehensively explored in [4],[5]. However, the number of satisfactory higher-dimensional parametric copulas are still scarce and some cannot account for features like heavy tails and asymmetry [6]. The ever-increasing demand for modelling high-dimensional dependence using hierarchical copula-based structures has motivated the innovation of more sophisticated structures.

Vine copulas, or pair-copula constructions, have emerged as one of the most promising tools for building the dependence structure between high-dimensional financial assets [7]. Vine copulas decompose complex high-dimensional dependence into unconditional and conditional bivariate copulas often called pair-copulas. Since the parameters of each pair-copula can be estimated in a different way, vine copulas permit each pair to have a different structure and strength of dependence. Hence, vine copula models results in flexible multivariate copula models which are often superior to other multivariate copula models [8], [9]. In particular, the vine copulas have become popular for modelling dependence between financial assets due to their simplicity, flexibility in modeling combinations of tail dependencies and the possibility of sequentially estimating parameters. The class of vine copula is generally broad and comprises of a large number of probable pair-copula constructions. However, two particular classes of regular vines namely canonical-Vine (C-Vine) and a drawable vine (D-Vine) are the most commonly utilized in many applications [33]. Graphically, C-Vine follows a star-like structure with a root node in each tree while D-Vine follows a path structure with the first tree having nodes with degree two or less. This implies that C-Vines are very practical for multivariate data where the significance of the variables can be ordered. For more information on the D-Vine and C-Vine see [11].

In the last decade, vine copulas and in particular regular vines have gained acceptance in many fields of research and are used in modelling the dependence structure of multivariate random variables. The list of references is increasingly growing in statistical modelling problems and financial applications ever since [7] inaugurated their inferential insights that motivated applications of the vine copula in diverse fields. These vine copula models have been applied to problems in hydrology ([12], [13], [14], [15]), biology ([16]; [17]), sociology ([18]), econometrics ([19], [20], [36]) and finance ([23], [24]), [25], [26] and [27] only to list a few recent articles. For an extensive survey that review vine copulas and their applications in finance see [21].

In this paper, we apply the regular vine-copula based approach to model the multivariate dependence dynamics among the six currency exchange rates namely; the British Pound (GBP), European Euro (EUR), Japanese Yen (JPY), Swiss Franc (CHF), Canadian Dollar (CAD) and Australian Dollar (AUD) all against the US Dollar (USD) for the period starting from January 2, 2001, to April 20, 2018. The pair copula construction approach with a regular vine structure specification is computed following the sequential selection algorithm [36].

The remainder of the paper is structured as follows: Section 2 introduces bivariate dependence measures of association and measures of tail-dependence that are commonly used in copula applications. Section 3 describes the basic principles of copula modeling and commonly used bivariate copulas as well as selected two parameter copulas. The regular vine copulas as well as the regular vine copula specification and the sequential selection procedure are discussed in Section 4. Section 5 reports empirical results and comparative performance between different R-Vine copula specifications. Finally, Section 6 gives the conclusion.

2 Bivariate measures of dependence

This section introduces measures of bivariate dependence that are commonly used in copula applications. The focus is on measures association and tail-dependence that have intuitive interpretations and are extremely appropriate in many statistical applications.

2.1 Measures of Correlation

Pearson's ρ. Pearson's correlation coefficient $\rho \in [-1, 1]$ is the most commonly used measure of dependence. Let $(X, Y)^T$ be vector of random variables with nonzero finite variances. The linear correlation coefficient between X and

Y is defined as

$$\rho(X,Y) = \frac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)}\sqrt{\mathrm{Var}(Y)}} \tag{1}$$

Kendall's τ. Let (X_1, Y_1) and (X_2, Y_2) be two independent pairs of random variables with a joint distribution F and marginal distributions F_X and F_Y. Then Kendall's τ is defined by

$$\begin{aligned}\rho_\tau(X,Y) &= P\left\{(X_1 - X_2)(Y_1 - Y_2) > 0\right\} - P\left\{(X_1 - X_2)(Y_1 - Y_2) < 0\right\} \\ &= P(X_1 < X_2, Y_1 > Y_2) - P(X_1 > X_2, Y_1 < Y_2), \end{aligned} \tag{2}$$

Spearman's ρ. Let (X_1, Y_1), (X_2, Y_2) and (X_3, Y_3) be three independent pairs of random variables with a common joint distribution F and marginal distributions F_X and F_Y. Then Spearman's ρ is defined by

$$\rho_S(X,Y) = 3\left(P\left\{(X_1 - X_2)(Y_1 - Y_3) > 0\right\} - P\left\{(X_1 - X_2)(Y_1 - Y_3) < 0\right\}\right), \tag{3}$$

2.2 Measures of Tail-dependence

The bivariate tail-dependence measures describe the dependence levels between extremal events in the upper, lower or both quadrant tails of a bivariate distribution. The bivariate tail dependence has been studied extensively in literature, see [4]. The extremal tail dependence of a bivariate distribution can be illustrated by the coefficients of tail dependence parameters of its copula. Tail dependence measures the joint probability of extreme movements that occur in the left (lower) quadrant tail or right (upper) quadrant tail or both tails of a 2-dimensional distribution. The coefficients of tail dependence are limits (if they exist) that are defined as follow:

$$\lambda_U = \lim_{u \to 1} P\left(Y \geq F_Y^{-1}(u) | X \geq F_X^{-1}(u)\right) = \lim_{u \to 1} \frac{1 - 2u + C(u,u)}{1-u} \tag{4}$$

$$\lambda_L = \lim_{u \to 0} P\left(Y \leq F_Y^{-1}(u) | X \leq F_X^{-1}(u)\right) = \lim_{u \to 0} \frac{C(u,u)}{u}. \tag{5}$$

Table 1: The coefficients of tail dependence for different copula families

Copula	**Lower tail-dependence**	**Upper tail-dependence**
Gaussian	–	–
Student's t	$2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$	$2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{\frac{1-\rho}{1+\rho}}\right)$
Clayton	$2^{-1/\theta}$	–
Gumbel	–	$2-2^{1/\theta}$
Frank	–	–
Joe	–	$2-2^{1/\delta}$
BB1	$2^{-1/(\delta\theta)}$	$2-2^{1/\delta}$
BB6	–	$2-2^{\frac{1}{\theta\delta}}$
BB7	$2^{-1/\delta}$	$2-2^{1/\theta}$
BB8	–	$2-2^{1/\theta}$, if $\delta=1$, 0 otherwise

It is important to note that, when λ_U exists and $\lambda_U \in (0,1]$, then copula C exhibit upper tail dependence coefficient, and no upper tail dependence coefficient if $\lambda_U = 0$. Also, if λ_L exists and $\lambda_L \in (0,1]$, then copula C exhibit lower tail dependence coefficient, and no lower tail dependence coefficient if $\lambda_L = 0$. The tail dependence depends only upon the underlying copula, not the marginal distributions. Table 1 illustrates the lower (upper) tail dependence coefficients for bivariate copula families and selected two parameter copulas.

3 Copulas

Copulas describe the intrinsic dependence formation between random variables. By Sklar's theorem, every multivariate joint distribution function can be decomposed into its marginal distribution functions and a copula function that captures the complete dependence structure between underlying variables. Let $X = (x_1, \ldots, x_n)^T$ be a vector of random variables with joint distribution functions $F(x_1, \ldots, x_n)$ and continuous marginal distributions $F_1(x_1), \ldots, F_n(x_n)$, there exists a unique copula C such that:

$$F(x_1, \ldots, x_n) = C\left(F_1(x_1), \ldots, F_n(x_n)\right), \qquad x_1, \ldots, x_n \in \mathbb{R}. \tag{6}$$

Conversely, if all the marginal distribution functions, $F_i(x_i)$ for $i = 1, \ldots, n$

are continuous, then the copula C is unique and is expressed as:

$$C(u_1, \ldots, u_n) = F\left(F_1^{-1}(u_1), \ldots, F_n^{-1}(u_n)\right), \qquad u_1, \ldots, u_n \in [0,1], \tag{7}$$

where $u_i = F_i(x_i)$ and $F_i^{-1}(u_i)$ are inverse distribution functions of the margins.

In addition, the copula density, $c(u_1, \ldots, u_n)$ is given by

$$c(u_1, \ldots, u_n) = \frac{\partial^n C(u_1, \ldots, u_n)}{\partial u_1, \ldots, \partial u_n}, \tag{8}$$

Therefore, as a result of Eqn.(6), the density of $X = (x_1, \ldots, x_n)^T$, satisfies the following result:

$$f(x_1, \cdots, x_n) = c(u_1, \ldots, u_n) \prod_{i=1}^{n} f_i(x_i) \tag{9}$$

where $f_i(x_i)$ is the marginal density of x_i, $i = 1, \ldots, n$.

The result in (9) is important in copula modelling since it allows defining a multivariate density as the product of marginal densities and a copula function which captures the dependence between the random variables.

3.1 Bivariate Copulas

In this section, the parametric form of well-known bivariate copula families are introduced. The bivariate copulas, i.e., two-dimensional copulas are the most commonly used in modelling the dependence between random variables. For a comprehensive list of copula families (see [4], [28], [5]). A few bivariate copulas are selected from among the elliptical, Archimedean and two parameter copula families that support the most important dependence characteristics: independence, positive and negative dependence, lower and upper tail-dependence, tail independence, symmetry and asymmetry.

Independence copula. The independence copula describes two independent random variables U_1 and U_2. Its copula density function $c(u_1, u_2) = 1$ is constant for $u_1 \in [0,1]$, $u_2 \in [0,1]$. The independence copula has no upper or lower tail-dependence.

Gaussian copula. The Gaussian (or normal) copula is symmetric, has no tail-dependence, and is defined as:

$$C_N(u_1, u_2) = \Phi_\rho\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2)\right),$$

The density function is given by

$$c(u_1, u_2; \rho) = \frac{1}{\sqrt{1-\rho^2}} \exp\left(-\frac{\rho^2(x_1^2 + x_2^2) - 2\rho x_1 x_2}{2(1-\rho^2)}\right),$$

where $x_1 = \Phi^{-1}(u_1)$ and $x_2 = \Phi^{-1}(u_2)$ are the inverse cumulative distribution function of a standard normal distribution.

Student-t copula. The Student-t copula is symmetric, has both upper and lower tail-dependence, and is given by

$$C_t(u_1, u_2; \rho, \nu) = t_{\nu,\rho}\left(t_\nu^{-1}(u_1), t_\nu^{-1}(u_2)\right)$$

Its density function is

$$c(u_1, u_2; \rho, \nu) = \frac{1}{2\pi\sqrt{1-\rho^2}} \frac{1}{dt(x_1, \nu) dt(x_2, \nu)} \left\{1 + \frac{x_1^2 + x_2^2 - 2\rho x_1 x_2}{\nu(1-\rho^2)}\right\}^{-\frac{\nu+2}{2}},$$

where

$$dt\left(x_i, \nu\right) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{\pi\nu}} \left(1 + \frac{x_i^2}{\nu}\right)^{-\frac{\nu+1}{2}}, \quad i = 1, 2$$

is the density of the univariate t-distribution with ν degrees of freedom and $\Gamma(\cdot)$ is the gamma function, $x_1 = t_\nu^{-1}(u_1)$, $x_2 = t_\nu^{-1}(u_2)$ and $t_\nu^{-1}(\cdot)$ is the quantile function of the univariate standard t-distribution with ν degrees of freedom. The Student-t copula has two parameters ν the degrees of freedom and $\rho \in (-1, 1)$ the coefficient of correlation.

Clayton copula. The Clayton copula is asymmetric, exhibits lower tail-dependence, but no upper tail-dependence, and its given by

$$C_C(u_1, u_2) = \max\left\{\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}}, 0\right\} \tag{10}$$

Its density function is

$$c(u_1, u_2; \theta) = (1+\theta)\left(u_1 u_2\right)^{-1-\theta}\left(u_1^{-\theta} + u_2^{-\theta} - 1\right)^{-\frac{1}{\theta}-2}$$

The Clayton copula has a single parameter $\theta \geq 0$, and can only characterize negative monotone dependence. For $\theta = 1$ the Clayton copula reduces to the independence copula; for $\theta > 0$, the Clayton copula exhibit negative dependence; and as $\theta \to \infty$, the Clayton displays perfect monotone dependence.

Gumbel copula. The Gumbel copula is asymmetric, contains no lower tail-dependence, but exhibit upper tail-dependence and is defined as:

$$C_G(u_1, u_2) = \exp\left[-\left((-\ln u_1)^\theta + (-\ln u_2)^\theta\right)^{\frac{1}{\theta}}\right]. \tag{11}$$

Its density function is

$$\begin{aligned} c(u_1, u_2) = {} & C(u_1, u_2)(u_1 u_2)^{-1}\left((-\ln u_1)^\theta + (-\ln u_2)^\theta\right)^{-2+\frac{2}{\theta}}(\ln u_1 \ln u_2)^{\theta-1} \\ & \times\left(1 + (\theta - 1)\left((-\ln u_1)^\theta + (-\ln u_2)^\theta\right)^{-\frac{1}{\theta}}\right), \end{aligned}$$

The Gumbel copula has a single parameter $\theta \geq 1$, and can only characterize positive monotone dependence. When $\theta = 1$ the Gumbel copula simplifies to the independence copula; for $\theta > 0$, the Gumbel copula exhibit positive dependence; and as $\theta \to \infty$, the Gumbel displays perfect monotone dependence.

Frank copula. The Frank copula is symmetric, has no tail-dependence, and is defined as:

$$C_F(u_1, u_2) = -\frac{1}{\theta}\ln\left(1 + \frac{(\exp(-\theta u_1) - 1)(\exp(-\theta u_2) - 1)}{\exp(-\theta) - 1}\right) \tag{12}$$

Its density function is

$$c(u_1, u_2) = \theta(1 - e^{-\theta})e^{-\theta(u_1+u_2)}\left[\left(1 - e^{-\theta}\right) - 1\left(1 - e^{-\theta u_1}\right)\left(1 - e^{-\theta u_2}\right)\right]^{-2},$$

The Frank copula has a single parameter $\theta \in (0, \infty)$. For $\theta = 0$, the Frank copula simplifies to the independence copula; and for $\theta \to \infty$, the Frank copula achieve maximal dependence.

Joe copula. The Joe copula is asymmetric, contains no lower tail-dependence, but exhibit upper tail-dependence and is defined as:

$$C(u_1, u_2) = 1 - \left((1 - u_1)^\theta + (1 - u_2)^\theta - (1 - u_1)^\theta(1 - u_2)^\theta\right)^{\frac{1}{\theta}}. \tag{13}$$

The copula density is

$$c(u_1, u_2) = \left((1-u_1)^\theta + (1-u_2)^\theta - (1-u_1)^\theta(1-u_2)^\theta\right)^{\frac{1}{\theta}-2} \cdot (1-u_1)^{\theta-1}(1-u_2)^{\theta-1} \\ \cdot \left[\theta - 1 + (1-u_1)^\theta + (1-u_2)^\theta - (1-u_1)^\theta(1-u_2)^\theta\right].$$

The Joe copula has a single parameter $\theta \geq 1$, and can only characterize positive monotone dependence. For $\theta = 1$ the Joe copula simplifies to the independence copula; for $\theta > 0$, the Joe copula exhibit positive dependence; and as $\theta \to \infty$, the Joe displays perfect monotone dependence.

BB1 (Clayton-Gumbel) copula. The BB1 (Clayton-Gumbel) copula is defined by

$$\begin{aligned} C(u, v; \theta, \delta) &= \left\{1 + [(u^{-\theta} - 1)^\delta + (v^{-\theta} - 1)^\delta]^{\frac{1}{\delta}}\right\}^{-\frac{1}{\theta}} \\ &= \eta(\eta^{-1}(u) + \eta^{-1}(v)), \quad \theta > 0, \delta \geq 1, \end{aligned} \tag{14}$$

where $\eta(s) = \eta_{\theta,\delta}(s) = \left(1 + s^{\frac{1}{\delta}}\right)^{-\frac{1}{\theta}}$.

BB6 (Joe-Gumbel) copula. The BB6 (Joe-Gumbel) copula is given by

$$C(u_1, u_2; \theta, \delta) = 1 - \left(1 - \exp\left\{-\left[\left(-\log\left(1-u_1^\theta\right)\right)^\delta + \left(-\log\left(1-u_2^\theta\right)\right)^\delta\right]^{\frac{1}{\delta}}\right\}\right)^{\frac{1}{\theta}}, \tag{15}$$

with $\theta \in [1, \infty)$ and $\delta \in [1, \infty)$

BB7 (Joe-Clayton) copula. The BB7 (Joe-Clayton) copula has the generator $\phi(s; \theta, \delta) = [1 - (1-s)^\theta]^{-\delta} - 1$ and it is given by

$$\begin{aligned} C_{JC}(u, v; \theta, \delta) &= 1 - \left(1 - \left[\left(1 - (1-u)^\theta\right)^{-\delta} + \left[1 - (1-v)^\theta\right]^{-\delta} - 1\right]^{-\frac{1}{\delta}}\right)^{\frac{1}{\theta}}, \\ &= \eta\left(\eta^{-1}(u) + \eta^{-1}(v)\right), \qquad \theta \geq 1, \delta > 0. \end{aligned} \tag{16}$$

where $\eta(s) = \eta_{\theta,\delta}(s) = 1 - \left[1 - (1+s)^{-\frac{1}{\delta}}\right]^{\frac{1}{\theta}}$,

BB8 (Frank-Joe) copula. The BB8 (Frank-Joe) copula is given by

$$C(u_1, u_2; \theta, \delta) = \frac{1}{\delta}\left(1 - \left[1 - \frac{1}{1-(1-\delta)^\theta}\left(1 - (1-\delta u_1)^\theta\right)\left(1 - (1-\delta u_2)^\theta\right)\right]^{\frac{1}{\theta}}\right), \tag{17}$$

with $\theta \in [1, \infty)$ and $\delta \in (0, 1]$.

3.2 Pair-copula constructions

A pair copula construction (PCC) is a multivariate copula based on an idea initially proposed by Joe [4] and further developed by Bedford and Cooke ([31], [32]), Kurowicka and Cooke [33]. The fundamental idea of PCC is the construction of higher-dimensional copulas through bivariate copulas which constitute a flexible class of dependence models. To illustrate the concept of PCC it is necessary to introduce the pair-copula decomposition of a multivariate density function. Let $X_1, \ldots, X_n$ be a vector of random variables with joint multivariate density function $f(x_1, \ldots, x_n)$. The multivariate density function can be factorized into a series of univariate (conditional) densities as follows:

$$f(x_1, \ldots, x_n) = f(x_1) \cdot f(x_2|x_1) \cdot f(x_3|x_1, x_2) \ldots f(x_n|x_1, \ldots, x_{n-1}) \tag{18}$$

where $f(\cdot|\cdot)$ denotes the conditional density function.

Next, we demonstrate pair-copula constructions based on a three dimensional vector of random variables. For three random variables, X_1, X_2 and X_3, the joint three-dimensional density can be decomposed as follows

$$f(x_1, x_2, x_3) = f(x_1) \cdot f_{2|1}(x_2|x_1) \cdot f_{3|1,2}(x_3|x_1, x_2) \tag{19}$$

The conditional density $f(x_2|x_1)$ in (19) can be expressed as:

$$f(x_2|x_1) = \frac{f(x_1, x_2)}{f_1(x_1)} = c_{1,2}\left(F_1(x_1), F_2(x_2)\right) f_2(x_2) \tag{20}$$

Note that $f(x_1, x_2) = c_{12}\left(F_1(x_1), F_2(x_2)\right) f_1(x_1) f_2(x_2)$ following Sklar's theorem, where $c_{1,2}(\cdot, \cdot)$ denote the appropriate pair-copula density for the pair of transformed variables $F_1(x_1)$ and $F_2(x_2)$.

The second conditional density $f(x_3|x_1, x_2)$ in (19) can also be expressed as:

$$f(x_3|x_1, x_2) = \frac{f(x_1, x_2, x_3)}{f(x_1, x_2)} = \frac{f_{23|1}(x_2, x_3|x_1)}{f_{2|1}(x_2|x_1)} \tag{21}$$

Following Sklar's theorem, the conditional density $f_{23|1}(x_2, x_3|x_1)$ in (21), is expressed as:

$$f_{23|1}(x_2, x_3|x_1) = c_{23|1}\left(F(x_2|x_1), F(x_3|x_1)\right) \cdot f_{2|1}(x_2|x_1) \cdot f_{3|1}(x_3|x_1), \tag{22}$$

where $c_{23|1}$ is a conditional copula.

Thus,

$$\begin{aligned} f(x_3|x_1,x_2) &= c_{2,3|1}\left(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1)\right) \cdot f_3(x_3|x_1) \\ &= c_{23|1}\left(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1)\right) \cdot c_{13}\left(F_1(x_1), F_3(x_3)\right) \cdot f_x(x_3) \end{aligned} \tag{23}$$

Therefore, substituting the terms (20) and (23) in (19) yields the full decomposition for a three dimensional density which can be expressed as:

$$\begin{aligned} f(x_1,x_2,x_3) = \prod_{i=1}^{3} & f_i(x_i) \cdot c_{12}\left(F_1(x_1), F_2(x_2)\right) \cdot c_{13}\left(F_1(x_1), F_3(x_3)\right) \\ & \cdot c_{23|1}\left(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1)\right) \end{aligned} \tag{24}$$

Thus, a pair-copula construction for three dimensional density, $f(x_1,x_2,x_3)$, is expressed as the product of the marginal densities and three pair-copulas: two unconditional (c_{12} and c_{13}) and one conditional ($c_{23|1}$), hence the expression pair-copula decomposition. However, this PCC decomposition is not unique anymore since one could use x_2 instead of x_1 for the conditional variable in (20). The choices of different conditional variables lead to three different pair-copula constructions.

In general, the conditional densities in (18) may be decomposed into the appropriate pair-copula times a conditional marginal density, using the general formula

$$f(x_i|\nu) = c_{x_i x_j|\upsilon_{-j}}\ F(x_i|\upsilon_{-j}), F(x_j|\upsilon_{-j}) \cdot f(x_i|\upsilon_{-j}), \tag{25}$$

for $i,j = 1,\ldots,n$ and where υ denote an arbitrary set of $x_1,\ldots,x_n$ with x_j in it but not x_i. Then υ_{-j} denotes the n-dimensional vector υ excluding the jth component.

Following [7] the pair-copula construction also involves different marginal conditional distributions functions of the form $F(x|\upsilon)$. Using the notations from Eqn.(25), for every j, Joe [4] showed that

$$F(x_i|\upsilon) = \frac{\partial C_{x_i,x_j|\upsilon_{-j}}\left(F(x_i|\upsilon_{-j}), F(x_j|\upsilon_{-j})\right)}{\partial F(x_j|\upsilon_{-j})}, \tag{26}$$

where $C_{x_i x_j|\upsilon_{-j}}$ is the bivariate copula distribution function. An iterative application of Eqn.(26) allows the representation of all conditional distribution functions as nested partial differentiated copulas and marginal distributions.

In conclusion, a multivariate density can be decomposed into a product of several different conditional probability distributions and pair copulas under appropriate regulatory conditions. The decomposition is also iterative in nature and is not unique, therefore in a given specific factorization, there still exists a large number of possible pair copula constrictions. For example, its possible to construct 240 different pair-copula constructions for a five dimensional distribution function [7].

3.3 Regular Vines

Bedford and Cooke [31] introduced a graphical structure, called regular vines that classifies all possible pair-copula decompositions. Regular vines are flexible graphical structures for modelling the multi-dimensional dependence using a cascade of conditional bivariate pair-copulas as building blocks. This concept was developed further and extended by Kurowicka and Cooke [31], Kurowicka [33]. Aas [7] developed their statistical inference. The following definitions are obtained from Kurowicka [33].

Definition 3.1. Tree *Let $N = 1, 2, \ldots, n$ be nodes and E be edges, $T = (N, E)$ is a tree that is connected where E is a subset of unordered pairs of N with no cycle; that is, there does not exist a sequence $a_1, \ldots, a_k$ $(k > 2)$ of elements of N such that*

$$a_1, a_2 \in E, \ldots, a_{k-1}, a_k \in E, a_k, a_1 \in E.$$

The degree of node $a_i \in N$ is the number $(a_j \in N | a_i, a_j \in E)$; that is, the number of edges attached to a_i.

The trees that are used to describe the dependence structures in high-dimensional distributions are called dependence trees.

Definition 3.2. (Regular vine)
$\mathcal{V}$ is a regular vine on n elements if the following conditions hold

1. $\mathcal{V} = (T_1, \ldots, T_{n-1})$

2. *$T_1 = (N_1, E_1)$ is a connected tree with nodes $N_1 = \{1, \ldots, n\}$, and edges E_1; for $i = 2, \ldots, n-1$, T_i is a connected tree with nodes $N_i = E_{i-1}$.*

3. *For $i = 2, \ldots, n-1$, if $\{a, b\}$ are nodes of T_i connected by an edge, where $a = \{a_1, a_2\}$ and $b = \{b_1, b_2\}$, then exactly one of the a_i equals one of the b_i (proximity condition).*

Proximity condition means that if there is an edge in tree T_i (where $i \geq 2$) connecting two nodes then those two nodes are edges in tree T and they share a common node (note that nodes in one tree are edges in the next tree). A regular vine on n variable is a vine in which two edges in tree j are connected by an edge in tree $j+1$ only if these edges share a common node, for $j = 1, \ldots, n-2$. There are $n(n-1)/2$ edges on a regular vine on n variables.

The class of regular vine is generally broad and comprises of a large number of probable pair-copula constructions. However, two particular classes of regular vines namely canonical-Vine (C-Vine) and a drawable vine (D-Vine) are the most commonly utilized in many applications (Kurowicka and Cooke [33]).

Definition 3.3. ***(D-Vine, C-Vine).*** *An R-Vine is called a*

(i) ***D-Vine*** *if each node in T_1 has a degree of at most 2.*

(ii) ***C-Vine*** *if each tree T_i has a unique node of degree $n-i$. The node with maximal degree in T_1 is the root node.*

Graphically, C-Vine follows a star-like structure with a root node in each tree while D-Vine follows a path structure with the first tree having nodes with degree two or less. This implies that C-Vines are very practical for multivariate data where the significance of the variables can be ordered. The C-Vine and D-Vine structures are somehow the two extreme contrary cases of an R-Vine. A node in an R-Vine can be of degree 1 to $n-1$ and therefore ranges between the C-Vine and D-Vine limitations. For more detailed information about these special cases see Aas et al.[7].

3.4 Regular vine copula specification

The graphical structure of regular vines is used to specify necessary copulas for the pair-copula constructions. To build an R-Vine copula one must specify the $n-1$ unconditional bivariate copulas between variables indexed by the conditioned sets of the edges in the first tree of the R-Vine. For the second and subsequent trees of the R-Vine one needs to specify the bivariate copulas between variables indexed by the conditioned sets conditional on variables indexed by the conditioning sets of edges of R-Vine. Therefore the R-Vine copula specification is formally defined corresponding to an R-Vine in Bedford and Cooke [33].

Definition 3.4. (Regular vine copula specification)
A regular vine copula specification on n *variables is a multivariate distribution function defined as* $C = (\mathcal{V}, B(\mathcal{V}), \theta(B(\mathcal{V})))$

(i) $\mathcal{V}$ *is a vine tree structure on* n *variables;*

(ii) $B(\mathcal{V}) = \{B_e | i = 1, \ldots, n-1,\ e \in E_i\}$ *is a set of* $n(n-1)/2$ *bivariate copulas; and*

(iii) $\theta(B(\mathcal{V})) = \left\{\theta_{e(a),e(b)|D_e} | e \in E_i,\ i = 1, \ldots, n-1\right\}$ *is the set of parameters corresponding to the copula family in* $B(\mathcal{V})$*.*

Following the definition of regular vine specification, the full specification of a regular vine copula has three components: the vine tree structure $\mathcal{V}$, the pair-copula family set $B(\mathcal{V})$, and the corresponding copula parameters $(B(\mathcal{V}))$ and marginal distribution functions. The statistical inference on regular vines to a given data set involves the implementation of three tasks: (a) selecting the corresponding vine structure with all its trees, (b) choosing a copula family for each of the $n(n-1)/2$ pair-copulas, and (c) estimating the corresponding parameters of each copula.

3.4.1 Tree structure construction

The regular vine tree structure is the dependence structure which connects all bivariate copula together. To determine the appropriate tree structure of the regular vine, the idea is to prioritize strongest dependencies in the first trees, because pair-copulas specified in first tree often have the greatest influence and dependence tends to be strongest in Tree 1 [36]. For precision of the model, the strongest dependencies are typically the most important and vice versa (copula distribution functions for parameters close to independence are similar). This type of modelling has its drawbacks, for example the solution is not necessarily global optimum, because each tree is analysed separately. This stepwise tree-by tree inference is a sequential method. However, it is a computationally fast and effective method comparing to alternatives. More about alternative ways to model with regular vine copulas can be found in [34], where different regular vine copula applications are reviewed and the advantages and disadvantages of each discussed. The Kendall's τ is used to measure the dependence and solving the optimization problem for each tree in order to find the so called maximum spanning tree (a tree that maximizes cumulative pairwise dependencies. After determining the regular vine tree structure, the next task is to fit pair-copulas to all edges of the regular vines (edges represent conditional and unconditional variable pairs).

3.4.2 Copula selection

In order to select an adequate copula for each pair-copula, a variety of bivariate copulas are considered for selection including; Gaussian (G), Student's t (t), Clayton (C), Gumbel (G), Frank (F), Joe (J), BB1 (Clayton-Gumbel), BB6 (Joe-Gumbel), BB7 (Joe-Clayton), BB8 (Frank-Joe), Survival Clayton, Survival Gumbel, Survival Joe, Survival BB1, Survival BB6, Survival BB7 and Survival BB8 copulas for every pair of currency exchange rates. The most appropriate copula model are selected independently for each pair-copula using the AIC selection criteria. This model selection method rewards goodness-of-fit of a model and penalizes increasing the number of parameters. Another possibility is to use Bayesian Information Criterion (BIC) instead of AIC. Both AIC and BIC use maximum likelihood, however, AIC depends on sample size and BIC does not. There is also the question of whether likelihood based model selection methods appropriately take into account tail dependence. The problem with maximum likelihood is that it mostly fits the distribution in the "middle" and its tail has little impact. After choosing the best fitting copula families for the conditional and unconditional variable pairs determined by the edges in regular vine, we can proceed to estimating the parameters of the pair-copulas.

3.4.3 Parameter estimation

The estimation of copula parameters and margin specifications is implemented using a two-step estimation procedure proposed by [29], the inference functions for margins (IFM) method that rely on maximum likelihood estimation (MLE). In the first step, the marginal parameters are estimated and in the second step the copula parameters are estimated. This procedure is commonly used in applications of regular vine copula specification and involves the selection of pair-copula types and estimation of the copula parameters to be done simultaneously [30].

4 Data and empirical results

4.1 Data description

The data set consists of a portfolio of six currency exchange rates covering the period from January 2, 2001 to April 20, 2018, yielding a total of 4511 daily observations (exclusive of public holidays and weekends). The currency exchange rates considered include: British pound (GBP), European Euro (EUR), Japanese Yen (JPY), Swiss Franc (CHF), Canadian Dollar (CAD) and Australian Dollar (AUD) all against the US Dollar (USD). The data were downloaded from (https://www.investing.com) website. All daily currency exchange rates are transformed into logarithm returns using the formula $r_{t,i} = \log(P_{t,i}/P_{t-1,i})$, where $P_{t,i}$ denote price at time t of i-th currency exchange rate. Figure 1 illustrate daily return plots of different currency exchange rates. Each plot illustrate some instances of high volatility clustering alternating with periods of relative tranquility. The volatile behaviour exhibited by returns suggests the presence of volatility clustering and conditional heteroscedasticity in the data.

Table 2 presents the summary statistics of the daily currency returns over the full sample period from January 2, 2001, to April 20, 2018, for the GBP, EUR, JPY, CHF, CAD and AUD. The mean values of all daily currency returns are relatively close to zero and high volatility is evident with significantly high standard deviations for all currency returns. The excess kurtosis values reported suggest that all currency return series distributions are heavy tailed and exhibit leptokurtic behaviour beyond that of the normal distribution. Moreover, the values for skewness imply that currency return series for the Euro, Japanese Yen and Swiss Frank are negatively skewed while other currency return series are positively skewed. Jarque-Bera (JB) test the normality of the unconditional distribution of currency returns and the results reject the normality hypothesis for each currency return series confirming that all the series are non-normal distributed. The Augmented Dickey-Fuller (ADF (k)) test for a unit root against a trend stationary alternative augmented with k lagged difference terms. The (ADF) results reject unit root hypothesis for all return series, which implies that the currency return series are assumed to be stationary, as using logarithm returns amounts to a variance stabilizing transformation. Ljung-Box (LB) portmanteau Q-test assess the null hypothesis of no serial autocorrelations in the squared returns at k lags. LB(Q) statistic values reported for squared return series are significantly high, thus we reject the null hypothesis of no serial autocorrelation up to 20th lag at every level of significance for all the currency return series. Finally, Engle's Lagrange multiplier (LM (k)) test is used for testing the presence of ARCH effects on k lags. The ARCH-LM test rejects the no ARCH effect hypothesis, thus confirming the presence of volatility clustering and conditional heteroscedasticity in currency exchange returns series. The asymmetric conditional heteroscedastic specification may be considered to be more practical in the presence of leverage effect.

4.2 The results for marginal specifications

In order to account for the stylized facts about financial returns, we employ a GARCH-type specifications to model volatility dynamics assuming the innovations term follows a skewed-student-t distribution. Formally, let $r_t = \ln(P_t/P_{t-1})$ the logarithmic return at time t, the E-GARCH specification originally proposed by Nelson [35] is utilized to account for asymmetries in the currency return series. The conditional mean component is given as follows:

$$r_t = \phi_0 + \sum_{i=1}^{m} \phi_i r_{t-i} - \sum_{j=1}^{n} \varphi_j \varepsilon_{t-j} + \varepsilon_t, \tag{27}$$

where ϕ_0 is a constant, ϕ_i and φ_j are the autoregressive (AR) and moving

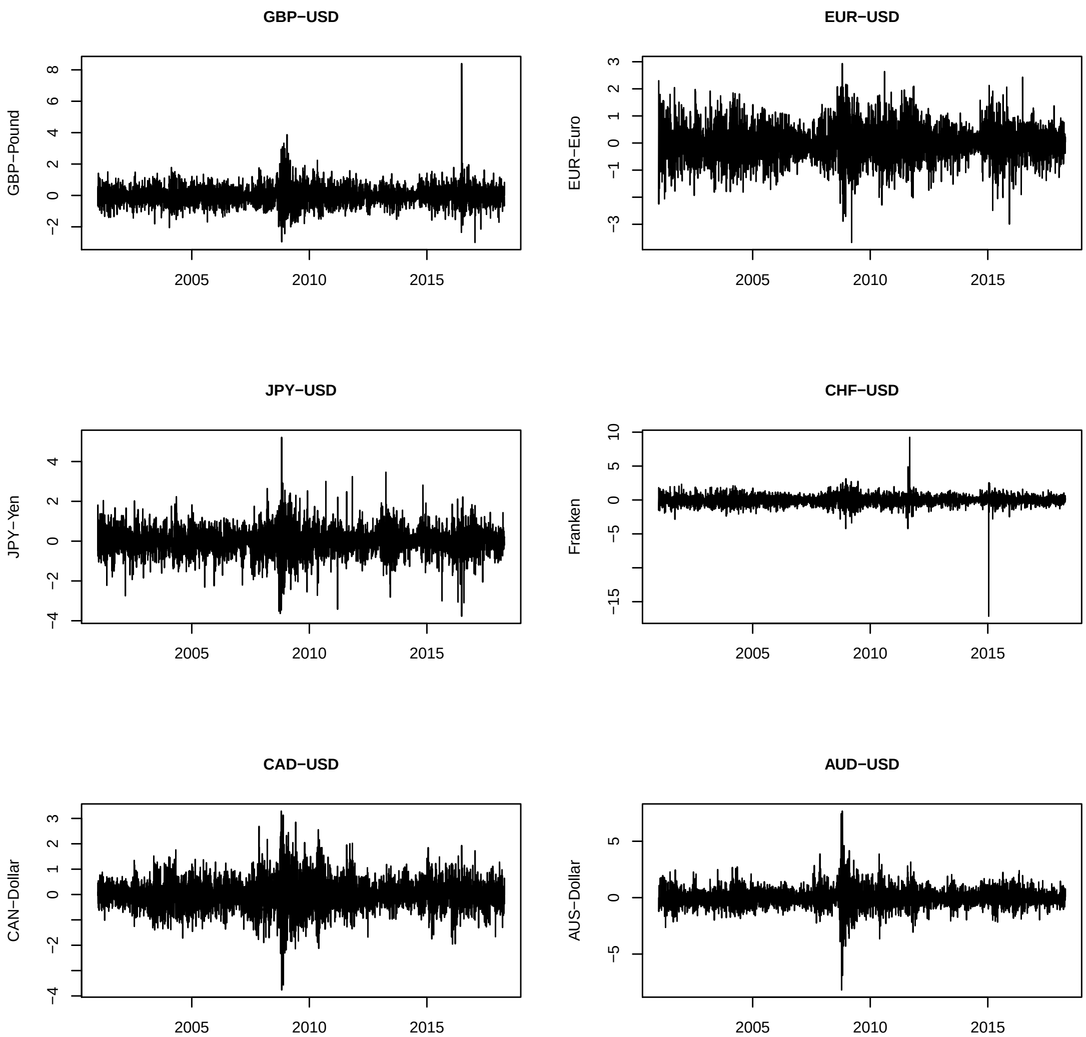

Figure 1: Currency exchange returns between January 2, 2001 and April 20, 2018

average (MA) parameters with m and n lags, respectively. $\varepsilon_t = \sigma_t z_t$, is a stochastic process with z_t as iid sequence assumed to follow a skewed-t distribution with ν degrees of freedom and σ_t is the conditional volatility. The conditional variance σ_t^2 is given by:

$$\sigma_t^2 = \omega + \sum_{i=1}^{p} \alpha_i \epsilon_{t-i}^2 + \sum_{i=1}^{p} \gamma_i \psi\left(\epsilon_{t-i}\right) \epsilon_{t-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^2, \tag{28}$$

where ω is a constant, α_i and β_j are the ARCH and GARCH effect parameters,

Table 2: Summary statistics of the daily currency exchange returns

	GBP	EUR	JPY	CHF	CAD	AUD
Nobs	4511	4511	4511	4511	4511	4511
Minimum	-2.996	$-3,672$	-3.772	-17.145	-3.766	-8.187
Maximum	8.400	2.933	5.216	9.239	3.290	7.674
Mean	0.001	-0.006	-0.001	-0.011	-0.004	-0.007
Stdev	0.585	0.619	0.647	0.719	0.573	0.826
Skewness	0.901	-0.023	-0.083	-2.574	0.121	0.439
Kurtosis	11.703	1.664	3.713	78.421	2.829	11.751
Normality and Stationarity Tests						
JB	26382.44	522.06	2600.24	1161960.36	1518.55	26129.47
ADF (15)	-15.835	-15.719	-16.612	-17.216	-16.685	-15.938
Heteroscedasticity Tests						
LBQ (1)	133.79	71.05	73.08	0.017	236.50	344.54
LBQ (5)	234.98	363.41	387.27	4.781	1421.20	2891.90
LBQ (10)	323.31	646.45	592.25	7.869	2640.70	4880.90
LBQ (20)	464.43	1212.1	1171.40	11.456	4884.5	7137.70
LM (10)	282.76	343.18	465.67	281.76	547.27	421.99
LM (20)	646.04	701.62	809.42	547.63	1093.03	847.87

The critical values of Ljung-Box test and LM test are 18.307 (lag 10), and 31.410 (lag 20) at 5%.

respectively. γ_i captures the leverage effect: $\psi(\epsilon_t) = \psi(z_t) = 1$ in case $z_t < 0$ and 0 if $z_t \geq 0$. The specification for the marginal distribution plays a pivotal role for dependence modelling since they filter any serial autocorrelation, heteroscedasticity and leverage effects from the return series hence yielding appropriate input data for the copula estimation.

The parameter estimates of the fitted marginal specifications given by Equations (27) and (28) for currency exchange returns are estimated by maximum likelihood estimation method. The parameters m, n, p and q can take different combinations of values starting from zero to two lags for brevity purposes. To select the most appropriate univariate model for each of the currency exchange rates series, the Akaike and the Bayesian information criteria were employed. Table 3 reports full sample estimation results of the most appropriate ARMA-EGARCH(1,1) specification model assuming skewed Student's-t distribution for the innovations (with standard errors enclosed in parenthesis) over the entire sample period from January 2, 2001, to April 20, 2018. Based on the results in Table 3, most of the parameter estimates for both the conditional mean and variance equations are confirmed to be statistically significant at 1% significance level. In fact, all the α_1 parameters are significant except for CHF while the β parameters are found to be close to one though most of them are not statistically significant except for JPY at 1%. In addition, most leverage effect parameter γ reported are significant except for the EUR series. Finally, the values of the degrees of freedom of the skewed Student-t distributions ranges from the smallest value of 5.8 to a maximum of 13.6 and all are statistically significant. Thus, the use of heavy-tailed innovations distribution seems to be justified to account for skewness and excess kurtosis in all currency return series. Ljung-Box portmanteau Q-test assessing the null hypothesis of no serial autocorrelations for standardized squared residuals fails to detect any serial correlation. Engle's Lagrange multiplier (LM (k)) test is used for testing the presence of ARCH effects up to 20 lags also fails to detect the presence of ARCH effects. Therefore, we confirm the ARMA-EGARCH(1, 1) specification satisfactorily filters any serial autocorrelation, heteroscedasticity and leverage effects in each currency return series. In order to analyze the dependence structure between currency exchange rates in a better way the standardized residuals are transformed into the unit square normal variates $[0, 1]$ by utilizing the probability integral transform (PIT), a necessary condition to implement copula estimation.

4.3 The results for dependence models

Following the sequential selection procedure by Dissmann et al. [36] described in section 3, we first determine the pairwise dependence dynamics between currency exchange return series using the Kendall's τ rank correlation coefficient to select the optimal R-Vine specification. Figure 2 illustrates the pairwise scatter plots for the resulting copula data on the top-right side of the figure and their corresponding estimated Kendall's τ values on the bottom-left side of the figure for the different exchange rates representing different mag-

Table 3: Parameter estimates for the fitted ARMA-EGARCH(1, 1) specification with skewed Student's-t innovations distribution for the entire sample period starting from January 2, 2001 to April 20, 2018

	GBP	EUR	JPY	CHF	CAD	AUD
Conditional mean parameter estimates						
μ	−0.005	−0.09	0.004	−0.004	−0.004	−0.017
	(0.007)	(0.009)	(0.007)	(0.008)	(0.007)	(0.000)
ϕ				−0.047	−0.030	−0.828
				(0.014)	(0.015)	(0.044)
θ		−0.032	−0.032			0.798
		(0.015)	(0.015)			(0.047)
Conditional volatility parameter estimates						
ω	−0.009	−0.005	−0.017	−0.006	−0.009	−0.007
	(0.002)	(0.001)	(0.005)	(0.001)	(0.002)	(0.002)
α_1	0.013	0.011	−0.018	−0.008	0.009	0.027
	(0.006)	(0.005)	(0.009)	(0.001)	(0.007)	(0.008)
β_1	0.993	0.996	0.982	0.994	0.993	0.991
	(0.002)	(0.000)	(0.005)	(0.000)	(0.001)	(0.001)
γ_1	0.084	0.075	0.143	0.066	0.107	0.117
	(0.015)	(0.000)	(0.016)	(0.007)	(0.011)	(0.013)
λ	9.216	10.133	5.849	6.922	13.593	10.164
	(1.160)	(1.426)	(0.493)	(0.650)	(2.596)	(1.452)
Heteroscedasticity tests						
LBQ(10)	0.005	0.916	0.916	1.000	0.584	0.054
LBQ(20)	0.156	0.890	0.981	1.000	0.195	0.052
LM(10)	0.006	0.043	0.195	0.539	0.349	0.174
LM(20)	0.016	0.180	0.053	0.656	0.419	0.181

nitude and direction of pair wise dependencies. Similar to the unconditional correlation measures, the conditional correlation based on Kendall's τ indicates that some currency returns generally exhibit higher dependencies than others, for example; EUR, CHF and AUD.

Table 4 also shows different pairwise dependencies based on Kendall's τ values reported in the correlation matrix between pairs of uniform-transformed

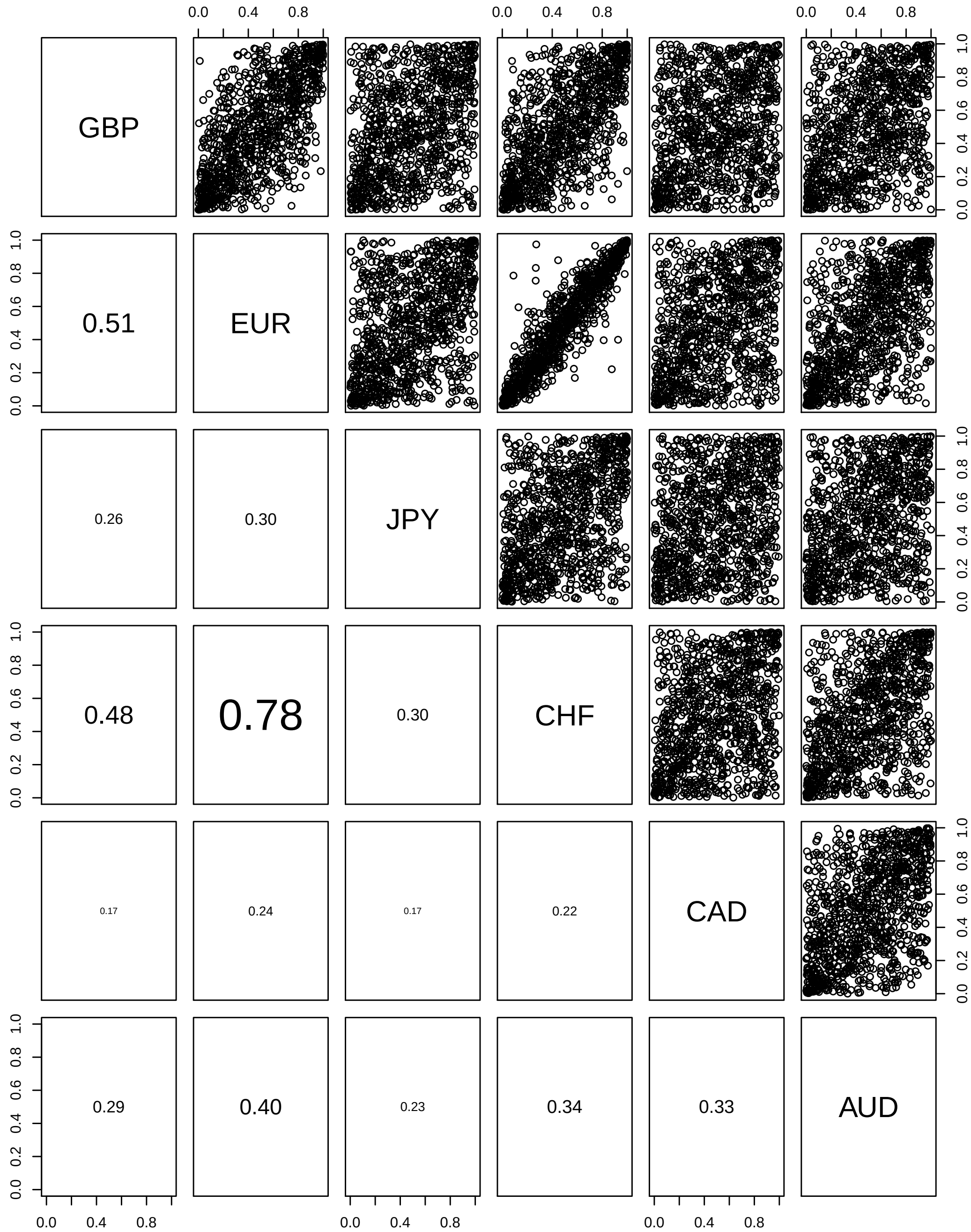

Figure 2: Pairs-plots and Kendall's taus each currency exchange rates; the pairplots (top-right) and the corresponding Kendall's τ vslues (bottom-left).

standardized residuals and the total over each row of the currency exchange rates. The summation of the pairwise correlations over the EUR row gives the highest value while the EUR-CHF pair records the strongest pairwise correlation.

Table 4: The empirical Kendall's τ values and the total over each row of the currency exchange rates.

	GBP	EUR	JPY	CHF	CAD	AUD	**Sum**
GBP	1	0.4568	0.1602	0.3911	0.2733	0.3468	2.6282
EUR	0.4568	1	0.2355	**0.6682**	0.3037	0.3974	**3.0616**
JPY	0.1602	0.2355	1	0.3069	0.0482	0.1320	1.8828
CHF	0.3911	**0.6682**	0.3069	1	0.2296	0.3042	2.9000
CAD	0.2733	0.3037	0.0482	0.2296	1	0.4225	2.2773
AUD	0.3468	0.3974	0.1320	0.3042	0.4225	1	2.6029

The regular vine (R-Vine) tree structure specification is selected by tree-wise selection procedure described in Section 3. The selection algorithm follows a step-wise approach that selects each tree T_i, $i = 1, 2, \ldots, n-1$ as the maximum spanning tree based on the (empirical) absolute Kendall's τ value of the variable pairs as edge weights see [36]. The first and second trees (levels) of the estimated R-Vine specification are illustrated in Figure 3. The letters and numbers reported on the edges in between the nodes represent the bivariate copulas chosen to model the dependence between the currency exchange rates, while the numbers correspond to subsequent Kendall's τ correlation value between the two variables. For example in the EUR and CHF pair, the selected copula is the Student-t, with 0.63 the Kendall's τ value. The tree structure graph shows a significant positive correlation between the currencies.

In order to determine the C-Vine structure, the root node must be selected in every level (tree). Analogous to the approach of fitting an R-Vine structure, the root node of the C-Vine is selected by summing Kendall's τ's values over each row and picking the node that maximizes the sum of absolute pairwise dependencies to this node which is measured by the Kendall's τ coefficient as the root node [37]. For the first (level) tree of the C-Vine, the EUR currency exchange rate is selected as the root node, since it has the maximum absolute Kendall's tau value compared to the other nodes. The root nodes for other (levels) trees of the C-Vine selected as described in [38] for the currency

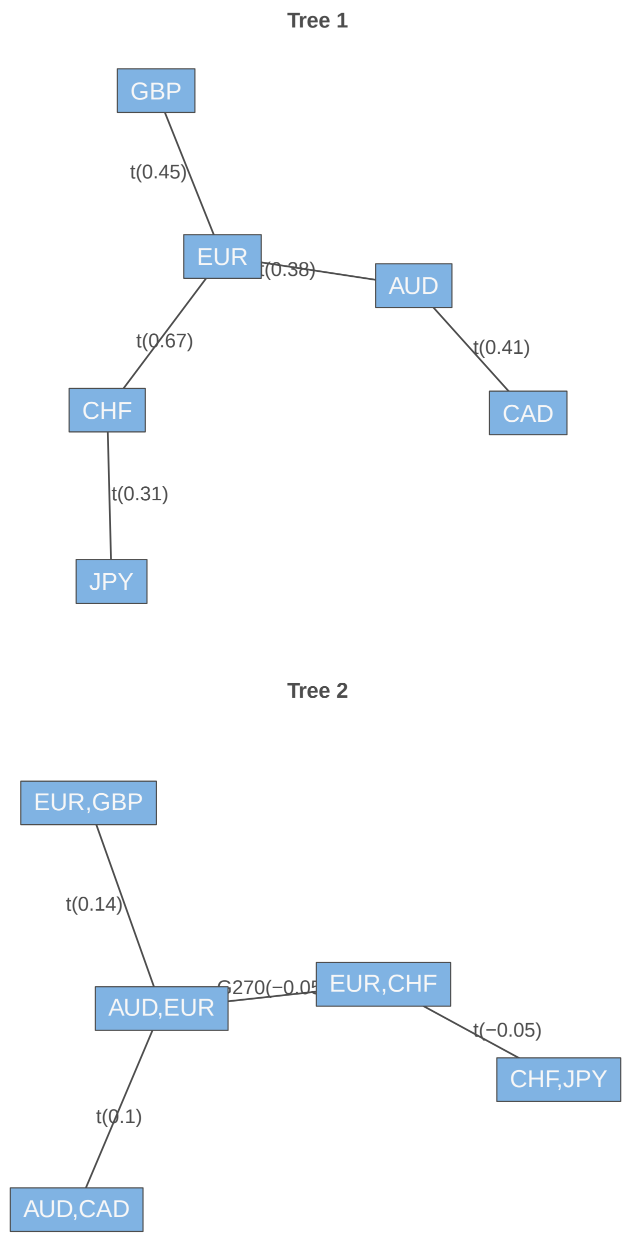

Figure 3: Tree 1 and Tree 2 for the estimated R-Vine specification of exchange rates data.

exchange returns data are presented in Table 5.

The first and second (level) tree structure for both the C-Vine and D-Vine specifications are illustrated in Figure 4 and 5 respectively. For T_1 of the C-

Table 5: Root nodes for the C-Vine structure of the currency exchange rates

Tree	Root-node
Tree 1	*EUR*
Tree 2	*AUD, EUR*
Tree 3	*AUD, JPY; EUR*
Tree 4	*JPY, GBP; AUD, EUR*
Tree 5	*CAD, GBP; JPY, AUD, EUR* or *GBP, CHF; JPY, AUD, EUR*

Vine all the other five currencies exchange rates are linked to the EUR which is the root node at the center of this tree diagram. The dependencies for possible pairs among the currency exchange rates in the first tree are positive and significant. The CHF-EUR pair has the highest absolute value of Kendall's τ representing the strongest correlation between the currencies. Other pair currencies that represent significant correlations are between EUR, GBP, CHF and AUD while JPY and CAD demonstrate lower dependence with other currency exchange rates.

Having selected the appropriate tree structures, the next step is to choose the appropriate pair-copula for each currency exchange pair linked to the R-, C- or D-Vine structures. The bivariate copulas considered in this paper include; Gaussian (G), Student-t (t), Clayton (C), Gumbel (G), Frank (F), Joe, BB1, BB6, BB7, BB8, Survival Clayton, Survival Gumbel, Survival Joe, Survival BB1, Survival BB6, Survival BB7 and Survival BB8 copula. The copula parameters are estimated using maximum likelihood estimation method. Finally, the most appropriate copula specification of each pair-copula is selected using the Akaike and the Bayesian information criteria. The results of the sequential selection procedure provide a tree structure, corresponding pair-copula types and parameter estimates.

Table 6 reports parameter estimates of the regular vine (R-Vine) copula, AIC, BIC, Log-likelihood, Kendall's τ, the upper (λ_U) and lower (λ_L) tail dependence values respectively. The corresponding p-values for the estimated parameters are enclosed in parentheses. The parameter estimates of the conditional and unconditional pair-copulas are observed to be significant at 1% level of significance. The results also demonstrate that all the pair dependencies in the first tree of the regular vine are modelled by Student-t copula, signifying the presence of symmetric tail dependence. The degrees of freedom for the

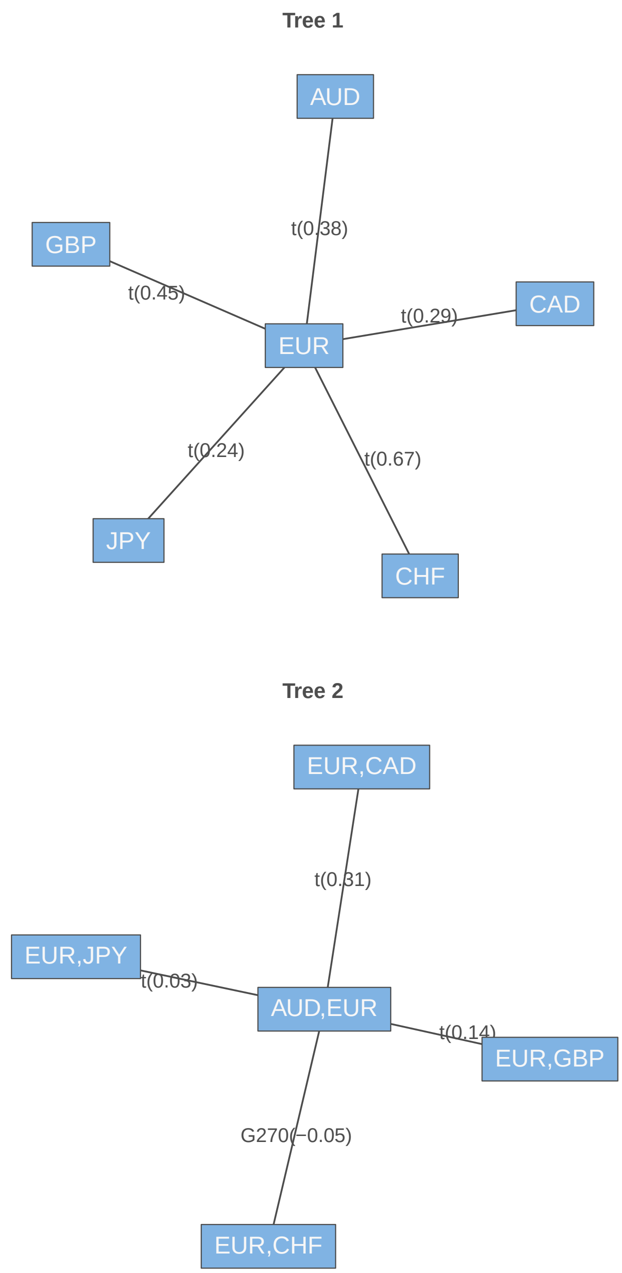

Figure 4: Tree 1 and Tree 2 for the C-Vine specification of the currency exchange rates dataset

Student-t distribution demonstrate presence of heavy-tailed distribution for the six currency exchange returns. Therefore, in general the results suggest that movements in currency exchange rates are inclined in the same direction

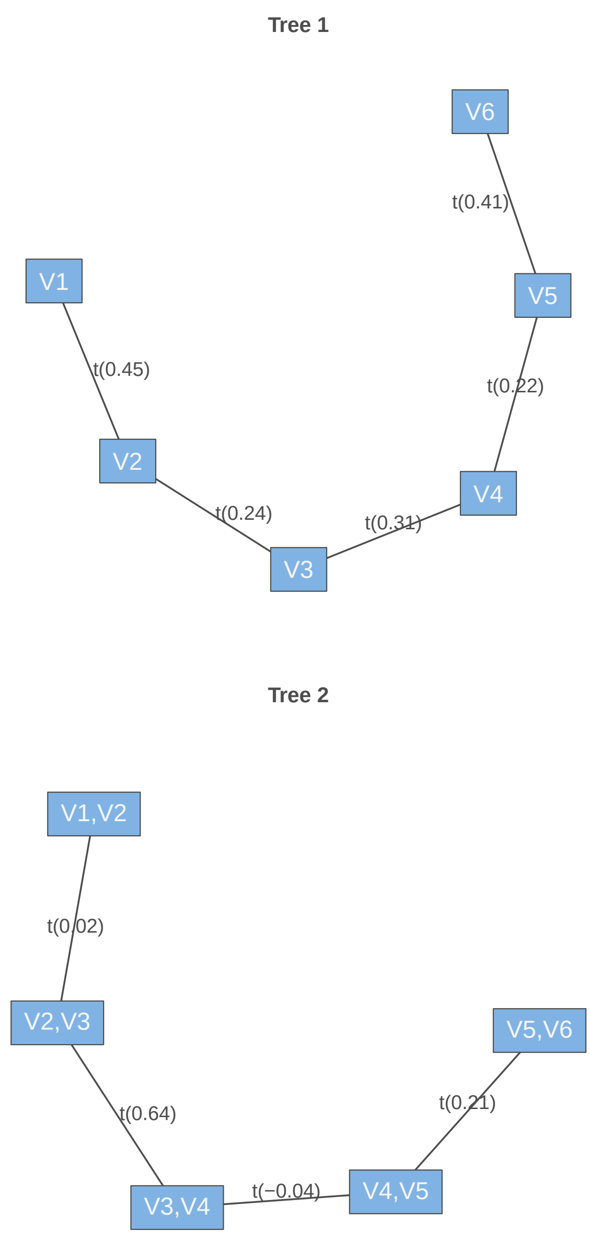

Figure 5: Tree 1 and Tree 2 for the D-Vine specification of the exchange rates dataset.

with different levels of price margins.

Correspondingly, the C-Vine and D-Vine copula parameter estimates, AIC, BIC, log-likelihood, Kendall's τ, the upper (λ_U) and lower (λ_L) tail dependence

Table 6: Parameter estimates for six-dimensional R-Vine copula

Tree	Pair	Copula	$\hat{\rho}$	$\hat{\nu}$	$\hat{\tau}$	λ_U	λ_L
1	*AUD,CAD*	t	0.60 (0.01)	10.90 (1.96)	0.41	0.11	0.11
	EUR,GBP	t	0.65 (0.01)	10.48 (1.69)	0.45	0.15	0.15
	CHF,JPY	t	0.47 (0.01)	5.52 (0.54)	0.31	0.17	0.17
	EUR,CHF	t	0.87 (0.00)	2.62 (0.16)	0.67	0.64	0.64
	AUD,EUR	t	0.56 (0.01)	7.71 (0.94)	0.38	0.15	0.15
2	*EUR,CAD;AUD*	t	0.16 (0.02)	17.56 (4.79)	0.10	0.00	0.00
	AUD,GBP;EUR	t	0.22 (0.01)	13.72 (2.77)	0.14	0.01	0.01
	EUR,JPY;CHF	t	−0.08 (0.02)	11.21 (1.86)	−0.05	0.00	0.00
	AUD,CHF;EUR	$G270$	−1.05 (0.01)	–	0.05	–	–
3	*GBP,CAD;EUR,AUD*	t	0.08 (0.02)	23.40 (8.00)	0.05	0.00	0.00
	CHF,GBP;AUD,EUR	t	0.07 (0.02)	19.92 (5.57)	0.04	0.00	0.00
	AUD,JPY;EUR,AUD	t	0.06 (0.02)	8.29 (1.11)	0.04	0.02	0.02
4	*CHF,CAD;GBP,EUR,AUD*	F	−0.02 (0.08)	–	−0.20	–	–
	JPY,GBP;CHF,AUD,EUR	t	0.01 (0.02)	20.93 (6.46)	0.01	0.00	0.00
5	*JPY,CAD;CHF,GBP,EUR,AUD*	F	−0.05 (0.08)	–	−0.05	–	–

For the Student's-t copula, the first and second parameter components are the correlation coefficient and degrees of freedom parameters respectively.

values are also presented in Tables 7 and 8 respectively. Similar to the first tree of the R-Vine copula, the parameter estimates of the unconditional and conditional pair-copulas are significant at 1% level of significance and Student-t copula is again selected as the most appropriate fit for most pairs of currency returns in the first and subsequent tree of both C-Vine and D-Vine, signifying the presence of symmetric tail dependence. The Kendall's τ values are also highest in the first tree and reduces significantly in magnitude in the higher levels starting from the third tree. The lower and upper tail dependence values are also significant for the first two levels of the regular copulas.

4.4 Comparison of the Vine copulas

For purposes of comparison, we explore the benefits of using different regular vine copula specifications (R-Vine, C-Vine and D-Vine), with the pair-copula families chosen independently from a variety of bivariate copulas and evaluate their overall performance compared to regular vines fitting all pair-copulas with only Student-t or Gaussian bivariate copulas. The six different regular vine specifications compared include;

- R-Vine specification with pair-copulas selected independently from a list of bivariate copula families listed in Section 3.

Table 7: Parameter estimates for six-dimensional C-Vine copula

Tree	Pair	Copula	$\hat{\rho}$	$\hat{\nu}$	$\hat{\tau}$	λ_U	λ_L
1	EUR,CHF	t	0.87 (0.00)	2.62 (0.16)	0.67	0.64	0.64
	EUR,GBP	t	0.65 (0.01)	10.48 (1.69)	0.45	0.15	0.15
	EUR,JPY	t	0.37 (0.01)	4.67 (0.39)	0.24	0.16	0.16
	EUR,CAD	t	0.44 (0.01)	9.77 (1.57)	0.29	0.06	0.06
	EUR,AUD	t	0.56 (0.01)	7.71 (0.94)	0.38	0.15	0.15
2	AUD,CHF;EUR	$G270$	−1.05 (0.01)	–	−0.05	–	–
	AUD,GBP;EUR	t	0.22 (0.01)	13.72 (2.77)	0.14	0.01	0.01
	AUD,JPY;EUR	t	0.04 (0.02)	6.59 (0.71)	0.03	0.03	0.03
	AUD,CAD;EUR	t	0.47 (0.01)	10.95 (1.92)	0.31	0.06	0.06
3	JPY,CHF;AUD,EUR	t	0.32 (0.01)	20.27 (5.68)	0.20	0.00	0.00
	JPY,GBP;AUD,EUR	t	0.03 (0.02)	14.49 (3.24)	0.02	0.00	0.00
	JPY,CAD;AUD,EUR	F	−0.53 (0.08)	–	−0.06	–	–
4	GBP,CHF;JPY,AUD,EUR	F	0.44 (0.08)	–	0.05	–	–
	CAD,GBP;JPY,AUD,EUR	t	0.09 (0.02)	23.52 (8.00)	0.05	0.00	0.00
5	CAD,CHF;GBP,JPY,AUD,EUR	F	−0.06 (0.08)	–	−0.01	–	–

Table 8: Parameter estimates for six-dimensional D-Vine copula

Tree	Pair	Copula	$\hat{\rho}$	$\hat{\nu}$	$\hat{\tau}$	λ_U	λ_L
1	CAD,AUD	t	0.60	11.02	0.41	0.11	0.11
	CHF,CAD	t	0.34	8.61	0.22	0.06	0.06
	JPY,CHF	t	0.47	5.52	0.31	0.17	0.17
	EUR,JPY	t	0.37	4.67	0.24	0.16	0.16
	GBP,EUR	t	0.65	10.63	0.45	0.14	0.14
2	CHF,AUD;CAD	t	0.32	13.71	0.21	0.01	0.01
	JPY,CAD;CHF	t	−0.06	11.19	−0.04	0.00	0.00
	EUR,CHF;JPY	t	0.85	3.00	0.64	0.59	0.59
	GBP,JPY;EUR	t	0.04	9.70	0.02	0.01	0.01
3	JPY,AUD;CHF,CAD	t	0.07	11.58	0.04	0.01	0.01
	EUR,CAD;JPY,CHF	t	0.27	28.52	0.18	0.00	0.00
	GBP,CHF;EUR,JPY	t	0.04	25.17	0.03	0.00	0.00
4	EUR,AUD;JPY,CHF,CAD	F	1.84	0.00	0.20	–	–
	GBP,CAD;EUR,JPY,CHF	t	0.18	15.50	0.11	0.00	0.00
5	GBP,AUD;EUR,JPY,CHF,CAD	BB8	2.14	0.52	0.01	–	–

- R-Vine specification with all pair-copula selected as Student-t copula. The student-t copula models both the lower and upper tail dependence.
- R-Vine specification with all pair-copula selected as Gaussian copula. This corresponds to the elliptical multivariate normal copula.

- C-Vine specification with pair-copulas selected independently from a list of bivariate copula families (as above).

- D-Vine specification with pair-copulas selected independently from a list of bivariate copula families (as above).

The [39] and [40] tests are used to test the hypothesis that the regular vine copula specification with independently selected pair-copulas compared to the other regular copula specifications fit the data more appropriately. In particular, we apply AIC, BIC, the log-likelihood and results of the Vuong and Clarke tests to select among six different regular vine specifications. Table 9 reports the log-likelihoods, AIC, BIC, numbers of parameters, number of fitted pair-copulas and goodness-of-fit tests for all regular vine copula specifications. The first row shows AIC, BIC, the log likelihood of the selected regular vine copula specification and number of estimated parameters for the selected pair-copulas families. The second rows lists the number of different pair-copula families selected. The third and fourth rows give the results of the Vuong and Clarke tests (corresponding p-value in parenthesis), with and exclusive of Akaike and Schwarz corrections, respectively, testing the regular vine specification with pair-copulas selected independently against the other specifications represented in the particular columns. The different pair-copula families for the fitted regular vine specifications are distributed as follows; for the R-Vine mixed copula; 12 Student-t, 2 Frank and 1 G270 copulas requiring 27 parameter estimates. The R-Vine (Student-t) has 30 parameters to be estimated for the 15 Student-t copulas while the R-Vine (Gaussian) has 15 parameters for the 15 Gaussian copulas. When the selection for a pairwise independent pair-copula is permitted, the results remain the same as regular mixed vine model. For the C-Vine mixed copula there are; 11 Student-t, 3 Frank and G270 pair-copulas. Finally, for the D-Vine mixed copula there are; 13 Student-t, 1 Frank and 1 BB8 pair-copulas. The Student-t copula is selected the highest number of time for all the regular vine specification. Note that the number of parameters to be estimated can reduce when using different copulas. The most appropriate regular vine copula specification can be selected based on the AIC, BIC and the log-likelihood values. From Table 9, the log-likelihood value of the regular vine (R-Vine) is 7325.70, the log-likelihood value of C-Vine specification is 7335.86 and the log-likelihood value of D-Vine specification is 7263.40.

Table 9: AIC, BIC, Log-Likelihoods, numbers of parameters, and of copulas for all regular vine specifications as well as results of the Vuong and Clarke tests for all copula specifications with corresponding p-values in parentheses

		R-Vine mixed	R-Vine all t	R-Vine all Gauss	R-Vine Indep.	C-Vine mixed	D-Vine mixed
AIC		−14597.40	−14577.79	−13102.33	−14597.40	−14619.73	−14468.80
BIC		−14424.21	−14385.36	−13006.11	−14424.21	−14452.96	−14282.79
Log Likelihood		7325.70	7318.89	6566.16	7325.70	7335.86	7263.40
No. of parameters		27	30	15	27	26	28
	Indep	0	0	0	0	0	0
	Gaussian	0	0	15	0	0	0
	Student t	12	15	0	12	11	13
No. of	Frank	2	0	0	2	3	1
copulas	G270	1	0	0	1	1	0
	BB8	0	0	0	0	0	1
	No correction		1.233	10.918	0.000	−1.319	3.820
	p-value		(0.218)	(0.000)	(1.000)	(0.187)	(0.000)
Vuong	Akaike corr.		1.776	10.745	0.000	−1.448	3.943
tests	p-value		(0.076)	(0.000)	(1.000)	(0.148)	(0.000)
	Schwarz corr.		3.519	10.192	0.000	−1.864	4.336
	p-value		(0.000)	(0.000)	(1.000)	(0.062)	(0.000)
	No correction		2149	2885	0.000	2382	2293
	p-value		(0.002)	(0.000)	(1.000)	(0.000)	(0.271)
Clarke	Akaike corr.		2184	2874	0.000	2374	2303
tests	p-value		(0.035)	(0.000)	(1.000)	(0.000)	(0.162)
	Schwarz corr.		2267	2846	0.000	2350	2320
	p-value		(0.743)	(0.000)	(1.000)	(0.005)	(0.057)

The table presents AIC, BIC, log-likelihoods, numbers of parameters, and of copulas for all models as well as results of the Vuong and Clarke tests (test statistics and p-values in parentheses) comparing the R-vine model with mixed copulas to all other models. The positive values of Vuong test statistics indicate that the test favours the R-vine model over the respective alternative model

Thus, the C-Vine copula specification is the most appropriate fit, with minor discrepancy compared to the R-Vine copula specification, demonstrating evidence of both upper and lower tail dependence. However, when comparing an R-Vine specification with independently selected copulas to the other R-Vine specifications like the C-Vine and D-Vine, the likelihood cannot be used since the models are non-nested. The Vuong and Clarke's likelihood-ratio tests are used to compare non-nested models.

The R-Vine copula specifications are compared here to determine which specification fits the currency exchange data better compared to the others. The null hypothesis is “The mixed R-Vine copula specification fits the data more appropriately than all other copula specifications under consideration". Based on the Vuong and Clarke tests results it is observed that the regular vine-copula specification with independently selected pair-copulas is favoured over the D-Vine specification and the multivariate Gaussian copula. In this cases the hypothesis cannot be reject given the corresponding high p-value. There is no statistical significant difference between regular vine copula specification with independently selected pair-copulas to the independent R-Vine. The negative test statistic values for the Voung tests and the corresponding high p-value confirms that the C-Vine specification is favoured over the regular vine copula specification with independently selected pair-copulas. This is as a result of the dependence structure exhibited by currency markets which are most likely to experience a crash and boom together thus, concluding that currency markets are integrated due to the nature of the global financial systems.

5 Conclusion

Understanding and modelling high-dimensional dependence behaviour between currency exchange rates can be a challenging task. The vine copula-based approach offers superior flexibility that facilitates modelling complex asymmetric dependence patterns common in multivariate financial variables. In this paper, a general regular vine copula model selection approach is provided to choose sequentially the vine tree structure, the copula families for each pair-copula term from a wide variety of bivariate copula classes and estimating their corresponding parameters. The selection approach employs [41] sequential algorithm which determines a maximum spanning tree and the absolute Kendall's tau empirical values are used as weights. The results of the sequential selection procedure provide a tree structure, corresponding pair-copula types and parameter estimates for all the regular vine-copula specifications. For purposes of comparison, we explore the benefits of using different regular vine copula specifications using the Voung and Clarke tests. The C-Vine copula specification is favoured over all the other regular vine copula specifications in modelling the dependence dynamics between currency exchange rates. In future research, we propose to implement the matrix representation of an R-vine copula specification and also investigate further the model selection problem to include the choice of other weights other than Kendall's tau. The regular vine-copula based model can also be used to explore the practical application of R-Vine copulas in estimating portfolio Value-at-Risk.

References

[1] C. Donnelly and P. Embrechts, The devil is in the tails: actuarial mathematics and the subprime mortgage crisis, *ASTIN Bulletin: The Journal of the IAA*, **40**(1), (2010), 1–33.

[2] M. Sklar, Fonctions de repartition an dimensions et leurs marges, *Publ. Inst. Statist. Univ. Paris*, **8**, (1959), 229–231.

[3] A. J. Patton, Modelling asymmetric exchange rate dependence, *International economic review*, **47**, (2006), 527–556.

[4] H. Joe, Harry, *Multivariate models and multivariate dependence concepts*, CRC Press, 1997.

[5] R. Nelsen, An introduction to copulas, 2nd edition, *Springer International*, New York, 2006.

[6] C. Genest, B. Rémillard, and D. Beaudoin, Goodness-of-fit tests for copulas: A review and a power study, *Insurance: Mathematics and economics*, **44**, (2009), 199–213.

[7] K. Aas, C. Czado, A. Frigessi, & H. Bakken, Pair-copula constructions of multiple dependence, *Insurance: Mathematics and economics*, **44**(2), (2009), 182–198.

[8] K. Aas and D. Berg, Models for construction of multivariate dependence: a comparison study, *The European Journal of Finance*, **15**, (2009), 639–659.

[9] M. Fischer, C. Köck, S. Schlüter, and F. Weigert, An empirical analysis of multivariate copula models,*Quantitative Finance*, **9**, (2009), 839–854.

[10] D. Kurowicka and R. M. Cooke, Uncertainty analysis with high dimensional dependence modelling, John Wiley & Sons, 2006.

[11] C. Czado, S. Jeske, and M. Hofmann, Selection strategies for regular vine copulae, *Journal de la Société Fracaise de Statistique*, **154**, (2013), 174–191.

[12] T. M. Erhardt, C. Czado, and U. Schepsmeier, R-vine models for spatial time series with an application to daily mean temperature, *Biometrics*, **71**, (2015), 323–332.

[13] I. H. Haff and J. Segers, Nonparametric estimation of pair-copula constructions with the empirical pair-copula, *Computational Statistics & Data Analysis*, **84**, (2015), 1–13.

[14] M. Killches and C.Czado, Block-maxima of vines, *Extreme Value Modelling and Risk Analysis: Methods and Applications*, (2015), 109-130.

[15] G. Pereira and Á. Veiga, Par (p)-vine copula based model for stochastic streamflow scenario generation, *Stochastic Environmental Research and Risk Assessment*, **32**, (2018), 833–842.

[16] N. Barthel, C. Geerdens, M. Killiches, P. Janssen, and C. Czado, Vine copula based inference of multivariate event time data, *arXiv preprint*, (2016).

[17] C. Schellhase and F. Spanhel, Estimating non-simplified vine copulas using penalized splines, *Statistics and Computing*, **28**, (2018), 387–409.

[18] R. M. Cooke, H. Joe, and B. Chang, Vine regression, (2015).

[19] D. Schirmacher and E. Schirmacher, Multivariate dependence modeling using pair-copulas, *tech. rep.*, Citeseer, 2008.

[20] L. Chollete, A. Heinen, and A. Valdesogo, Modeling international financial returns with a multivariate regime-switching copula, *Journal of financial econometrics*, **7**, (2009), 437–480.

[21] K. Aas, Pair-copula constructions for financial applications: A review, *Econometrics*, **4**, (2016), 43.

[22] J. Dissmann, E. C. Brechmann, C. Czado, and D. Kurowicka, Selecting and estimating regular vine copulae and application to financial returns, *Computational Statistics & Data Analysis*, **59**, (2013), 52–69.

[23] L. Maya, R. Albeiro, J. E. Gomez-Gonzalez, and L. F. Melo Velandia, Latin american exchange rate dependencies: A regular vine copula approach, *Contemporary Economic Policy*, **33** (2015), 535–549.

[24] C. Almeida, C. Czado, and H. Manner, Modeling high-dimensional timevarying dependence using dynamic d-vine models, *Applied Stochastic Models in Business and Industry*, **32** (2016), 621–638.

[25] D. Kraus and C. Czado, Growing simplified vine copula trees: improving dissmann's algorithm, *arXiv preprint arXiv:1703.05203*, (2017).

[26] D. E. Allen, M. McAleer, and A. K. Singh, Risk measurement and risk modelling using applications of vine copulas, *Sustainability*, **9**, (2017), 1762.

[27] M. Geidosch and M. Fischer, Application of vine copulas to credit portfolio risk modeling, *Journal of Risk and Financial Management*, **9**, (2016), 4.

[28] H. Joe, *Multivariate models and multivariate dependence concepts*, CRC Press, 2014.

[29] H. Joe and J.J. Xu, The estimation method of inference functions for margins for

multivariate models, (1996).

[30] Aas, K. Pair-copula constructions for nancial applications: A review. *Econometrics*, **4**(4), (2016), 43.

[31] T. Bedford and R. M. Cooke, Probability density decomposition for conditionally dependent random variables modeled by vines, *Annals of Mathematics and Artificial intelligence*, **32**, (2001), 245–268.

[32] T. Bedford and R. M. Cooke, Vines: A new graphical model for dependent random variables, *Annals of Statistics*, (2002), 1031–1068.

[33] D. Kurowicka and R. M. Cooke, Uncertainty analysis with high dimensional dependence modelling, John Wiley & Sons, 2006.

[34] L. Gruber and C. Czado, Sequential bayesian model selection of regular vine copulas, *Bayesian Analysis*, **10**(4), (2015), 937–963.

[35] D. B. Nelson and C. Q. Cao, Inequality constraints in the univariate garch model, *Journal of Business & Economic Statistics*, **10** (1992), 229–235.

[36] J. Dissmann, E. C. Brechmann, C. Czado and D. Kurowicka, Selecting and estimating regular vine copulae and application to financial returns, *Computational Statistics & Data Analysis*, **59** (2013), 52–69.

[37] U. Schepsmeier, Maximum likelihood estimation of c-vine pair-copula constructions on bivariate copulas from different families, *Center of Mathematical Sciences, Munich University of Technology, Garching bei München*, (2010).

[38] C. Czado, U. Schepsmeier, and A. Min, Maximum likelihood estimation of mixed c-vines with application to exchange rates, *Statistical Modelling*, **12** (2012), 229–255.

[39] Q. H. Vuong, Likelihood ratio tests for model selection and non-nested hypotheses, *Econometrica: Journal of the Econometric Society*, (1989), 307–333.

[40] K. A. Clarke, A simple distribution-free test for nonnested model selection, *Political Analysis* **15** (2007), 347–363.

[41] J. Dissmann, E. C. Brechmann, C. Czado, and D. Kurowicka, Selecting and estimating regular vine copulae and application to financial returns, *Computational Statistics & Data Analysis*, **59** (2013), 52–69.

[42] E. C. Brechmann and C. Czado, Risk management with high-dimensional vine copulas: An analysis of the euro stoxx 50, *Statistics & Risk Modeling*, **30** (2013), 307–342.